A Conflict-Free Learning Approach for MILP and WCSP Optimization

SLOPPY Workshop

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05/11/2024

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Conflict-free learning in MILP

Conflict-free learning

Objective

- Learn the lower bounds of subproblems visited during search

Example

$$
\begin{aligned}
& \min 3x_1 + 4x_2 + 6x_3 + 8x_4 + x_5 + 6x_6 \\
s.t \\
& 2x_1 + 2x_2 + 3x_3 + 4x_4 + x_5 + 6x_6 \geq 10 \\
& x_1 + x_2 = 1 \\
& x_i \leq 1 && \forall i \in [1, 6] \\
& x_i \in \{0, 1\} && \forall i \in [1, 6]
\end{aligned}
$$

MILP standard form

$$\min 3x_1 + 4x_2 + 6x_3 + 8x_4 + x_5 + 6x_6$$

$s.t$

$$2x_1 + 2x_2 + 3x_3 + 4x_4 + x_5 + 6x_6 - s = 10$$

$$x_1 + x_2 = 1$$

$$x_i + \overline{x_i} = 1 \quad \forall i \in [1, 6]$$

$$x_i \in \{0, 1\} \quad \forall i \in [1, 6]$$

$$\overline{x_i} \geq 0 \quad \forall i \in [1, 6]$$

$$s \geq 0$$

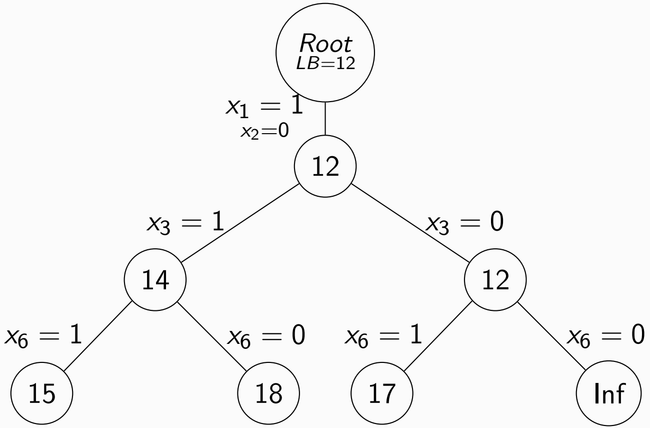
Domain restriction

To remove a variable we increase its coefficient in the objective function by a large value.
Removing x_1:

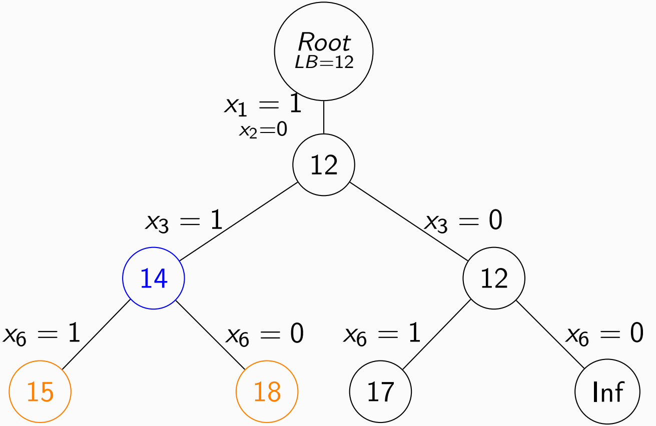
$$\min 100x_1 + 4x_2 + 6x_3 + 8x_4 + x_5 + 6x_6$$

→ Sets of primal constraints/dual variables are the same at every node of the search tree.

Example

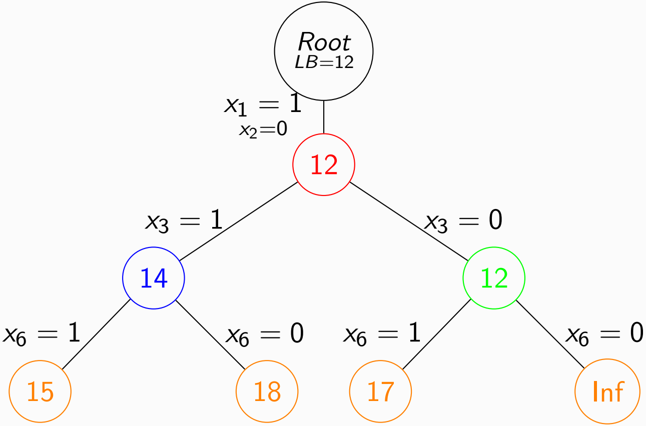


Example



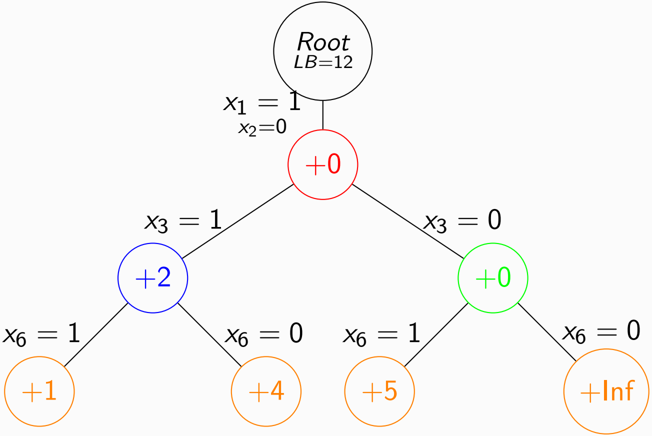
Learn a constraint capturing that the LB of the blue node is 15.

Example



Learn a constraint capturing that the LB of the blue node is 15.
Learn a constraint capturing that the LB of the green node is 17.
Learn a constraint capturing that the LB of the red node is 15.

Example



Learn constraint capturing that we can increase the LB by 3 at blue node.
Learn constraint capturing that we can increase the LB by 5 at green node.
Learn constraint capturing that we can increase the LB by 3 at red node.

Conflict-free learning

Defined for MILP

Farkas constraint [Farkas, 1902]

- Linear combination of primal constraints guided by an LP (dual) solution
- Explain a lower bound

Conflict-free learning [Witzig, 2022]

- Modify the Farkas constraint according to information obtained deeper in the tree $\rightarrow$ tighten the Farkas constraint
- Learned constraints are useful only to prune more values

Memo Constraint

Definition of Memo Constraint

- A constraint $w^T z = b$ is a memo constraint if $w \leq obj$
- A memo constraint $w^T z = b$ explains an LB of b

There exists a family of Farkas constraints that are memo constraints.

Example

$\min 3x_1 + 4x_2 + 6x_3 + 8x_4 + x_5 + 6x_6$	
$2\, x_1 + 2x_2 + 3x_3 + 4x_4 + x_5 + 6x_6 - s = 10$	2
$x_1 + \overline{x_1} = 1$	-1
$x_2 + \overline{x_2} = 1$	
$x_3 + \overline{x_3} = 1$	
$x_4 + \overline{x_4} = 1$	
$x_5 + \overline{x_5} = 1$	-1
$x_6 + \overline{x_6} = 1$	-6

Example

$\min 3x_1 + 4x_2 + 6x_3 + 8x_4 + x_5 + 6x_6$	
$2\ x_1 + 2x_2 + 3x_3 + 4x_4 + x_5 + 6x_6 - s = 10$	2
$x_1 + \overline{x_1} = 1$	-1
$x_2 + \overline{x_2} = 1$	
$x_3 + \overline{x_3} = 1$	
$x_4 + \overline{x_4} = 1$	
$x_5 + \overline{x_5} = 1$	-1
$x_6 + \overline{x_6} = 1$	-6

Farkas Constraint: $3x_1 + 3x_2 + 6x_3 + 8x_4 + x_5 - \overline{x_5} + 6x_6 - 6\overline{x_6} - 2s = 12$

Capturing the increase of lower bound ?

From the LP optimal solution rewrite the objective to obtain an **equivalent** problem
$\rightarrow$ We use the reduced costs

- The Farkas constraint saves a subset of costs justifying the increase of LB
- Then we remove that information from the objective function

Example

Problem P :

$$\min 3x_1 + 4x_2 + 6x_3 + 8x_4 + x_5 + 6x_6$$

$s.t$

$$2x_1 + 2x_2 + 3x_3 + 4x_4 + x_5 + 6x_6 - s = 10$$
$$x_1 + x_2 = 1$$
$$x_i + \overline{x_i} = 1$$
$$x_i \in \{0,1\}$$
$$\overline{x_i} \geq 0$$
$$s \geq 0$$

Problem P' :

$$\min 0x_1 + x_2 + \overline{x_5} + 6\overline{x_6} + 2s + 12$$

$s.t$

$$2x_1 + 2x_2 + 3x_3 + 4x_4 + x_5 + 6x_6 - s = 10$$
$$x_1 + x_2 = 1$$
$$x_i + \overline{x_i} = 1 \quad \forall i \in [1,6]$$
$$x_i \in \{0,1\} \quad \forall i \in [1,6]$$
$$\overline{x_i} \geq 0 \quad \forall i \in [1,6]$$
$$s \geq 0$$

Farkas constraint: $3x_1 + 3x_2 + 6x_3 + 8x_4 + x_5 - \overline{x_5} + 6x_6 - 6\overline{x_6} - 2s = 12$

Memo Resolution

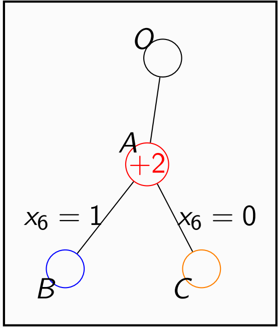
Fusion Resolution[Gocht, Nordstrom, and Buss,2021]

$$\frac{\overline{x_1} + 2x_3 + 4x_4 - x_5 \geq b \qquad x_1 + 2x_3 + 4x_4 - x_5 \geq b'}{2x_3 + 4x_4 - x_5 \geq \min(b, b')}$$

Memo Resolution (simplified)

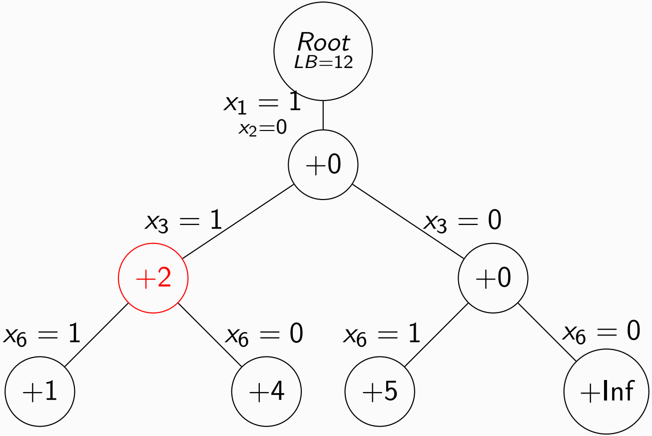
$$\frac{w_j\overline{x_j} + \sum_{i \neq j} w_i x_i = b, \quad w'_j x_j + \sum_{i \neq j} w'_i x_i = b'}{\sum_{i \neq j} \max(w_i, w'_i) x_i \geq \min(b, b')}$$

General Idea



1. Compute a Farkas constraint c_A at node A and transform the objective function. Then compute memo constraints for B and C.
2. Apply memo resolution on c_B and c_C
3. Sum the resulting constraint with the memo constraint c_A
4. Learn the resulting **memo** constraint and return it to node O

Example



Example

1) Compute memo constraint c_A.

$$\min 100x_2 + 100\overline{x_3} + \overline{x_5} + 6\overline{x_6} + 2s + 12$$

$s.t$

$$2x_1 + 2x_2 + 3x_3 + 4x_4 + x_5 + 6x_6 - s = 10$$

$$x_1 + x_2 = 1$$

$$x_i + \overline{x_i} = 1 \quad \forall i \in [1, 6]$$

$$x_i \geq 0 \quad \forall i \in [1, 6]$$

$$\overline{x_i} \geq 0 \quad \forall i \in [1, 6]$$

$$s \geq 0$$

$$c_A : \quad 3\overline{x_3} + \overline{x_5} + 6\overline{x_6} - 4x_4 + s = 2$$

Example

1) Compute memo constraint c_A, and transform the objective function:

$$
\begin{aligned}
&\min 100x_2 + 100\overline{x_3} + 4x_4 + s + 14 \\
s.t \\
&2x_1 + 2x_2 + 3x_3 + 4x_4 + x_5 + 6x_6 - s = 10 \\
&x_1 + x_2 = 1 \\
&x_i + \overline{x_i} = 1 && \forall i \in [1, 6] \\
&x_i \geq 0 && \forall i \in [1, 6] \\
&\overline{x_i} \geq 0 && \forall i \in [1, 6] \\
&s \geq 0
\end{aligned}
$$

$$c_A : \quad 3\overline{x_3} + \overline{x_5} + 6\overline{x_6} - 4x_4 + s = 2$$

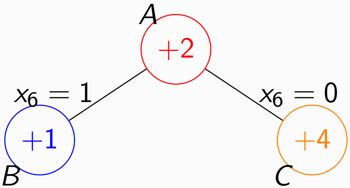
General Idea

1) Compute memo constraints c_B, c_C at nodes B and C

$$c_A : \quad 3\overline{x_3} + \overline{x_5} + 6\overline{x_6} - 4x_4 + s = 2$$

$$c_B : \quad 6\overline{x_6} + 3\overline{x_3} - 4x_4 - x_5 + s = 1$$

$$c_C : \quad 6x_6 + x_2 + 4x_4 - 2\overline{x_1} - 3\overline{x_3} - \overline{x_5} - s = 4$$



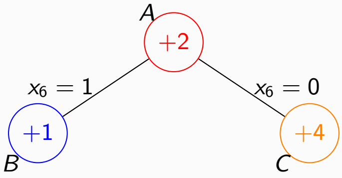
General Idea

2) Apply memo resolution on c_B and c_C.

$$\frac{c_B : {\color{red}6\overline{x_6}} + 3\overline{x_3} - 4x_4 - x_5 + s = 1 \quad c_C : {\color{red}6x_6} + x_2 + 4x_4 - 2\overline{x_1} - 3\overline{x_3} - \overline{x_5} - s = 4}{c_{BC} : 3\overline{x_3} + x_2 + 4x_4 + s \geq 1}$$

$$c_{BC} : \quad 3\overline{x_3} + x_2 + 4x_4 + s - s_1 = 1$$

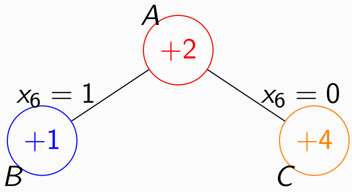
c_{BC} is a memo constraint at node A' (**after objective reformulation**).



General Idea

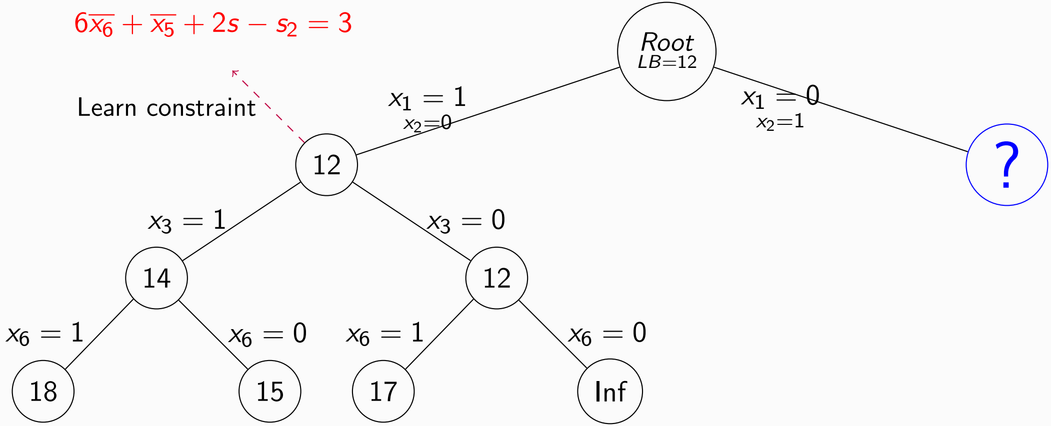
3) Sum the resulting **memo** constraint with the constraint c_A

$$
\begin{array}{rr}
c_{BC}: & 3\overline{x_3} + x_2 + 4x_4 + s - s_1 = 1 \\
 & + \\
c_A: & 3\overline{x_3} + \overline{x_5} + 6\overline{x_6} - 4x_4 + s = 2 \\
\hline
c_{ABC}: & 6\overline{x_3} + x_2 + \overline{x_5} + 6\overline{x_6} + 2s - s_1 = 3
\end{array}
$$



c_{ABC} is a memo constraint at node A.

Example


$6\overline{x_6} + \overline{x_5} + 2s - s_2 = 3$
Learn constraint
Root
LB=12
$x_1 = 1$
$x_2=0$
$x_1 = 0$
$x_2=1$
12
?
$x_3 = 1$
$x_3 = 0$
14
12
$x_6 = 1$
$x_6 = 0$
$x_6 = 1$
$x_6 = 0$
18
15
17
Inf

Example

$$
\begin{aligned}
& \min 100x_1 + x_2 + \overline{x_5} + 6\overline{x_6} + 2s + 12 \\
s.t \\
& 2x_1 + 2x_2 + 3x_3 + 4x_4 + x_5 + 6x_6 - s = 10 \\
& x_1 + x_2 = 1 \\
& \color{red}{6\overline{x_6} + \overline{x_5} + 2s - s_2 = 3} \\
& x_i + \overline{x_i} = 1 && \forall i \in [1, 6] \\
& \overline{x_i}, x_i \geq 0 && \forall i \in [1, 6] \\
& s, s_1 \geq 0
\end{aligned}
$$

Without the red constraint: OPT=13 $\rightarrow$ the search continue
With the red constraint: OPT=16 $\rightarrow$ end of search

Implementation

Python script based on CPLEX Python API

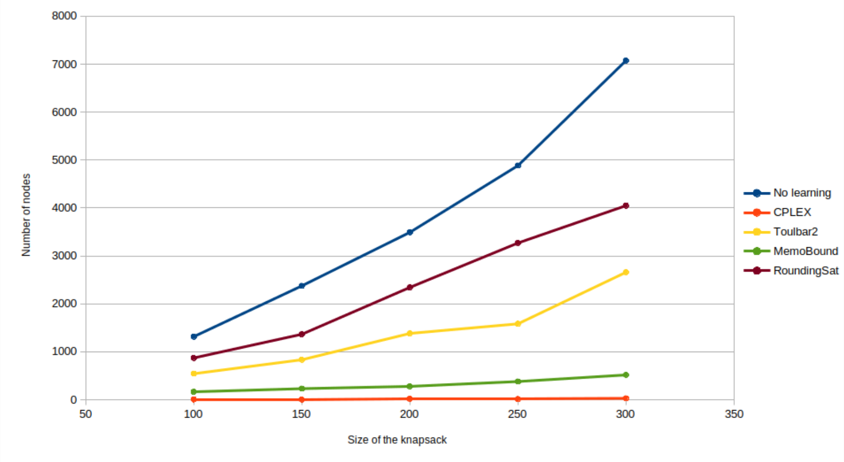
No learning	Depth-first search B&B Branch on the first fractional variable Value removal by bound propagation Solve the LP
MemoBound	+ Rewrite the objective function Value removal by node consistency Learn memo constraints

Results

Knapsack problem

- 100-300 variables
- Random weights and profits

Results



Number of nodes
8000
7000
6000
5000
4000
3000
2000
1000
0
50
100
150
200
250
300
350
Size of the knapsack
No learning
CPLEX
Toulbar2
MemoBound
RoundingSat

Experimental results

Knapsack Problem with Conflict Graph (KPCG)

- 120 or 250 Boolean variables.
- One linear constraint of size equal to the number of variables.
- Weights are uniformly distributed. Sometimes correlated to the profit (instance C).
- A conflict graph (binary constraints) of varying density (0.1-0.3).
- 10 instances in each class

KPCG

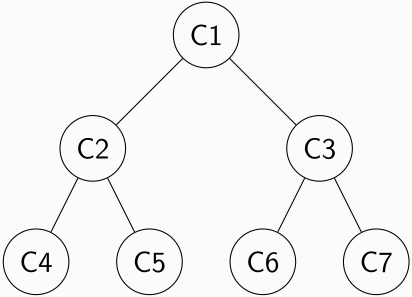
	No learning	MemoBound	TOULBAR2	ROUNDINGSAT	CPLEX
$R^1 120 - 0.1$	1030	226	213	2660	0.3
$R^1 120 - 0.2$	1054	270	260	2558	0
$R^1 120 - 0.3$	1004	298	270	2546	1.7
$R^1 250 - 0.1$	2123	418	522	8985	0
$R^3 120 - 0.1$	2272	472	476	5330	0.8
$R^3 120 - 0.2$	3064	906	1026	6061	39.2
$R^3 120 - 0.3$	3115	1487	1886	6650	66.8
$R^3 250 - 0.1$	9423	1223	1679	17532	10.3
$C^1 120 - 0.1$	10989	2646	1580	6064	0
$C^1 120 - 0.2$	8672	2292	1151	8779	5.8
$C^1 120 - 0.3$	6437	2156	1537	8043	73

Table: Average number of nodes developed to solve different configurations of the KPCG problem.

Results

Kbtree problem

- 44 to 188 Boolean variables
- 190 to 838 binary constraints
- Very specific structure (bounded tree-width)



- Each cluster is a complete graph.
- 2 clusters are connected by 2 *separators*.
- kb-7-3 indicates clusters of size 7 and tree height 3.

Results

	Variables	Constraints	No learning	MemoBound	TOULBAR2-BTD	ROUNDINGSAT	CPLEX
kb-7-3	44	190	7.79	5.9	5.58	4649	0 (573)
kb-7-4	92	406	43	17	16.33	64549	0 (1235)
kb-7-5	188	838	1240	262	37	-	0 (2556)
kb-8-3	51	246	13.89	8	12	7568	0 (758)
kb-8-4	107	526	128	40	40	153374	0 (1613)
kb-9-3	58	309	26	14	27	19153	0(979)
kb-9-4	122	661	457	156	95	-	0(2071)

Conclusion

- The proof of concept validates our theory

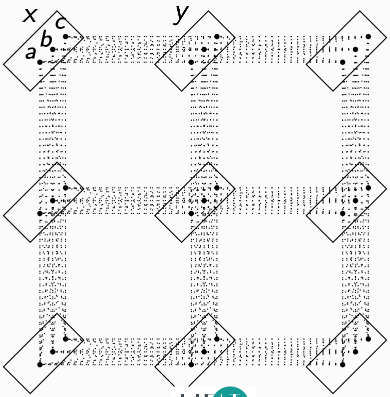
- Implementation in a fully functional solver?
- Heuristics?
  - Restart
  - Selecting the learned constraints
  - Constraint strengthening

- Theoretical results
  - Comparable to dynamic programming?

Conflict-free learning in WCSP

Graphical Models

- The nodes represent discrete domain variables
- (Hyper)-edges represent interactions between variables.



Different types of GM:

- Bayesian Networks (probabilities)
- Markov Random Fields (potentials)
- **Cost Function Networks** (costs)

Cost function

Definition: Cost function

- Scope A (a set of variables)
- Associate a cost to each tuple in the scope:
  - $f_A : \prod_{x \in A} D_x \rightarrow \mathbb{N} \bigcup \{\infty\}$

Unary cost function f_x

x	f_x
a	2
b	0

Binary cost function f_{xy}

x	y	f_{xy}
a	a	3
a	b	2
b	a	0
b	b	∞

Hard binary constraint f_{xz}

x	z	f_{xz}
a	a	∞
a	b	0
b	a	0
b	b	∞

Weighted Constraint Satisfaction Problem

Definition: Cost Function Network $P = (V, S, f)$

- Set V of discrete domain variables
- Set S of scopes
- For each scope $A \in S$, we define a cost function:
  - $f_A : \prod_{x \in A} D_x \rightarrow \mathbb{N} \bigcup \{\infty\}$

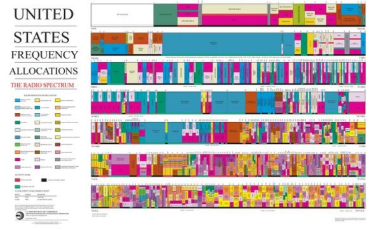
Objective:

Find a complete assignment v minimizing $\sum_{A \in S} f_A(v[A])$

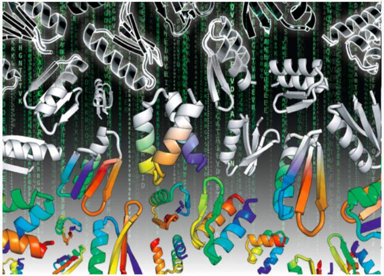
$\rightarrow$ NP-Hard Problem

Cost functions in real life

Frequency Assignment Problem

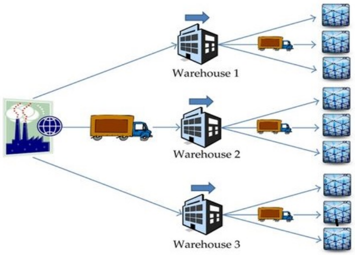


Protein Design



Grid operation-based outage maintenance planning

Warehouse location problem



Example

x	y	f_{xy}
a	a	3
a	b	2
b	a	0
b	b	∞

x	f_x
a	0
b	2

y	f_y
a	0
b	2

$$\min_{x,y} \quad f_x(x) + f_y(y) + f_{xy}(x, y)$$

Example

x	y	f_{xy}
a	a	3
a	b	2
b	a	0
b	b	∞

x	f_x
a	0
b	2

y	f_y
a	0
b	2

$$\min_{x,y} \quad f_x(x) + f_y(y) + f_{xy}(x, y)$$

$(x = b, y = a)$ is the optimal assignment with cost 2.

How to compute the lower bounds ?

- Soft arc consistency algorithms
- Equivalence preserving transformations

Equivalence Preserving Transformation

Equivalent WCSP

$P = (V, S, f)$ and $P' = (V, S, f')$ are equivalent if for any complete assignment v:

$$\sum_{A \in S} f_A(v[A]) = \sum_{A \in S} f'_A(v[A])$$

Equivalence Preserving Transformation (EPT)

Transform a WCSP P into an equivalent WCSP P' by moving costs from a cost function f_A to another cost function f_B.

Example

x	y	f_{xy}
a	a	3
a	b	2
b	a	0
b	b	∞

x	f_x
a	0
b	2

y	f_y
a	0
b	2

$$\min_{x,y} \quad f_x(x) + f_y(y) + f_{xy}(x, y)$$

Example

x	y	f_{xy}
a	a	3-2
a	b	2-2
b	a	0
b	b	∞

x	f_x
a	0+2
b	2

y	f_y
a	0
b	2

$$\min_{x,y} \quad f_x(x) + f_y(y) + f_{xy}(x, y)$$

Example

x	**y**	$\mathbf{f_{xy}}$
a	a	1
a	b	0
b	a	0
b	b	∞

x	$\mathbf{f_x}$
a	2
b	2

y	$\mathbf{f_y}$
a	0
b	2

$$\min_{x,y} \quad f_x(x) + f_y(y) + f_{xy}(x,y)$$

Example

x	y	f_{xy}
a	a	1
a	b	0
b	a	0
b	b	∞

x	f_x
a	2-2
b	2-2

y	f_y
a	0
b	2

$$f_\emptyset = 2$$

$$\min_{x,y} \quad f_x(x) + f_y(y) + f_{xy}(x,y) + f_\emptyset$$

Example

x	y	f_{xy}
a	a	1
a	b	0
b	a	0
b	b	∞

x	f_x
a	0
b	0

y	f_y
a	0
b	2

$$f_\emptyset = 2$$

$$\min_{x,y} \quad f_x(x) + f_y(y) + f_{xy}(x,y) + f_\emptyset$$

How to Solve a WCSP?

Branch&Bound Algorithm

At each node of the search tree produce a sequence of EPTs maximizing $f_\emptyset$

The optimal sequence (using rational costs) can be obtained from the optimal solution of a linear problem: **The Local Polytope**.
However, solving this LP to optimality is often **too expensive**

Soft Local Consistency Algorithms

Reason on a 'local' level by considering only a subset of cost functions.

- Prune locally inconsistent values
- Define a sequence of EPTs increasing $f_\emptyset$

Examples:

Node Consistency, Soft Arc Consistency, Existential Directional Arc Consistency, **Virtual Arc Consistency (VAC)**,...

Global constraints

Examples: alldiff, among, clique, grammar, Pseudo-Boolean...

Global constraints

1. Hard Global Constraint
   - Representation is implicit
   - Design a dedicated propagator
2. Soft global constraint

Conflict-free learning in CFNs

- We know a LP: **The Local Polytope**
- Soft consistency algorithms
  - Find solutions of the The Local Polytope
  - Natively reformulate the problem

Thanks

Thanks for your attention! Questions ?

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