Certified Branch-and-Bound MaxSAT Solving

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Lund

VUB
ARTIFICIAL INTELLIGENCE RESEARCH GROUP

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- Software testing doesn't suffice to resolve this problem
- Formal verification techniques cannot deal with complexity of modern solvers [BHI+23]

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Design certifying algorithms [ABM⁺11, MMNS11] that

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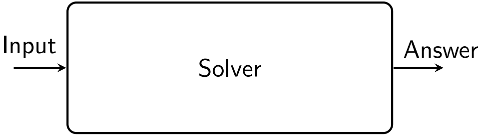
Proof logging should be done

- with **minimal overhead**
- without changing a **solver's reasoning**

CERTIFIED RESULTS WITH PROOF LOGGING



Workflow:

1. Run solver on problem input

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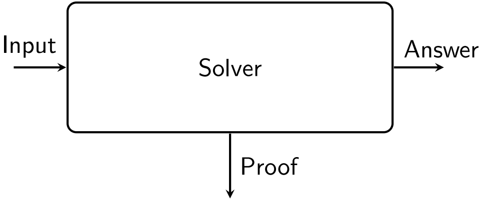

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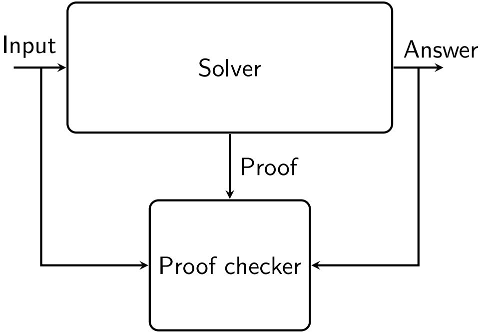
3. Feed input + answer + proof to proof checker



CERTIFIED RESULTS WITH PROOF LOGGING

Workflow:

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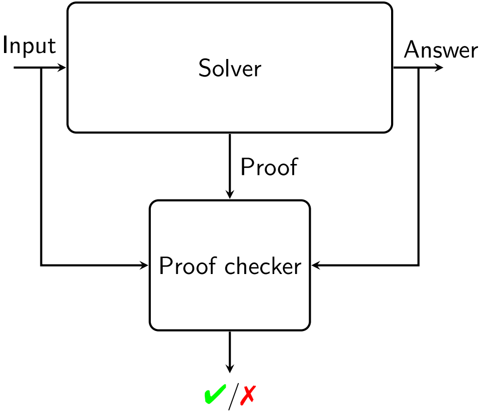
4. Check if proof checker says answer is correct



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But efficient proof logging has remained out of reach for other paradigms,
e.g. Maximum Satisfiability (MaxSAT)

OUTLINE OF THIS PRESENTATION

- MaxSAT and how to proof log it
- An introduction to the VeriPB proof system.
- MaxCDCL: Branch-and-Bound with clause learning
- Unweighted MaxCDCL revisited with literal unlocking
- Solution-Improving Constraint using Binary Decision Diagram (BDD) encoding
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PRELIMINARIES

Example:

$$F = \{x_1 \vee x_2,\ x_2 \vee x_3,\ x_1 \vee \overline{x_2} \vee x_3\}$$

- Boolean variable: x
- Assignment α: assigns variables true $(= 1)$ or false $(= 0)$
- Literal l: variable x (satisfied if $\alpha(x) = 1$) or its negation $\overline{x}$ (satisfied if $\alpha(x) = 0$)
- Clause C: Disjunction of literals $l_1 \vee \cdots \vee l_k$
  (C is satisfied by α if at least one literal in C is assigned true)
- Propositional formula in CNF: $F = C_1 \wedge \cdots \wedge C_n$
  (F is satisfied if all clauses C_i are satisfied)

THE MAXIMUM SATISFIABILITY PROBLEM

Example:

$F = \{x_1 \vee x_2,\ x_2 \vee x_3,\ x_1 \vee \overline{x_2} \vee x_3\}$

$\mathcal{O} = x_1 + x_2 + x_3$ *(min)*

Optimization variant of Satisfiability Problem.

A MaxSAT-instance is a tuple $(F, \mathcal{O})$ with:

- F a propositional formula
- $\mathcal{O}$ an integer linear objective over Boolean variables

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An optimal solution is a solution such that no other solution has lower objective value.

PROOF SYSTEMS FOR MAXSAT REASONING

Proof systems for MaxSAT are studied theoretically for proof complexity

- MaxSAT resolution [LH05, HL06, BLM06, BLM07]
- Tableaux reasoning [LMS16, LCH+22, LM22]
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No certified state-of-the-art MaxSAT solver using native proof system!

MAXSAT SOLVERS

Four main categories:

- Branch-and-Bound
- Solution-Improving
- Core-Guided
- Implicit Hitting Set

Different reasoning techniques!

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Only proves answer correct, not reasoning within solver!

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- Branch-and-Bound with clause learning
  - MaxCDCL – **This talk** (and a little bit about *Solution Improving Search*)

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VeriPB: A PROOF SYSTEM FOR PSEUDO-BOOLEAN OPTIMIZATION

VeriPB is a proof system for pseudo-Boolean optimization [BGMN22, EGMN20].

A pseudo-Boolean constraint is a 0–1 integer linear inequalities:

$$\sum_i a_i \ell_i \geq A$$

- $a_i, A \in \mathbb{Z}$
- literals ℓ_i: x_i or $\overline{x}_i$ (where $x_i + \overline{x}_i = 1$)

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  - generalisation of the RAT-rule [BT19]
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  - e.g. introducing "fresh" reification variables, such as $r \Leftrightarrow (\sum_i a_i l_i \geq A)$
- Support for Optimisation [BGMN22]
  - allows deriving solution-improving constraints ($\mathcal{O} < v^*$)
  - proving optimality by contradiction

PSEUDO-BOOLEAN REASONING: CUTTING PLANES [CCT87]

Input/model axioms

From the input

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Input/model axioms From the input

Literal axioms

$$\frac{}{\ell_i \geq 0}$$

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Input/model axioms From the input

Literal axioms

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Addition

$$\frac{\sum_i a_i \ell_i \geq A \qquad \sum_i b_i \ell_i \geq B}{\sum_i (a_i + b_i)\ell_i \geq A + B}$$

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Multiplication for any $c \in \mathbb{N}^+$

$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i c a_i \ell_i \geq cA}$$

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Division for any $c \in \mathbb{N}^+$
(assumes normalized form)

$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i \left\lceil \frac{a_i}{c} \right\rceil \ell_i \geq \left\lceil \frac{A}{c} \right\rceil}$$

CUTTING PLANES TOY EXAMPLE

$$w + 2x + y \geq 2$$

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Multiply by 2 $\dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}$

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$$\text{Multiply by } 2 \; \dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}$$

$$\text{Add} \; \dfrac{2w + 4x + 2y \geq 4 \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9}$$

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$$\text{Add}\ \dfrac{\text{Multiply by 2}\ \dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9} \qquad \overline{z} \geq 0$$

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Multiply by 2

$$\frac{\dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9}\ \text{Add} \qquad \frac{\overline{z} \geq 0}{2\overline{z} \geq 0}\ \text{Multiply by } 2$$

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$$\text{Multiply by } 2 \; \dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} \qquad w + 2x + 4y + 2z \geq 5$$

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$$\text{Add} \; \dfrac{3w + 6x + 6y + 2z \geq 9 \qquad 2\overline{z} \geq 0}{3w + 6x + 6y + 2z + 2\overline{z} \geq 9}$$

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Multiply by 2
$$\frac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}$$

$$w + 2x + 4y + 2z \geq 5$$

Add
$$3w + 6x + 6y + 2z \geq 9$$

Multiply by 2
$$\frac{\overline{z} \geq 0}{2\overline{z} \geq 0}$$

Add
$$3w + 6x + 6y + 2 \qquad \geq 9$$

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Multiply by 2 $\dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}$

Add: $\dfrac{2w + 4x + 2y \geq 4 \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9}$

$\dfrac{\overline{z} \geq 0}{2\overline{z} \geq 0}$ Multiply by 2

Add: $\dfrac{3w + 6x + 6y + 2z \geq 9 \qquad 2\overline{z} \geq 0}{3w + 6x + 6y \qquad\qquad \geq 7}$

CUTTING PLANES TOY EXAMPLE

Multiply by 2: from $w + 2x + y \geq 2$ derive $2w + 4x + 2y \geq 4$

Add: $2w + 4x + 2y \geq 4$ and $w + 2x + 4y + 2z \geq 5$ gives $3w + 6x + 6y + 2z \geq 9$

Multiply by 2: from $\overline{z} \geq 0$ derive $2\overline{z} \geq 0$

Add: $3w + 6x + 6y + 2z \geq 9$ and $2\overline{z} \geq 0$ gives $3w + 6x + 6y \geq 7$

Divide by 3: from $3w + 6x + 6y \geq 7$ derive $w + 2x + 2y \geq 2\frac{1}{3}$

CUTTING PLANES TOY EXAMPLE

Multiply by 2:
$$\frac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}$$

Multiply by 2:
$$\frac{\overline{z} \geq 0}{2\overline{z} \geq 0}$$

Add:
$$\frac{2w + 4x + 2y \geq 4 \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9}$$

Add:
$$\frac{3w + 6x + 6y + 2z \geq 9 \qquad 2\overline{z} \geq 0}{3w + 6x + 6y \geq 7}$$

Divide by 3:
$$\frac{3w + 6x + 6y \geq 7}{w + 2x + 2y \geq 3}$$

CUTTING PLANES TOY EXAMPLE

Multiply by 2:
$$\frac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}$$

Add $w + 2x + 4y + 2z \geq 5$:
$$\frac{2w + 4x + 2y \geq 4 \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9}$$

Multiply by 2:
$$\frac{\overline{z} \geq 0}{2\overline{z} \geq 0}$$

Add:
$$\frac{3w + 6x + 6y + 2z \geq 9 \qquad 2\overline{z} \geq 0}{3w + 6x + 6y \geq 7}$$

Divide by 3:
$$\frac{3w + 6x + 6y \geq 7}{w + 2x + 2y \geq 3}$$

Naming constraints by integers and literal axioms by the literal involved (with $\sim$ for negation) as

$$\begin{aligned} \text{Constraint 1} &\doteq 2x + y + w \geq 2 \\ \text{Constraint 2} &\doteq 2x + 4y + 2z + w \geq 5 \\ \sim\text{z} &\doteq \overline{z} \geq 0 \end{aligned}$$

CUTTING PLANES TOY EXAMPLE

Multiply by 2: $\dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}$

Add $2w + 4x + 2y \geq 4$ and $w + 2x + 4y + 2z \geq 5$: $3w + 6x + 6y + 2z \geq 9$

Multiply by 2: $\dfrac{\overline{z} \geq 0}{2\overline{z} \geq 0}$

Add $3w + 6x + 6y + 2z \geq 9$ and $2\overline{z} \geq 0$: $3w + 6x + 6y \geq 7$

Divide by 3: $\dfrac{3w + 6x + 6y \geq 7}{w + 2x + 2y \geq 3}$

Naming constraints by integers and literal axioms by the literal involved (with ~ for negation) as

$$\text{Constraint 1} \doteq 2x + y + w \geq 2$$
$$\text{Constraint 2} \doteq 2x + 4y + 2z + w \geq 5$$
$$\sim\text{z} \doteq \overline{z} \geq 0$$

such a calculation is written in the proof log in reverse Polish notation as

```
p 1 2 * 2 + ~z 2 * + 3 d
```

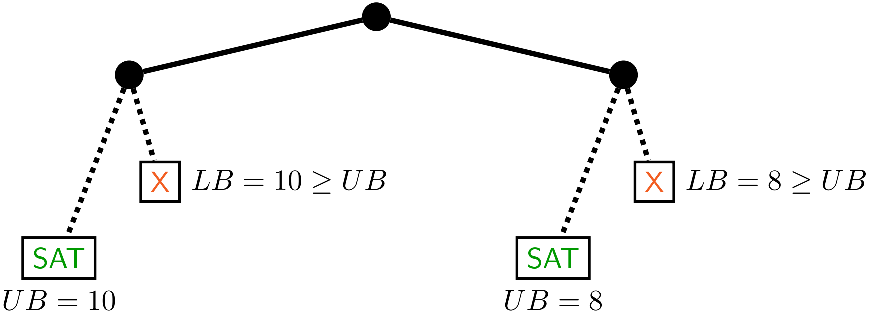
OUTLINE OF THIS PRESENTATION

- MaxSAT and how to proof log it
- An introduction to the VeriPB proof system.
- MaxCDCL: Branch-and-Bound with clause learning
- Unweighted MaxCDCL revisited with literal unlocking
- Solution-Improving Constraint using Binary Decision Diagram (BDD) encoding
- Conclusions & Future work

BRANCH AND BOUND

Branch and Bound:

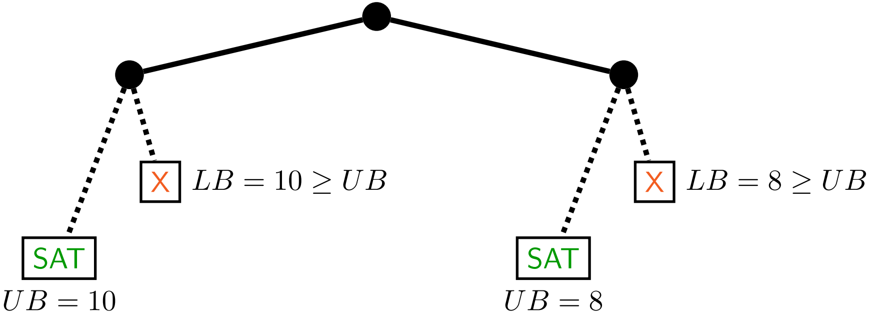
- Explore the search tree for solutions
- Update Upper Bound UB when solution with better objective value is found
- Underestimate Lower Bound LB at every node
- Prune branch when conflict found or when $LB \geq UB$



MAXCDCL AS BRANCH AND BOUND

Branch and Bound in MaxCDCL:

- Explore the search tree for solutions
- Update Upper Bound UB when solution with better objective value is found
- Underestimate Lower Bound LB at every node **using lookahead with UP**
- Prune branch when conflict found or when $LB \geq UB$ **and learn a clause**



MAXCDCL AS CDCL GENERALIZATION

MaxCDCL conflicts:

- **Hard conflict:**
  - A clause is falsified
- **Soft conflict:**
  - (underestimated) LB $\geq$ UB

MAXCDCL AS CDCL GENERALIZATION

MaxCDCL conflicts:

- **Hard conflict:**
  - A clause is falsified
- **Soft conflict:**
  - (underestimated) LB $\geq$ UB

In both cases: conflict analysis for learning new clause (CDCL)

LOOKAHEAD: LB UNDERESTIMATION (UNWEIGHTED CASE)

Lookahead with UP for underestimating LB:

1. Assume unassigned objective literals false and apply UP until:
   - A hard clause is falsified
   - Or a not yet assigned objective literal is assigned 1
2. We have found a **local** unsatisfiable core
3. Since unweighted case: Each **disjoint** core increases the LB by 1
4. When $LB \geq UB$, a soft conflict is found

SOFT CONFLICT DETECTION BY EXAMPLE (UNWEIGHTED CASE)

$\mathcal{O}^t = y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 \quad \mathbf{UB} = 3$

Trail: $x_1^d \; \overline{x_2}^p \; x_3^p \; \overline{x_4}^d \; x_5^p \; x_6^p \; x_7^p$

SOFT CONFLICT DETECTION BY EXAMPLE (UNWEIGHTED CASE)

$\mathcal{O}^t = \cancel{y_1} + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8$ $\quad \mathbf{UB} = 3$

Trail: $x_1^d \; \overline{x_2}^p \; x_3^p \; \overline{x_4}^d \; x_5^p \; x_6^p \; x_7^p$

Find one core:

$x_1^d \; \overline{x_2}^p \; x_3^p \; \overline{x_4}^d \; x_5^p \; x_6^p \; x_7^p \; \overline{y}_1^a \; x_9^p \; x_{10}^p$

SOFT CONFLICT DETECTION BY EXAMPLE (UNWEIGHTED CASE)

$\mathcal{O}^t = \cancel{y_1} + \cancel{y_2} + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 \qquad \mathbf{UB} = 3$

Trail: $x_1^d \; \overline{x_2}^p \; x_3^p \; \overline{x_4}^d \; x_5^p \; x_6^p \; x_7^p$

Find one core:

$x_1^d \; \overline{x_2}^p \; x_3^p \; \overline{x_4}^d \; x_5^p \; x_6^p \; x_7^p \; \overline{y}_1^a \; x_9^p \; x_{10}^p \; \overline{y}_2^a \; \overline{x_{11}}^p$

SOFT CONFLICT DETECTION BY EXAMPLE (UNWEIGHTED CASE)

$\mathcal{O}^t = \cancel{y_1} + \cancel{y_2} + \cancel{y_3} + y_4 + y_5 + y_6 + y_7 + y_8 \qquad \mathbf{UB} = 3$

Trail: $x_1^d \; \overline{x_2}^p \; x_3^p \; \overline{x_4}^d \; x_5^p \; x_6^p \; x_7^p$

Find one core:

$x_1^d \; \overline{x_2}^p \; x_3^p \; \overline{x_4}^d \; x_5^p \; x_6^p \; x_7^p \; \overline{y}_1^a \; x_9^p \; x_{10}^p \; \overline{y}_2^a \; \overline{x_{11}}^p \; \overline{y}_3^a$

SOFT CONFLICT DETECTION BY EXAMPLE (UNWEIGHTED CASE)

$\mathcal{O}^t = \cancel{y_1} + \cancel{y_2} + \cancel{y_3} + \cancel{y_4} + y_5 + y_6 + y_7 + y_8 \quad \mathbf{UB} = 3$

Trail: $x_1^d \ \overline{x_2}^p \ x_3^p \ \overline{x_4}^d \ x_5^p \ x_6^p \ x_7^p$

Find one core:

$x_1^d \ \overline{x_2}^p \ x_3^p \ \overline{x_4}^d \ x_5^p \ x_6^p \ x_7^p \ \overline{y}_1^a \ x_9^p \ x_{10}^p \ \overline{y}_2^a \ \overline{x_{11}}^p \ \overline{y}_3^a \ \overline{y}_4^a \ x_{12}^p$ ($\overline{x_{12}} \vee x_{11} \in F$ falsified)

SOFT CONFLICT DETECTION BY EXAMPLE (UNWEIGHTED CASE)

$\mathcal{O}^t = \cancel{y_1} + \cancel{y_2} + \cancel{y_3} + \cancel{y_4} + y_5 + y_6 + y_7 + y_8 \qquad \mathbf{UB} = 3$

Trail: $x_1^d \; \overline{x_2}^p \; x_3^p \; \overline{x_4}^d \; x_5^p \; x_6^p \; x_7^p$

Find one core:

$x_1^d \; \overline{x_2}^p \; x_3^p \; \overline{x_4}^d \; x_5^p \; x_6^p \; x_7^p \; \overline{y_1}^a \; x_9^p \; x_{10}^p \; \overline{y_2}^a \; \overline{x_{11}}^p \; \overline{y_3}^a \; \overline{y_4}^a \; x_{12}^p$ ($\overline{x_{12}} \vee x_{11} \in F$ falsified)

$x_1^d \; \overline{x_2}^p \; x_3^p \; \overline{x_4}^d \; x_5^p \; x_6^p \; x_7^p \; \overline{y_1}^a \; \overline{y_2}^a \; \overline{y_3}^a \; \overline{y_4}^a$ (Assumptions suffice)

SOFT CONFLICT DETECTION BY EXAMPLE (UNWEIGHTED CASE)

$\mathcal{O}^t = \cancel{y_1} + y_2 + y_3 + \cancel{y_4} + y_5 + y_6 + y_7 + y_8 \qquad \mathbf{UB} = 3$

Trail: $x_1^d\ \overline{x_2}^p\ x_3^p\ \overline{x_4}^d\ x_5^p\ x_6^p\ x_7^p$

Find one core:

$x_1^d\ \overline{x_2}^p\ x_3^p\ \overline{x_4}^d\ x_5^p\ x_6^p\ x_7^p\ \overline{y_1}^a\ x_9^p\ x_{10}^p\ \overline{y_2}^a\ \overline{x_{11}}^p\ \overline{y_3}^a\ \overline{y_4}^a\ x_{12}^p$ ($\overline{x_{12}} \vee x_{11} \in F$ falsified)

$x_1^d\ \overline{x_2}^p\ x_3^p\ \overline{x_4}^d\ x_5^p\ x_6^p\ x_7^p\ \overline{y_1}^a\ \overline{y_2}^a\ \overline{y_3}^a\ \overline{y_4}^a$ (Assumptions suffice)

$\overline{x_2}^p\ \overline{x_4}^d\ \overline{y_1}^a\ \overline{y_4}^a$ (Conflict analysis)

SOFT CONFLICT DETECTION BY EXAMPLE (UNWEIGHTED CASE)

$\mathcal{O}^t = \cancel{y_1} + y_2 + y_3 + \cancel{y_4} + y_5 + y_6 + y_7 + y_8 \qquad \mathbf{UB} = 3$

Trail: $x_1^d\ \overline{x_2}^p\ x_3^p\ \overline{x_4}^d\ x_5^p\ x_6^p\ x_7^p$

Find one core:

$x_1^d\ \overline{x_2}^p\ x_3^p\ \overline{x_4}^d\ x_5^p\ x_6^p\ x_7^p\ \overline{y}_1^a\ x_9^p\ x_{10}^p\ \overline{y}_2^a\ \overline{x_{11}}^p\ \overline{y}_3^a\ \overline{y}_4^a\ x_{12}^p$ ($\overline{x_{12}} \vee x_{11} \in F$ falsified)

$x_1^d\ \overline{x_2}^p\ x_3^p\ \overline{x_4}^d\ x_5^p\ x_6^p\ x_7^p\ \overline{y}_1^a\ \overline{y}_2^a\ \overline{y}_3^a\ \overline{y}_4^a$ (Assumptions suffice)

$\overline{x_2}^p\ \overline{x_4}^d\ \overline{y}_1^a\ \overline{y}_4^a$ (Conflict analysis)

Local core:

$\overline{x_2} \wedge \overline{x_4} \wedge \overline{y}_1 \wedge \overline{y}_4 \vdash_{\mathsf{UP}} \square$

$\overline{x_2} \wedge \overline{x_4} \rightarrow y_1 \vee y_4$ (Reasons $\rightarrow$ Core)

SOFT CONFLICT DETECTION BY EXAMPLE (UNWEIGHTED CASE)

$\mathcal{O} = y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 \qquad \mathbf{UB} = 3$

Trail: $x_1^d\ \overline{x_2}^p\ x_3^p\ \overline{x_4}^d\ x_5^p\ x_6^p\ x_7^p$

Found disjoint local cores

Core 1: $\overline{x_2} \wedge \overline{x_4} \rightarrow y_1 \vee y_4$

Core 2: $\overline{x_2} \wedge x_7 \rightarrow y_2 \vee y_3 \vee y_5$

Core 3: $x_1 \wedge \overline{x_4} \wedge x_7 \rightarrow y_6 \vee y_7$

SOFT CONFLICT DETECTION BY EXAMPLE (UNWEIGHTED CASE)

$\mathcal{O} = y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 \qquad \mathbf{UB} = 3$

Trail: $x_1^d \; \overline{x_2}^p \; x_3^p \; \overline{x_4}^d \; x_5^p \; x_6^p \; x_7^p$

Found disjoint local cores

Core 1: $\overline{x_2} \wedge \overline{x_4} \rightarrow y_1 \vee y_4$

Core 2: $\overline{x_2} \wedge x_7 \rightarrow y_2 \vee y_3 \vee y_5$

Core 3: $x_1 \wedge \overline{x_4} \wedge x_7 \rightarrow y_6 \vee y_7$

$x_1 \wedge \overline{x_2} \wedge \overline{x_4} \wedge x_7 \rightarrow (y_1 \vee y_4) \wedge (y_2 \vee y_3 \vee y_5) \wedge (y_6 \vee y_7)$

SOFT CONFLICT DETECTION BY EXAMPLE (UNWEIGHTED CASE)

$\mathcal{O} = y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8$ $\quad \mathbf{UB} = 3$

Trail: $x_1^d \; \overline{x_2}^p \; x_3^p \; \overline{x_4}^d \; x_5^p \; x_6^p \; x_7^p$

Found disjoint local cores

Core 1: $\overline{x_2} \wedge \overline{x_4} \rightarrow y_1 \vee y_4$

Core 2: $\overline{x_2} \wedge x_7 \rightarrow y_2 \vee y_3 \vee y_5$

Core 3: $x_1 \wedge \overline{x_4} \wedge x_7 \rightarrow y_6 \vee y_7$

$x_1 \wedge \overline{x_2} \wedge \overline{x_4} \wedge x_7 \rightarrow (y_1 \vee y_4) \wedge (y_2 \vee y_3 \vee y_5) \wedge (y_6 \vee y_7)$

$x_1 \wedge \overline{x_2} \wedge \overline{x_4} \wedge x_7 \rightarrow LB = 3 \geq 3 = UB$

SOFT CONFLICT DETECTION BY EXAMPLE (UNWEIGHTED CASE)

$\mathcal{O} = y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8$ $\quad \mathbf{UB} = 3$

Trail: $x_1^d\ \overline{x_2}^p\ x_3^p\ \overline{x_4}^d\ x_5^p\ x_6^p\ x_7^p$

Found disjoint local cores

Core 1: $\overline{x_2} \wedge \overline{x_4} \rightarrow y_1 \vee y_4$

Core 2: $\overline{x_2} \wedge x_7 \rightarrow y_2 \vee y_3 \vee y_5$

Core 3: $x_1 \wedge \overline{x_4} \wedge x_7 \rightarrow y_6 \vee y_7$

$x_1 \wedge \overline{x_2} \wedge \overline{x_4} \wedge x_7 \rightarrow (y_1 \vee y_4) \wedge (y_2 \vee y_3 \vee y_5) \wedge (y_6 \vee y_7)$

$x_1 \wedge \overline{x_2} \wedge \overline{x_4} \wedge x_7 \rightarrow LB = 3 \geq 3 = UB$

Soft conflict

Conflicting clause: $\overline{x_1} \vee x_2 \vee x_4 \vee \overline{x_7}$

SOFT CONFLICT DETECTION (WEIGHTED CASE)

Weighted MaxCDCL

- Weight of Local Core $\mathcal{K}$ = smallest coefficient of objective literals in $\mathcal{K}$
- Each objective literal can contribute to many cores
- The total contribution of an objective literal cannot exceed its coefficient

SOFT CONFLICT DETECTION (WEIGHTED CASE)

Weighted MaxCDCL

- Weight of Local Core $\mathcal{K}$ = smallest coefficient of objective literals in $\mathcal{K}$
- Each objective literal can contribute to many cores
- The total contribution of an objective literal cannot exceed its coefficient

$\mathcal{O}^t = 7y_1 + 2y_2 + 1y_3 + 1y_4 + 1y_5 + 4y_6 + 1y_7 + 3y_8 \quad \mathbf{UB} = 4$

Trail: $x_1^d\ \overline{x_2}^p\ x_3^p\ \overline{x_4}^d\ x_5^p\ x_6^p\ x_7^p$

Found local cores

SOFT CONFLICT DETECTION (WEIGHTED CASE)

Weighted MaxCDCL

- Weight of Local Core $\mathcal{K}$ = smallest coefficient of objective literals in $\mathcal{K}$
- Each objective literal can contribute to many cores
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$\mathcal{O}^t = 7y_1 + 2y_2 + 1y_3 + 1y_4 + 1y_5 + 4y_6 + 1y_7 + 3y_8 \quad \mathbf{UB} = 4$

Trail: $x_1^d\ \overline{x_2}^p\ x_3^p\ \overline{x_4}^d\ x_5^p\ x_6^p\ x_7^p$

Found local cores

Core 1: $\overline{x_2} \wedge \overline{x_4} \rightarrow y_1 \vee y_2$ (weight 2)

SOFT CONFLICT DETECTION (WEIGHTED CASE)

Weighted MaxCDCL

- Weight of Local Core $\mathcal{K}$ = smallest coefficient of objective literals in $\mathcal{K}$
- Each objective literal can contribute to many cores
- The total contribution of an objective literal cannot exceed its coefficient

$\mathcal{O}^t = \not{7}\ 5y_1 + \not{2}\ 0y_2 + 1y_3 + 1y_4 + 1y_5 + 4y_6 + 1y_7 + 3y_8 \quad \mathbf{UB} = 4$

Trail: $x_1^d\ \overline{x_2}^p\ x_3^p\ \overline{x_4}^d\ x_5^p\ x_6^p\ x_7^p$

Found local cores

Core 1: $\overline{x_2} \land \overline{x_4} \rightarrow y_1 \lor y_2$ (weight 2)

SOFT CONFLICT DETECTION (WEIGHTED CASE)

Weighted MaxCDCL

- Weight of Local Core $\mathcal{K}$ = smallest coefficient of objective literals in $\mathcal{K}$
- Each objective literal can contribute to many cores
- The total contribution of an objective literal cannot exceed its coefficient

$\mathcal{O}^t = \not{7}\ 5y_1 + \not{2}\ 0y_2 + 1y_3 + 1y_4 + 1y_5 + 4y_6 + 1y_7 + 3y_8 \quad \textbf{UB} = 4$

Trail: $x_1^d\ \overline{x_2}^p\ x_3^p\ \overline{x_4}^d\ x_5^p\ x_6^p\ x_7^p$

Found local cores

Core 1: $\overline{x_2} \wedge \overline{x_4} \rightarrow y_1 \vee y_2$ (weight 2)

Core 2: $x_3 \wedge \overline{x_4} \rightarrow y_1 \vee y_5$ (weight 1)

SOFT CONFLICT DETECTION (WEIGHTED CASE)

Weighted MaxCDCL

- Weight of Local Core $\mathcal{K}$ = smallest coefficient of objective literals in $\mathcal{K}$
- Each objective literal can contribute to many cores
- The total contribution of an objective literal cannot exceed its coefficient

$\mathcal{O}^t = \not{7}\ \not{5}\ 4y_1 + \not{2}\ 0y_2 + 1y_3 + 1y_4 + \not{1}\ 0y_5 + 4y_6 + 1y_7 + 3y_8 \quad \mathbf{UB} = 4$

Trail: $x_1^d\ \overline{x_2}^p\ x_3^p\ \overline{x_4}^d\ x_5^p\ x_6^p\ x_7^p$

Found local cores

Core 1: $\overline{x_2} \wedge \overline{x_4} \rightarrow y_1 \vee y_2$ (weight 2)

Core 2: $x_3 \wedge \overline{x_4} \rightarrow y_1 \vee y_5$ (weight 1)

SOFT CONFLICT DETECTION (WEIGHTED CASE)

Weighted MaxCDCL

- Weight of Local Core $\mathcal{K}$ = smallest coefficient of objective literals in $\mathcal{K}$
- Each objective literal can contribute to many cores
- The total contribution of an objective literal cannot exceed its coefficient

$\mathcal{O}^t = \not{7}\ \not{5}\ \not{4}\ 1y_1 + \not{2}\ 0y_2 + 1y_3 + 1y_4 + \not{1}\ 0y_5 + \not{4}\ 1y_6 + 1y_7 + \not{3}\ 0y_8 \quad \mathbf{UB} = 4$

Trail: $x_1^d\ \overline{x_2}^p\ x_3^p\ \overline{x_4}^d\ x_5^p\ x_6^p\ x_7^p$

Found local cores

Core 1: $\overline{x_2} \wedge \overline{x_4} \rightarrow y_1 \vee y_2$ (weight 2)

Core 2: $x_3 \wedge \overline{x_4} \rightarrow y_1 \vee y_5$ (weight 1)

Core 3: $x_1 \rightarrow y_1 \vee y_6 \vee y_8$ (weight 3)

SOFT CONFLICT DETECTION (WEIGHTED CASE)

Weighted MaxCDCL

- Weight of Local Core $\mathcal{K}$ = smallest coefficient of objective literals in $\mathcal{K}$
- Each objective literal can contribute to many cores
- The total contribution of an objective literal cannot exceed its coefficient

$\mathcal{O}^t = \cancel{7}\ \cancel{5}\ 2y_1 + \cancel{2}\ 0y_2 + 1y_3 + 1y_4 + 1y_5 + \cancel{4}\ 1y_6 + 1y_7 + \cancel{3}\ 0y_8 \qquad \textbf{UB} = 4$

Trail: $x_1^d\ \overline{x_2}^p\ x_3^p\ \overline{x_4}^d\ x_5^p\ x_6^p\ x_7^p$

Found local cores

Core 1: $\overline{x_2} \wedge \overline{x_4} \rightarrow y_1 \vee y_2$ (weight 2)

~~Core 2: $x_3 \wedge \overline{x_4} \rightarrow y_1 \vee y_5$ (weight 1)~~

Core 3: $x_1 \rightarrow y_1 \vee y_6 \vee y_8$ (weight 3)

SOFT CONFLICT DETECTION (WEIGHTED CASE)

Weighted MaxCDCL

- Weight of Local Core $\mathcal{K}$ = smallest coefficient of objective literals in $\mathcal{K}$
- Each objective literal can contribute to many cores
- The total contribution of an objective literal cannot exceed its coefficient

$\mathcal{O}^t = \not{7}\ \not{5}\ 2y_1 + \not{2}\ 0y_2 + 1y_3 + 1y_4 + 1y_5 + \not{4}\ 1y_6 + 1y_7 + \not{3}\ 0y_8 \quad \mathbf{UB} = 4$

Trail: $x_1^d\ \overline{x_2}^p\ x_3^p\ \overline{x_4}^d\ x_5^p\ x_6^p\ x_7^p$

Found local cores

Core 1: $\overline{x_2} \land \overline{x_4} \rightarrow y_1 \lor y_2$ (weight 2)

Core 3: $x_1 \rightarrow y_1 \lor y_6 \lor y_8$ (weight 3)

SOFT CONFLICT DETECTION (WEIGHTED CASE)

Weighted MaxCDCL

- Weight of Local Core $\mathcal{K}$ = smallest coefficient of objective literals in $\mathcal{K}$
- Each objective literal can contribute to many cores
- The total contribution of an objective literal cannot exceed its coefficient

$\mathcal{O}^t = \cancel{7}\ \cancel{5}\ 2y_1 + \cancel{2}\ 0y_2 + 1y_3 + 1y_4 + 1y_5 + \cancel{4}\ 1y_6 + 1y_7 + \cancel{3}\ 0y_8$ $\quad$ $\mathbf{UB} = 4$

Trail: $x_1^d\ \overline{x_2}^p\ x_3^p\ \overline{x_4}^d\ x_5^p\ x_6^p\ x_7^p$

Found local cores

Core 1: $\overline{x_2} \wedge \overline{x_4} \rightarrow y_1 \vee y_2$ (weight 2)

Core 3: $x_1 \rightarrow y_1 \vee y_6 \vee y_8$ (weight 3)

Conclusion $x_1 \wedge \overline{x_2} \wedge \overline{x_4} \rightarrow LB = 5 \geq 4 = UB$

SOFT CONFLICT DETECTION (WEIGHTED CASE)

Weighted MaxCDCL

- Weight of Local Core $\mathcal{K}$ = smallest coefficient of objective literals in $\mathcal{K}$
- Each objective literal can contribute to many cores
- The total contribution of an objective literal cannot exceed its coefficient

$\mathcal{O}^t = \not 7\ \not 5\ 2y_1 + \not 2\ 0y_2 + 1y_3 + 1y_4 + 1y_5 + \not 4\ 1y_6 + 1y_7 + \not 3\ 0y_8 \qquad \mathbf{UB} = 4$

Trail: $x_1^d\ \overline{x_2}^p\ x_3^p\ \overline{x_4}^d\ x_5^p\ x_6^p\ x_7^p$

Found local cores

Core 1: $\overline{x_2} \wedge \overline{x_4} \rightarrow y_1 \vee y_2$ (weight 2)

Core 3: $x_1 \rightarrow y_1 \vee y_6 \vee y_8$ (weight 3)

Conclusion $x_1 \wedge \overline{x_2} \wedge \overline{x_4} \rightarrow LB = 5 \geq 4 = UB$

Soft conflict Conflicting clause: $\overline{x}_1 \vee x_2 \vee x_4$

PROOF LOGGING SOFT CONFLICTS

To Derive: $\overline{x}_1 + x_2 + x_4 \geq 1 \quad \mathbf{UB} = 4$

PROOF LOGGING SOFT CONFLICTS

To Derive: $\overline{x}_1 + x_2 + x_4 \geq 1$ $\quad \mathbf{UB} = 4$

Found "disjoint" cores

Core 1: $\overline{x_2} \wedge \overline{x_4} \rightarrow y_1 \vee y_2$ (2)

Core 2: $x_1 \rightarrow y_1 \vee y_6 \vee y_8$ (3)

PROOF LOGGING SOFT CONFLICTS

To Derive: $\overline{x}_1 + x_2 + x_4 \geq 1$ $\quad \mathbf{UB} = 4$

Found "disjoint" cores

Core 1: $\overline{x_2} \wedge \overline{x_4} \rightarrow y_1 \vee y_2$ (2)

PB: $x_2 + x_4 + y_1 + y_2 \geq 1$

Core 2: $x_1 \rightarrow y_1 \vee y_6 \vee y_8$ (3)

PB: $\overline{x}_1 + y_1 + y_6 + y_8 \geq 1$

PROOF LOGGING SOFT CONFLICTS

To Derive: $\overline{x}_1 + x_2 + x_4 \geq 1$ $\quad \mathbf{UB} = 4$

Found "disjoint" cores (RUP)

Core 1: $\overline{x_2} \wedge \overline{x_4} \rightarrow y_1 \vee y_2$ (2)

PB: $x_2 + x_4 + y_1 + y_2 \geq 1$

Core 2: $x_1 \rightarrow y_1 \vee y_6 \vee y_8$ (3)

PB: $\overline{x}_1 + y_1 + y_6 + y_8 \geq 1$

PROOF LOGGING SOFT CONFLICTS

To Derive: $\overline{x}_1 + x_2 + x_4 \geq 1$ $\quad \mathbf{UB} = 4$

Found "disjoint" cores (RUP)

Core 1: $\overline{x_2} \wedge \overline{x_4} \rightarrow y_1 \vee y_2$ (2)

PB: $2x_2 + 2x_4 + 2y_1 + 2y_2 \geq 2 \cancel{1}$

Core 2: $x_1 \rightarrow y_1 \vee y_6 \vee y_8$ (3)

PB: $3\overline{x}_1 + 3y_1 + 3y_6 + 3y_8 \geq 3 \cancel{1}$

Multiplication by their weight

PROOF LOGGING SOFT CONFLICTS

To Derive: $\overline{x}_1 + x_2 + x_4 \geq 1$ $\quad \mathbf{UB} = 4$

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PB: $2x_2 + 2x_4 + 2y_1 + 2y_2 \geq 2 \not{1}$

Core 2: $x_1 \rightarrow y_1 \vee y_6 \vee y_8$ (3)

PB: $3\overline{x}_1 + 3y_1 + 3y_6 + 3y_8 \geq 3 \not{1}$

Multiplication by their weight and addition:

$3\overline{x}_1 + 2x_2 + 2x_4 + 5y_1 + 2y_2 + 3y_6 + 3y_8 \geq 5$

PROOF LOGGING SOFT CONFLICTS

To Derive: $\overline{x}_1 + x_2 + x_4 \geq 1$ $\quad \mathbf{UB} = 4$

Found "disjoint" cores (RUP)

Core 1: $\overline{x_2} \wedge \overline{x_4} \rightarrow y_1 \vee y_2$ (2)

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Core 2: $x_1 \rightarrow y_1 \vee y_6 \vee y_8$ (3)

PB: $3\overline{x}_1 + 3y_1 + 3y_6 + 3y_8 \geq 3 \not{1}$

Multiplication by their weight and addition:

$3\overline{x}_1 + 2x_2 + 2x_4 + 5y_1 + 2y_2 + 3y_6 + 3y_8 \geq 5$

Model improving constraint

$7y_1 + 2y_2 + 1y_3 + 1y_4 + 1y_5 + 4y_6 + 1y_7 + 3y_8 \leq 3$

PROOF LOGGING SOFT CONFLICTS

To Derive: $\overline{x}_1 + x_2 + x_4 \geq 1$ $\mathbf{UB} = 4$

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$3\overline{x}_1 + 2x_2 + 2x_4 + 5y_1 + 2y_2 + 3y_6 + 3y_8 \geq 5$

Model improving constraint

$7y_1 + 2y_2 + 1y_3 + 1y_4 + 1y_5 + 4y_6 + 1y_7 + 3y_8 \leq 3$

In normalized form:

$7\overline{y}_1 + 2\overline{y}_2 + 1\overline{y}_3 + 1\overline{y}_4 + 1\overline{y}_5 + 4\overline{y}_6 + 1\overline{y}_7 + 3\overline{y}_8 \geq 20 - 3$

PROOF LOGGING SOFT CONFLICTS

To Derive: $\overline{x}_1 + x_2 + x_4 \geq 1$ $\quad \mathbf{UB} = 4$

Found "disjoint" cores (RUP)

Core 1: $\overline{x_2} \wedge \overline{x_4} \rightarrow y_1 \vee y_2$ (2)

PB: $2x_2 + 2x_4 + 2y_1 + 2y_2 \geq 2$ ~~1~~

Core 2: $x_1 \rightarrow y_1 \vee y_6 \vee y_8$ (3)

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$3\overline{x}_1 + 2x_2 + 2x_4 + 5y_1 + 2y_2 + 3y_6 + 3y_8 \geq 5$

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By adding literal axioms:

$5\overline{y}_1 + 2\overline{y}_2 + 3\overline{y}_6 + 3\overline{y}_8 \geq 13 - 3$

PROOF LOGGING SOFT CONFLICTS

To Derive: $\overline{x}_1 + x_2 + x_4 \geq 1$ $\quad \mathbf{UB} = 4$

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$3\overline{x}_1 + 2x_2 + 2x_4 + 5y_1 + 2y_2 + 3y_6 + 3y_8 \geq 5$

Model improving constraint

$7y_1 + 2y_2 + 1y_3 + 1y_4 + 1y_5 + 4y_6 + 1y_7 + 3y_8 \leq 3$

In normalized form:

$7\overline{y}_1 + 2\overline{y}_2 + 1\overline{y}_3 + 1\overline{y}_4 + 1\overline{y}_5 + 4\overline{y}_6 + 1\overline{y}_7 + 3\overline{y}_8 \geq 20 - 3$

By adding literal axioms:

$5\overline{y}_1 + 2\overline{y}_2 + 3\overline{y}_6 + 3\overline{y}_8 \geq 13 - 3$

Addition:

$$3\overline{x}_1 + 2x_2 + 2x_4 + 5y_1 + 5\overline{y}_1 + 2y_2 + 2\overline{y}_2 + 3y_6 + 3\overline{y}_6 + 3y_8 + 3\overline{y}_8 \geq 13 + 5 - 3$$

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$5\overline{y}_1 + 2\overline{y}_2 + 3\overline{y}_6 + 3\overline{y}_8 \geq 13 - 3$

Addition:

$$3\overline{x}_1 + 2x_2 + 2x_4 + \cancel{5y_1 + 5\overline{y}_1 + 2y_2 + 2\overline{y}_2 + 3y_6 + 3\overline{y}_6 + 3y_8 + 3\overline{y}_8} \geq \cancel{13} + 5 - 3$$

PROOF LOGGING SOFT CONFLICTS

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By adding literal axioms:

$5\overline{y}_1 + 2\overline{y}_2 + 3\overline{y}_6 + 3\overline{y}_8 \geq 13 - 3$

Addition:

$3\overline{x}_1 + 2x_2 + 2x_4 + \cancel{5y_1 + 5\overline{y}_1 + 2y_2 + 2\overline{y}_2 + 3y_6 + 3\overline{y}_6 + 3y_8 + 3\overline{y}_8} \geq \cancel{13} + 5 - 3$

Division by a large enough number (and rounding up): $\overline{x}_1 + x_2 + x_4 \geq 1$

PROOF LOGGING MAXCDCL

Proof logging Learned clause after conflict analysis RUP

PROOF LOGGING MAXCDCL

Proof logging Learned clause after conflict analysis RUP

Proof logging Optimality:

- Unit propagation in MaxCDCL derives conflict at DL $= 0$
- Proof: RUP $0 \geq 1$

OUTLINE OF THIS PRESENTATION

- MaxSAT and how to proof log it
- An introduction to the VeriPB proof system.
- MaxCDCL: Branch-and-Bound with clause learning
- Unweighted MaxCDCL revisited with literal unlocking
- Solution-Improving Constraint using Binary Decision Diagram (BDD) encoding
- Conclusions & Future work

UNWEIGHTED MAXCDCL REVISITED

Unweighted MaxCDCL searches for set $\mathcal{L}$ of tuples (b, L) such that

1. Each L is a set of objective literals
2. For each (b, L) in $\mathcal{L}$, it holds that $F \wedge \alpha \models \sum_{\ell \in L} \ell \geq b$.
3. For each pair (b, L) and (b', L') in $\mathcal{L}$, $L \cap L' = \emptyset$.
4. The total weight exceeds the current upper bound: $\sum_{(b,L) \in \mathcal{L}} b \geq \mathbf{UB}$.

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$\mathcal{O} = y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9 + \ldots \qquad \mathbf{UB} = 4$

Found disjoint local "cores"

Core 1: $\overline{x}_2 \wedge \overline{x}_4 \rightarrow y_1 + y_3 + y_5 + y_8 \geq 3$

Core 2: $x_4 \wedge \overline{x}_7 \wedge x_9 \rightarrow y_2 + y_4 + y_6 \geq 2$

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Core 2: $x_4 \wedge \overline{x}_7 \wedge x_9 \rightarrow y_2 + y_4 + y_6 \geq 2$

$\overline{x}_2 \wedge \overline{x}_4 \wedge \overline{x}_7 \wedge x_9 \rightarrow LB = 5 \geq 4 = UB$ **Soft conflict clause:** $x_2 \vee x_4 \vee x_7 \vee \overline{x}_9$

LOOKAHEAD WITH LITERAL UNLOCKING (BY EXAMPLE)

$\mathcal{O}^t = \cancel{y_1} + \cancel{y_2} + \cancel{y_3} + \cancel{y_4} + \cancel{y_5} + \cancel{y_6} + \cancel{y_7} + \cancel{y_8} + y_9 + \ldots$

Trail: $x_1^d\ \overline{x_2}^d\ x_3^p\ \overline{x_4}^d\ x_5^p$

Found disjoint local "cores"

Core 1: $\overline{x_2} \land \overline{x_4} \rightarrow y_3 + y_5 + y_6 \geq 1$

Core 2: $x_1 \land \overline{x_2} \rightarrow y_1 + y_2 + y_4 + y_7 + y_8 \geq 2$

LOOKAHEAD WITH LITERAL UNLOCKING (BY EXAMPLE)

$\mathcal{O}^t = \cancel{y_1} + \cancel{y_2} + \cancel{y_3} + \cancel{y_4} + \cancel{y_5} + \cancel{y_6} + \cancel{y_7} + \cancel{y_8} + y_9 + \ldots$

Trail: $x_1^d \; \overline{x_2}^d \; x_3^p \; \overline{x_4}^d \; x_5^p$

Found disjoint local "cores"

Core 1: $\cancel{\overline{x_2} \wedge \overline{x_4}} \rightarrow y_3 + y_5 + y_6 \geq 1$

Core 2: $\cancel{x_1 \wedge \overline{x_2}} \rightarrow y_1 + y_2 + y_4 + y_7 + y_8 \geq 2$

LOOKAHEAD WITH LITERAL UNLOCKING (BY EXAMPLE)

$\mathcal{O}^t = \cancel{y_1} + \cancel{y_2} + \cancel{y_3} + \cancel{y_4} + \cancel{y_5} + \cancel{y_6} + \cancel{y_7} + \cancel{y_8} + y_9 + \ldots$

Trail: $x_1^d \; \overline{x_2}^d \; x_3^p \; \overline{x_4}^d \; x_5^p$

Found disjoint local "cores"

Core 1: $y_3 + y_5 + y_6 \geq 1$

Core 2: $y_1 + y_2 + y_4 + y_7 + y_8 \geq 2$

LOOKAHEAD WITH LITERAL UNLOCKING (BY EXAMPLE)

$\mathcal{O}^t = \cancel{y_1} + \cancel{y_2} + \cancel{y_3} + \cancel{y_4} + \cancel{y_5} + \cancel{y_6} + \cancel{y_7} + \cancel{y_8} + \cancel{y_9} + \dots$

Trail: $x_1^d\ \overline{x_2}^d\ x_3^p\ \overline{x_4}^d\ x_5^p\ \overline{y_9}^a\ y_1^p\ y_3^p$

Found disjoint local "cores"

Core 1: $y_3 + y_5 + y_6 \geq 1$

Core 2: $y_1 + y_2 + y_4 + y_7 + y_8 \geq 2$

LOOKAHEAD WITH LITERAL UNLOCKING (BY EXAMPLE)

$\mathcal{O}^t = \cancel{y_1} + \cancel{y_2} + \cancel{y_3} + \cancel{y_4} + y_5 + y_6 + \cancel{y_7} + \cancel{y_8} + \cancel{y_9} + \dots$

Trail: $x_1^d\ \overline{x_2}^d\ x_3^p\ \overline{x_4}^d\ x_5^p\ \overline{y_9}^a\ y_1^p\ y_3^p$

Found disjoint local "cores"

Core 1: $y_3 + \cancel{y_5} + \cancel{y_6} \geq 1$

"$\{y_9\}$ unlocks Core 1 on $\{y_3\}$"

Core 2: $y_1 + y_2 + y_4 + y_7 + y_8 \geq 2$

LOOKAHEAD WITH LITERAL UNLOCKING (BY EXAMPLE)

$\mathcal{O}^t = \not{y_1} + \not{y_2} + \not{y_3} + \not{y_4} + \not{y_5} + \not{y_6} + \not{y_7} + \not{y_8} + \not{y_9} + \ldots$

Trail: $x_1^d\ \overline{x_2}^d\ x_3^p\ \overline{x_4}^d\ x_5^p\ \overline{y_9}^a\ y_1^p\ y_3^p\ \overline{y_5}^a$

Found disjoint local "cores"

Core 1: $y_3 + \not{y_5} + \not{y_6} \geq 1$

"$\{y_9\}$ unlocks Core 1 on $\{y_3\}$"

Core 2: $y_1 + y_2 + y_4 + y_7 + y_8 \geq 2$

LOOKAHEAD WITH LITERAL UNLOCKING (BY EXAMPLE)

$\mathcal{O}^t = \cancel{y_1} + \cancel{y_2} + \cancel{y_3} + \cancel{y_4} + \cancel{y_5} + \cancel{y_6} + \cancel{y_7} + \cancel{y_8} + \cancel{y_9} + \ldots$

Trail: $x_1^d \; \overline{x_2}^d \; x_3^p \; \overline{x_4}^d \; x_5^p \; \overline{y_9}^a \; y_1^p \; y_3^p \; \overline{y_5}^a \; \overline{y_6}^a \; y_7^p$

Found disjoint local "cores"

Core 1: $y_3 + \cancel{y_5} + \cancel{y_6} \geq 1$

"$\{y_9\}$ unlocks Core 1 on $\{y_3\}$"

Core 2: $y_1 + y_2 + y_4 + y_7 + y_8 \geq 2$

LOOKAHEAD WITH LITERAL UNLOCKING (BY EXAMPLE)

$\mathcal{O}^t = \cancel{y_1} + y_2 + \cancel{y_3} + y_4 + \cancel{y_5} + \cancel{y_6} + \cancel{y_7} + y_8 + \cancel{y_9} + \ldots$

Trail: $x_1^d \; \overline{x_2}^d \; x_3^p \; \overline{x_4}^d \; x_5^p \; \overline{y_9}^a \; y_1^p \; y_3^p \; \overline{y_5}^a \; \overline{y_6}^a \; y_7^p$

Found disjoint local "cores"

Core 1: $y_3 + \cancel{y_5} + \cancel{y_6} \geq 1$

"$\{y_9\}$ unlocks Core 1 on $\{y_3\}$"

Core 2: $y_1 + \cancel{y_2} + \cancel{y_4} + y_7 + \cancel{y_8} \geq 2$

"$\{y_9, y_5, y_6\}$ unlocks Core 2 on $\{y_1, y_7\}$"

LOOKAHEAD WITH LITERAL UNLOCKING (BY EXAMPLE)

$\mathcal{O}^t = \cancel{y_1} + \cancel{y_2} + \cancel{y_3} + \cancel{y_4} + \cancel{y_5} + \cancel{y_6} + \cancel{y_7} + y_8 + \cancel{y_9} + \ldots$

Trail: $x_1^d\ \overline{x_2}^d\ x_3^p\ \overline{x_4}^d\ x_5^p\ \overline{y_9}^a\ y_1^p\ y_3^p\ \overline{y_5}^a\ \overline{y_6}^a\ y_7^p\ \overline{y_2}^a\ \bot$

Found disjoint local "cores"

Core 1: $y_3 + \cancel{y_5} + \cancel{y_6} \geq 1$

"$\{y_9\}$ unlocks Core 1 on $\{y_3\}$"

Core 2: $y_1 + \cancel{y_2} + \cancel{y_4} + y_7 + \cancel{y_8} \geq 2$

"$\{y_9, y_5, y_6\}$ unlocks Core 2 on $\{y_1, y_7\}$"

New core: $y_9 + y_5 + y_6 + y_2 \geq 1$

LOOKAHEAD WITH LITERAL UNLOCKING (BY EXAMPLE)

$\mathcal{O}^t = \cancel{y_1} + \cancel{y_2} + \cancel{y_3} + \cancel{y_4} + \cancel{y_5} + \cancel{y_6} + \cancel{y_7} + \cancel{y_8} + \cancel{y_9} + \dots$

Trail: $x_1^d\ \overline{x_2}^d\ x_3^p\ \overline{x_4}^d\ x_5^p\ \overline{y_9}^a\ y_1^p\ y_3^p\ \overline{y_5}^a\ \overline{y_6}^a\ y_7^p\ \overline{y_2}^a\ \perp$

Found disjoint local "cores"

Core 1: $y_3 + \cancel{y_5} + \cancel{y_6} \geq 1$

"$\{y_9\}$ unlocks Core 1 on $\{y_3\}$"

Core 2: $y_1 + \cancel{y_2} + \cancel{y_4} + y_7 + \cancel{y_8} \geq 2$

"$\{y_9, y_5, y_6\}$ unlocks Core 2 on $\{y_1, y_7\}$"

New core: $y_9 + y_5 + y_6 + y_2 \geq 1$

Conclusion $\sum_{i=1}^{9} y_i \geq 4$ **?**

LOOKAHEAD WITH LITERAL UNLOCKING (BY EXAMPLE)

$\mathcal{O}^t = \cancel{y_1} + \cancel{y_2} + \cancel{y_3} + \cancel{y_4} + \cancel{y_5} + \cancel{y_6} + \cancel{y_7} + \cancel{y_8} + \cancel{y_9} + \dots$

Trail: $x_1^d\ \overline{x_2}^d\ x_3^p\ \overline{x_4}^d\ x_5^p\ \overline{y_9}^a\ y_1^p\ y_3^p\ \overline{y_5}^a\ \overline{y_6}^a\ y_7^p\ \overline{y_2}^a\ \perp$

Found disjoint local "cores"

Core 1: $y_3 + y_5 + y_6 \geq 1$

"$\{y_9\}$ unlocks Core 1 on $\{y_3\}$"

Core 2: $y_1 + y_2 + y_4 + y_7 + y_8 \geq 2$

"$\{y_9, y_5, y_6\}$ unlocks Core 2 on $\{y_1, y_7\}$"

New core: $y_9 + y_5 + y_6 + y_2 \geq 1$

Conclusion $\sum_{i=1}^{9} y_i \geq 4$ **?**

LOOKAHEAD WITH LITERAL UNLOCKING (BY EXAMPLE)

$\mathcal{O}^t = \cancel{y_1} + \cancel{y_2} + \cancel{y_3} + \cancel{y_4} + \cancel{y_5} + \cancel{y_6} + \cancel{y_7} + \cancel{y_8} + \cancel{y_9} + \ldots$

Trail: $x_1^d \; \overline{x_2}^d \; x_3^p \; \overline{x_4}^d \; x_5^p \; \overline{y_9}^a \; y_1^p \; y_3^p \; \overline{y_5}^a \; \overline{y_6}^a \; y_7^p \; \overline{y_2}^a \; \perp$

Found disjoint local "cores"

Core 1: $\overline{y_9} \rightarrow y_3 + y_5 + y_6 \geq 1$

"$\{y_9\}$ unlocks Core 1 on $\{y_3\}$"

Core 2: $y_1 + y_2 + y_4 + y_7 + y_8 \geq 2$

"$\{y_9, y_5, y_6\}$ unlocks Core 2 on $\{y_1, y_7\}$"

New core: $\cancel{y_9} + y_5 + y_6 + y_2 \geq 1$

Conclusion $\sum_{i=1}^{9} y_i \geq 4$ **?**

LOOKAHEAD WITH LITERAL UNLOCKING (BY EXAMPLE)

$\mathcal{O}^t = \not{y_1} + \not{y_2} + \not{y_3} + \not{y_4} + \not{y_5} + \not{y_6} + \not{y_7} + \not{y_8} + \not{y_9} + \ldots$

Trail: $x_1^d\ \overline{x_2}^d\ x_3^p\ \overline{x_4}^d\ x_5^p\ \overline{y_9}^a\ y_1^p\ y_3^p\ \overline{y_5}^a\ \overline{y_6}^a\ y_7^p\ \overline{y_2}^a\ \perp$

Found disjoint local “cores”

Core 1: $\overline{y_9} \rightarrow y_3 + \not{y_5} + \not{y_6} \geq 1$

“$\{y_9\}$ unlocks Core 1 on $\{y_3\}$”

Core 2: $\overline{y_9} \wedge \overline{y_5} + \overline{y_6} \rightarrow y_1 + y_2 + y_4 + y_7 + y_8 \geq 2$

“$\{y_9, y_5, y_6\}$ unlocks Core 2 on $\{y_1, y_7\}$”

New core: $\not{y_9} + \not{y_5} + \not{y_6} + y_2 \geq 1$

Conclusion $\sum_{i=1}^{9} y_i \geq 4$ **?**

LOOKAHEAD WITH LITERAL UNLOCKING (BY EXAMPLE)

$\mathcal{O}^t = \cancel{y_1} + \cancel{y_2} + \cancel{y_3} + \cancel{y_4} + \cancel{y_5} + \cancel{y_6} + \cancel{y_7} + \cancel{y_8} + \cancel{y_9} + \ldots$

Trail: $x_1^d\ \overline{x_2}^d\ x_3^p\ \overline{x_4}^d\ x_5^p\ \overline{y_9}^a\ y_1^p\ y_3^p\ \overline{y_5}^a\ \overline{y_6}^a\ y_7^p\ \overline{y_2}^a\ \bot$

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PROOF LOGGING LITERAL UNLOCKING

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To Derive: $\sum_{i=1}^{9} y_i \geq 4$

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Notation:

$L = \{y_9\}$

Found disjoint local "cores":

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$U_1 = \{y_3\},\ R_1 = \{y_5, y_6\}$

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"L unlocks Core 1 on U_1"

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"$\{y_9, y_5, y_6\}$ unlocks Core 2 on $\{y_1, y_7\}$"

"$L \cup R_1$ unlocks Core 2 on U_2"

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To Derive: $\sum_{i=1}^{9} y_i \geq 4$

$L + (\sum_i U_i + R_i) \geq \sum_i b_i + 1$

PROOF LOGGING LITERAL UNLOCKING

From the constraints

$$L_i \geq b_i \ (\forall 1 \leq i \leq k), \qquad L + \sum_{j<i} R_j + \ell \geq 1 \ (\forall 1 \leq i \leq k, \ell \in U_i), \qquad L + \sum_{j} R_j \geq 1$$

we derive

$$L + \sum_{j<i} R_j + \sum_{j \geq i} L_j \geq 1 + \sum_{j \geq i} b_j$$

for each $i \in \{1, \ldots, k+1\}$.

PROOF LOGGING LITERAL UNLOCKING

To Derive: $L + \sum_{j<i} R_j + \sum_{j\geq i} L_j \geq 1 + \sum_{j\geq i} b_j$.

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To Derive: $L + \sum_{j<i} R_j + \sum_{j \geq i} L_j \geq 1 + \sum_{j \geq i} b_j$. By **induction** on i.

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To Derive: $L + \sum_{j<i} R_j + \sum_{j \geq i} L_j \geq 1 + \sum_{j \geq i} b_j$. By **induction** on i.

For $i = k + 1$ (base case):

$$L + \sum_j R_j \geq 1$$

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For i between 1 and $k - 1$: Cutting Planes Derivation from IH:

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For $i = 1$ (New cardinality constraint!):

$$L + \sum_j L_j \geq 1 + \sum_j b_j$$

OUTLINE OF THIS PRESENTATION

- MaxSAT and how to proof log it
- An introduction to the VeriPB proof system.
- MaxCDCL: Branch-and-Bound with clause learning
- Unweighted MaxCDCL revisited with literal unlocking
- Solution-Improving Constraint using Binary Decision Diagram (BDD) encoding
- Conclusions & Future work

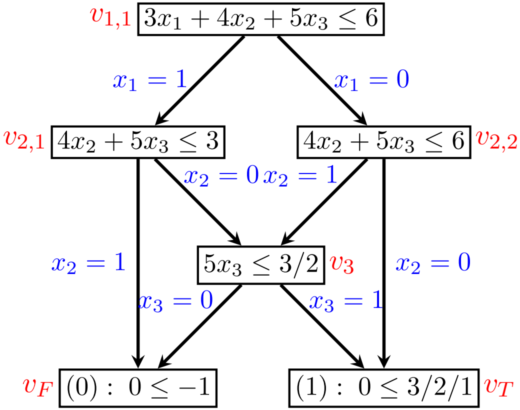
MAXCDCL'S USAGE OF BDDS

MaxCDCL $\cup$ Solution-Improving: MaxCDCL encodes solution-improving constraint

MAXCDCL'S USAGE OF BDDS

MaxCDCL ∪ Solution-Improving: MaxCDCL encodes solution-improving constraint

Binary Decision Diagram:

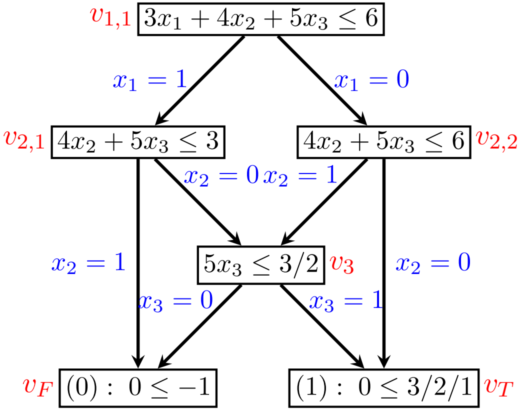


MAXCDCL'S USAGE OF BDDS

MaxCDCL ∪ Solution-Improving: MaxCDCL encodes solution-improving constraint

Binary Decision Diagram:

- Every node corresponds with part of the original PB constraint and,



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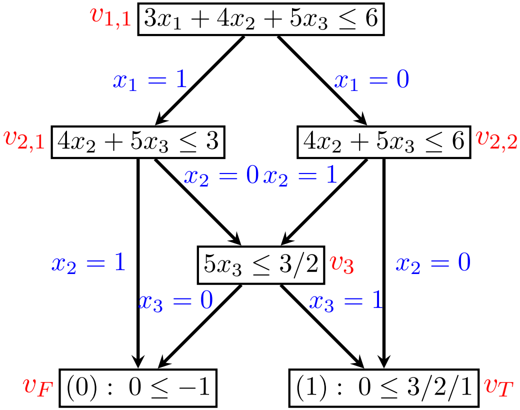

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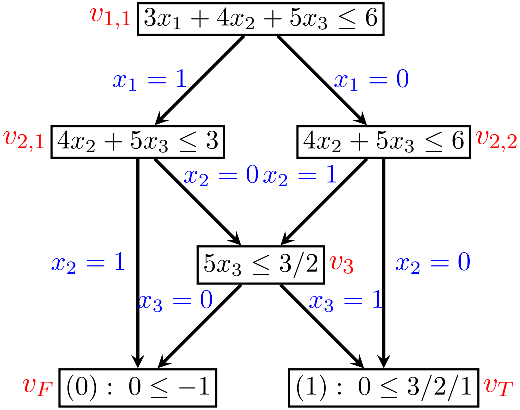
MaxCDCL ∪ Solution-Improving: MaxCDCL encodes solution-improving constraint

Binary Decision Diagram:

- Every node corresponds with part of the original PB constraint and,
- Every node propagates based on one decision literal.
- If v_F node is propagated true, then constraint in root is falsified.

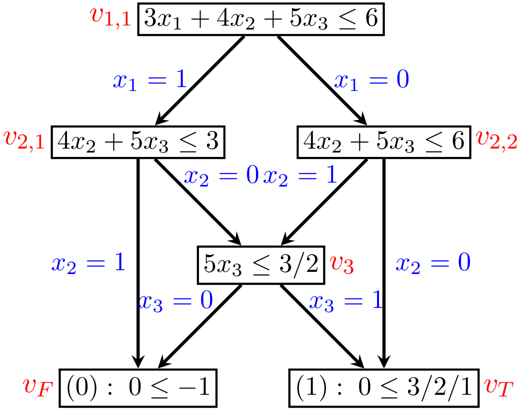


MAXCDCL'S USAGE OF BDDS

MaxCDCL ∪ Solution-Improving: MaxCDCL encodes solution-improving constraint

Introducing fresh variables for each node with meaning:

- E.g., $v_{2,2} \leftrightarrow 4x_2 + 5x_3 \leq 6$



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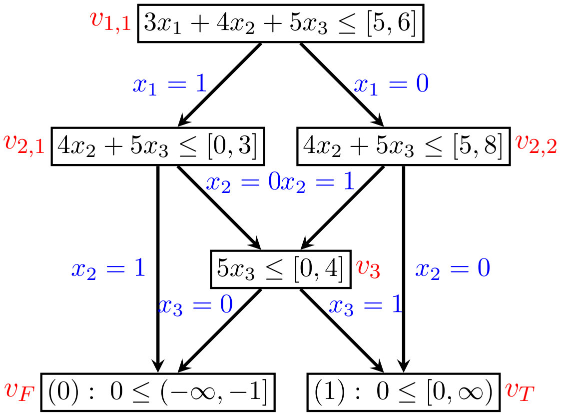

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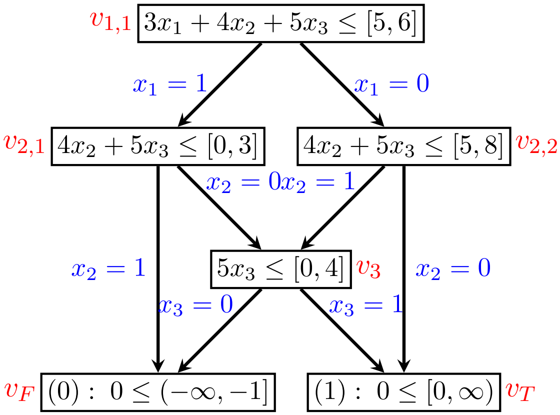

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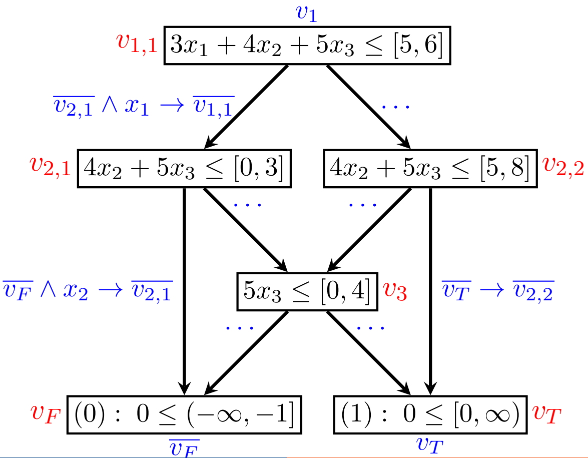
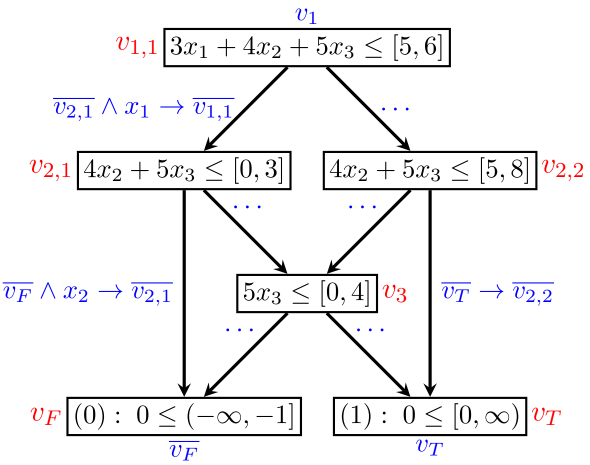
After introducing the variables, clauses are added to the solver.



HOW TO PROOF LOG BDDS?

Step 1: Derive reification of node variables. E.g.,

- $v_{2,2} \leftrightarrow 4x_2 + 5x_3 \leq [5, 8]$
  - $v_{2,2} \rightarrow 4x_2 + 5x_3 \leq 5$
  - $v_{2,2} \leftarrow 4x_2 + 5x_3 \leq 8$



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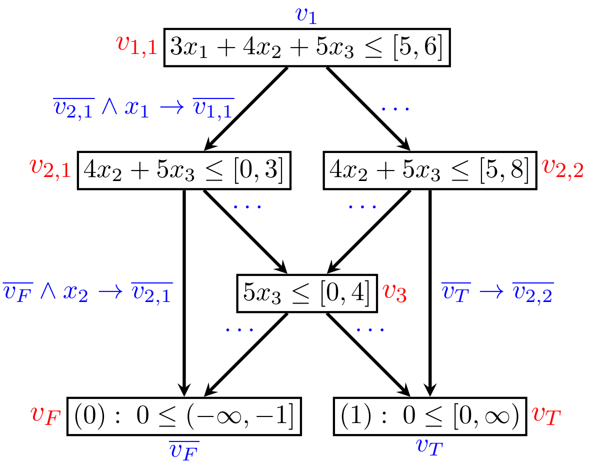
by introducing

- $v_{2,2} \leftrightarrow 4x_2 + 5x_3 \leq 5$
- $v'_{2,2} \leftrightarrow 4x_2 + 5x_3 \leq 8$ (only in proof)

and deriving

- $v'_{2,2} \rightarrow v_{2,2}$



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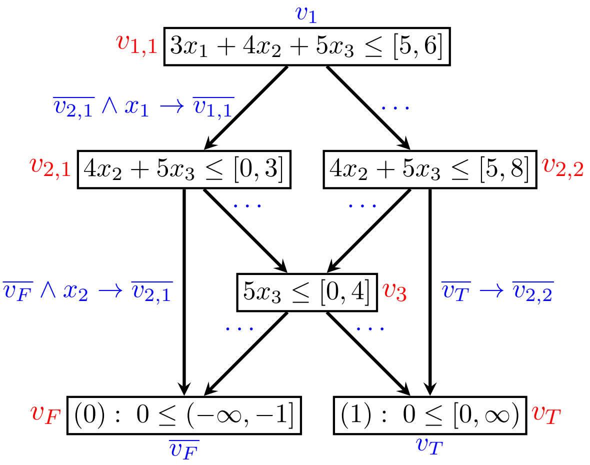
and deriving

- $v'_{2,2} \rightarrow v_{2,2}$

Step 2: Derive clauses.

- Straight-forward cutting planes derivation.



INTERMEZZO: PROOF BY CONTRADICTION

Remember definition of Redundance-Based Strengthening:

Definition

A constraint C is redundant with respect to the pseudo-Boolean formula F if there exists a substitution ω, called a witness, such that

$$F \wedge \neg C \models F|_\omega \wedge C|_\omega$$

Proof by contradiction — Take empty witness.

Condition to prove RBS becomes:

$$F \wedge \neg C \models F \wedge C$$

Only one non-trivial proof goal:

$$F \wedge \neg C \wedge \neg C \vdash 0 \geq 1$$

PROVING REIFICATION OF NODE VARIABLES

We have

- $v_{2,2} \rightarrow 4x_2 + 5x_3 \leq 5$
- $v'_{2,2} \leftarrow 4x_2 + 5x_3 \leq 8$

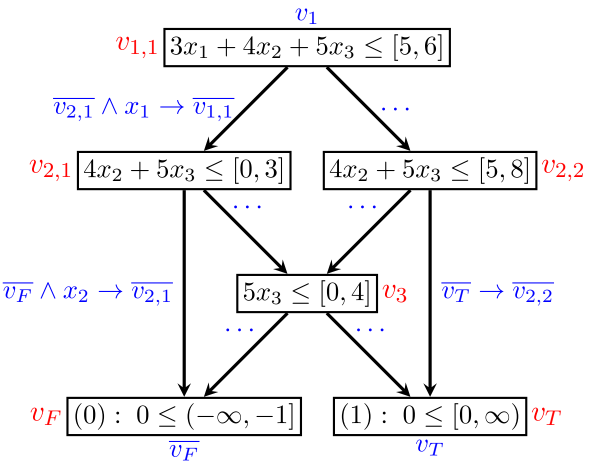
and we want to derive

- $v'_{2,2} \rightarrow v_{2,2}$

If we can prove

- $\overline{x}_2 + \overline{v}'_{2,2} + v_{2,2} \geq 1$
- $x_2 + \overline{v}'_{2,2} + v_{2,2} \geq 1$

then $\overline{v}'_{2,2} + v_{2,2} \geq 1$ follows.



PROVING REIFICATION OF NODE VARIABLES

To derive:

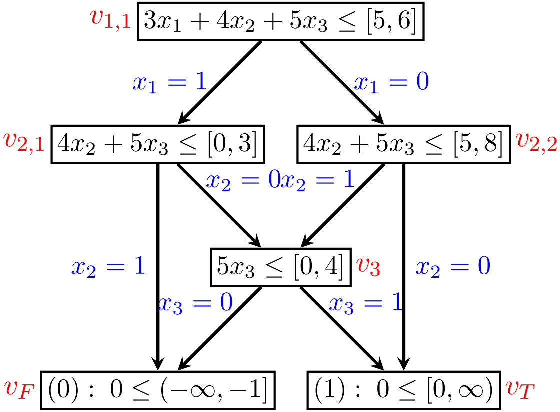
- $\overline{x}_2 + \overline{v}'_{2,2} + v_{2,2} \geq 1$

We have for node $v_{2,2}$:

- $v_{2,2} \leftrightarrow 4x_2 + 5x_3 \leq 5$
- $v'_{2,2} \leftrightarrow 4x_2 + 5x_3 \leq 8$

For node v_3:

- $v_3 \rightarrow 5x_3 \leq 0$
- $v_3 \leftarrow 5x_3 \leq 4$



PROVING REIFICATION OF NODE VARIABLES (BY CONTRADICTION)

To Derive: $\overline{x}_2 + \overline{v}'_{2,2} + v_{2,2} \geq 1$. We assume the negation, i.e.,

$$x_2 \geq 1, \qquad v'_{2,2} \geq 1, \qquad \overline{v}_{2,2} \geq 1$$

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$$x_2 \geq 1, \qquad v'_{2,2} \geq 1, \qquad \overline{v}_{2,2} \geq 1$$

Constraints already derived:

$$v'_{2,2} \leftrightarrow 4x_2 + 5x_3 \leq 8 \qquad v_{2,2} \leftrightarrow 4x_2 + 5x_3 \leq 5$$

$$v_3 \leftarrow 5x_3 \leq 4 \qquad v_3 \rightarrow 5x_3 \leq 0$$

PROVING REIFICATION OF NODE VARIABLES (BY CONTRADICTION)

To Derive: $\overline{x}_2 + \overline{v}'_{2,2} + v_{2,2} \geq 1$. We assume the negation, i.e.,

$$x_2 \geq 1, \qquad v'_{2,2} \geq 1, \qquad \overline{v}_{2,2} \geq 1$$

Constraints already derived:

$$v'_{2,2} \leftrightarrow 4x_2 + 5x_3 \leq 8 \qquad v_{2,2} \leftrightarrow 4x_2 + 5x_3 \leq 5$$

$$v_3 \leftarrow 5x_3 \leq 4 \qquad v_3 \rightarrow 5x_3 \leq 0$$

From $v'_{2,2} \geq 1$: $4x_2 + 5x_3 \leq 8$

Using $x_2 \geq 1$: $5x_3 \leq 4$

Using definition of v_3: $v_3 \geq 1$

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To Derive: $\overline{x}_2 + \overline{v}'_{2,2} + v_{2,2} \geq 1$. We assume the negation, i.e.,

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$$v_3 \leftarrow 5x_3 \leq 4 \qquad v_3 \rightarrow 5x_3 \leq 0$$

From $v'_{2,2} \geq 1$: $4x_2 + 5x_3 \leq 8$	From $\overline{v}_{2,2} \geq 1$: $4x_2 + 5x_3 \geq 5 + 1$
Using $x_2 \geq 1$: $5x_3 \leq 4$	Weakening x_2: $5x_3 \geq 2$
Using definition of v_3: $v_3 \geq 1$	Using definition of v_3: $\overline{v}_3 \geq 1$

PROVING REIFICATION OF NODE VARIABLES (BY CONTRADICTION)

To Derive: $\overline{x}_2 + \overline{v}'_{2,2} + v_{2,2} \geq 1$. We assume the negation, i.e.,

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From $v'_{2,2} \geq 1$: $4x_2 + 5x_3 \leq 8$

Using $x_2 \geq 1$: $5x_3 \leq 4$

Using definition of v_3: $v_3 \geq 1$

From $\overline{v}_{2,2} \geq 1$: $4x_2 + 5x_3 \geq 5 + 1$

Weakening x_2: $5x_3 \geq 2$

Using definition of v_3: $\overline{v}_3 \geq 1$

Contradiction.

PROVING REIFICATION OF NODE VARIABLES (BY CONTRADICTION)

To Derive: $\overline{x}_2 + \overline{v}'_{2,2} + v_{2,2} \geq 1$. We assume the negation, i.e.,

$$x_2 \geq 1, \qquad v'_{2,2} \geq 1, \qquad \overline{v}_{2,2} \geq 1$$

Constraints already derived:

$$v'_{2,2} \leftrightarrow 4x_2 + 5x_3 \leq 8 \qquad v_{2,2} \leftrightarrow 4x_2 + 5x_3 \leq 5$$

$$v_3 \leftarrow 5x_3 \leq 4 \qquad v_3 \rightarrow 5x_3 \leq 0$$

From $v'_{2,2} \geq 1$: $4x_2 + 5x_3 \leq 8$

Using $x_2 \geq 1$: $5x_3 \leq 4$

Using definition of v_3: $v_3 \geq 1$

From $\overline{v}_{2,2} \geq 1$: $4x_2 + 5x_3 \geq 5 + 1$

Weakening x_2: $5x_3 \geq 2$

Using definition of v_3: $\overline{v}_3 \geq 1$

Contradiction. Same reasoning to obtain $x_2 + \overline{v}'_{2,2} + v_{2,2} \geq 1$.

PROVING REIFICATION OF NODE VARIABLES

We have

- $v_{2,2} \rightarrow 4x_2 + 5x_3 \leq 5$
- $v'_{2,2} \leftarrow 4x_2 + 5x_3 \leq 8$

and we want to derive

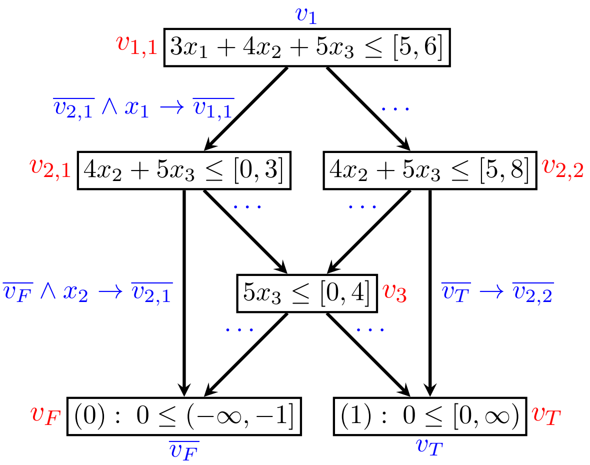
- $v'_{2,2} \rightarrow v_{2,2}$

If we can prove

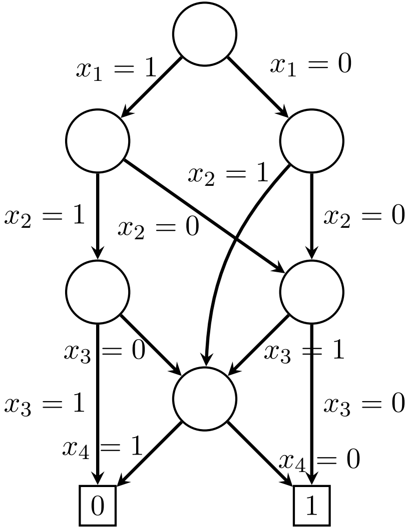
- $\overline{x}_2 + \overline{v}'_{2,2} + v_{2,2} \geq 1$
- $x_2 + \overline{v}'_{2,2} + v_{2,2} \geq 1$

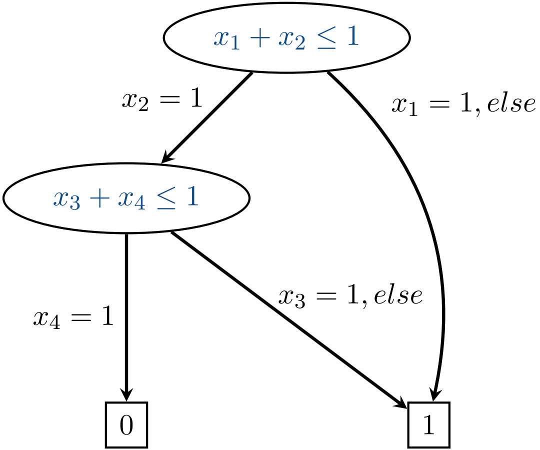
then $\overline{v}'_{2,2} + v_{2,2} \geq 1$ follows.

Clauses: Derived from reification constraints.



MULTI-VALUED DECISION DIAGRAM (MDD)





OUTLINE OF THIS PRESENTATION

- MaxSAT and how to proof log it
- An introduction to the VeriPB proof system.
- MaxCDCL: Branch-and-Bound with clause learning
- Unweighted MaxCDCL revisited with literal unlocking
- Solution-Improving Constraint using Binary Decision Diagram (BDD) encoding
- Conclusions & Future work

WRAPPING UP

Future work:

- Implementation & Experiments
- Implicit Hitting Set solvers

WRAPPING UP

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- Implementation & Experiments
- Implicit Hitting Set solvers

This talk:

- Proof logging for yet another MaxSAT Solver
  - MaxCDCL: MaxSAT solving by combining Branch-and-Bound and CDCL

WRAPPING UP

Future work:

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- Pseudo-Boolean reasoning helps to express MaxSAT algorithms

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- Proof logging for yet another MaxSAT Solver
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- Proof logging for MaxSAT is possible with VeriPB!!

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- Pseudo-Boolean reasoning helps to express MaxSAT algorithms
- Proof logging for MaxSAT is possible with VeriPB!!

Thank you for your attention!

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PROOF LOGGING LITERAL UNLOCKING

To Derive: $L + \sum_{j<i} R_j + \sum_{j \geq i} L_j \geq 1 + \sum_{j \geq i} b_j$.

PROOF LOGGING LITERAL UNLOCKING

To Derive: $L + \sum_{j<i} R_j + \sum_{j \geq i} L_j \geq 1 + \sum_{j \geq i} b_j$. By **induction** on i.

PROOF LOGGING LITERAL UNLOCKING

To Derive: $L + \sum_{j<i} R_j + \sum_{j \geq i} L_j \geq 1 + \sum_{j \geq i} b_j$. By **induction** on i.

For $i = k + 1$:

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To Derive: $L + \sum_{j<i} R_j + \sum_{j \geq i} L_j \geq 1 + \sum_{j \geq i} b_j$. By **induction** on i.

For $i = k + 1$:

$$L + \sum_{j} R_j \geq 1$$

PROOF LOGGING LITERAL UNLOCKING

To Derive: $L + \sum_{j<i} R_j + \sum_{j \geq i} L_j \geq 1 + \sum_{j \geq i} b_j$. By **induction** on i.

For i between 1 and $k-1$ (assuming already derived for $i+1$):

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Step 0. Induction Hypothesis

$$L + \sum_{j<i+1} R_j + \sum_{j\geq i+1} L_j \geq 1 + \sum_{j\geq i+1} b_j$$

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$$L + \sum_{j<i+1} R_j + \sum_{j \geq i+1} L_j \geq 1 + \sum_{j \geq i+1} b_j$$

Step 1. Addition of $L + \sum_{j<i} R_j + \ell \geq 1$ for every $\ell \in U_i$ results in

$$b_i L + b_i \sum_{j<i} R_j + U_i \geq b_i$$

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Step 2. Addition with IH gives :

$$(b_i + 1) \cdot L + (b_i + 1) \sum_{j<i} R_j + \sum_{j \geq i} U_j + \sum_{j \geq i} R_j \geq 1 + \sum_{j \geq i} b_j$$

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Step 2. Addition with IH gives (with $R_i \cup U_i = L_i$):

$$(b_i + 1) \cdot L + (b_i + 1) \sum_{j<i} R_j + \cancel{\sum_{j \geq i} U_j + \sum_{j \geq i} R_j} + \sum_{j \geq i} L_j \geq 1 + \sum_{j \geq i} b_j$$

PROOF LOGGING LITERAL UNLOCKING

To Derive: $L + \sum_{j<i} R_j + \sum_{j \geq i} L_j \geq 1 + \sum_{j \geq i} b_j$. By **induction** on i.

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Step 3. Multiplying all constraints $L_j \geq b_j$ for $j \geq i$ with b_i gives:

$$b_i \sum_{j \geq i} L_j \geq b_i \sum_{j \geq i} b_j$$

PROOF LOGGING LITERAL UNLOCKING

To Derive: $L + \sum_{j<i} R_j + \sum_{j \geq i} L_j \geq 1 + \sum_{j \geq i} b_j$. By **induction** on i.

For i between 1 and $k-1$ (assuming already derived for $i+1$):

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Step 5. Dividing this by $b_{i+1} + 1$ (and rounding the righthand-side up) yields

$$L + \sum_{j<i} R_j + \sum_{j \geq i} L_j \geq 1 + \sum_{j \geq i} b_j$$

INTERMEZZO: PROOF BY CONTRADICTION

Remember definition of Redundance-Based Strengthening:

Definition

A constraint C is redundant with respect to the pseudo-Boolean formula F if and only if there exists a substitution ω, called a witness, such that

$$F \wedge \neg C \models F|_{\omega} \wedge C|_{\omega}$$

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Proof by contradiction — Take empty witness.

Condition to prove RBS becomes:

$$F \wedge \neg C \models F \wedge C$$

Only one non-trivial proof goal:

$$F \wedge \neg C \wedge \neg C \vdash 0 \geq 1$$

INTERMEZZO: PROOF BY CASE SPLITTING

Suppose we have derived two constraints:

$$a \cdot x + \sum_i b_i l_i \geq B \qquad\qquad a \cdot \overline{x} + \sum_i b_i l_i \geq B$$

And we want to derive the constraint

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By contradiction. Needed: CP derivation that shows

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After normalization:

$$(a \cdot x + \sum_i b_i l_i \geq B) \wedge (a \cdot \overline{x} + \sum_i b_i l_i \geq B) \wedge (\sum_i b_i l_i \geq \sum_i b_i - B + 1) \vdash 0 \geq 1$$

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To show:

$$(a \cdot x + \sum_i b_i l_i \geq B) \land (a \cdot \overline{x} + \sum_i b_i l_i \geq B) \land (\sum_i b_i \bar{l}_i \geq \sum_i b_i - B + 1) \vdash 0 \geq 1$$

Addition of $(a \cdot x + \sum_i b_i l_i \geq B)$ with $(\sum_i b_i \bar{l}_i \geq \sum_i b_i - B + 1)$ gives

$$a \cdot x + \sum_i b_i l_i + \sum_i b_i \bar{l}_i \geq B + \sum_i b_i - B + 1$$

which is equal to

$$a \cdot x \geq 1$$

After saturation: $x \geq 1$.
Similarly, addition of $(a \cdot \overline{x} + \sum_i b_i l_i \geq B)$ and $(\sum_i b_i \bar{l}_i \geq \sum_i b_i - B + 1)$ and saturation gives

$$\overline{x} \geq 1$$

which is clearly contradiction with $x \geq 1$.