Certified symmetry breaking with VeriPB

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(Thanks to several co-conspirators)
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ARTIFICIAL
INTELLIGENCE
RESEARCH GROUP

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  A permutation σ of literals is a (syntactic) symmetry of a formula F if:
  - σ respects negation: $\overline{\sigma(x)} = \sigma(\overline{x})$
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Example

Consider the formula F:

$$a \lor \overline{b} \lor x \lor y \qquad b \lor c \lor x \lor y$$

$$\overline{c} \lor \overline{d} \lor x \lor y \qquad d \lor \overline{a} \lor x \lor y$$

The permutation

$$(ab\overline{c}d)(xy)(\overline{a}\overline{b}c\overline{d})(\overline{x}\overline{y})$$

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$$(ab\overline{c}d)(xy)(\overline{a}\overline{b}c\overline{d})(\overline{x}\overline{y})$$

is a symmetry of F since $F\restriction_\sigma$ is

$$b \vee c \vee y \vee x \qquad \overline{c} \vee \overline{d} \vee y \vee x$$
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    ($F\upharpoonright_\sigma$ is replacing each x by $\sigma(x)$ in F)
- Symmetric problems are often problematic for vanilla CDCL solvers (insert obligatory reference to PH principle here)

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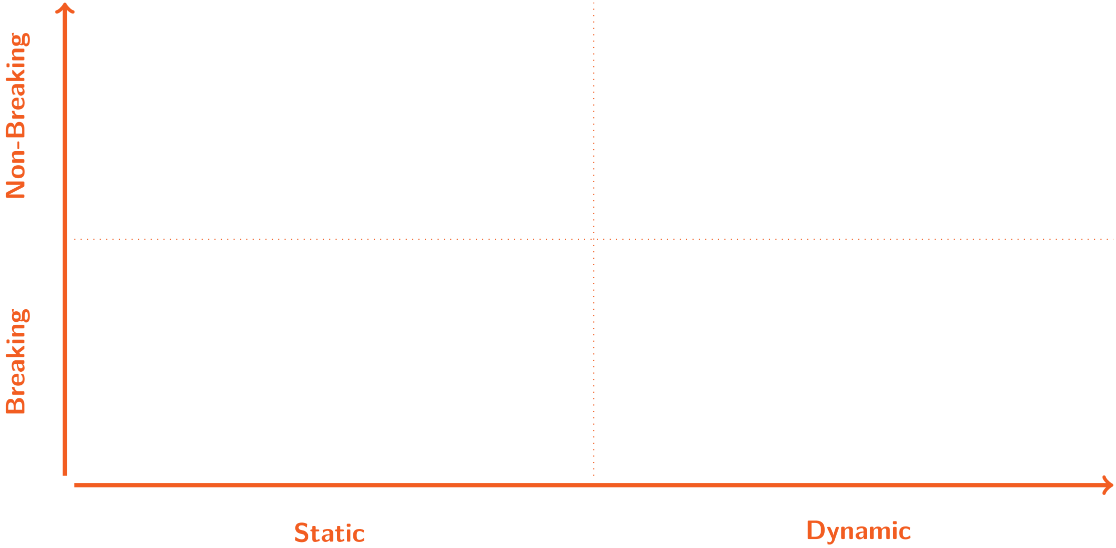
OUTLINE OF THIS TALK

SYMMETRY HANDLING TECHNIQUES



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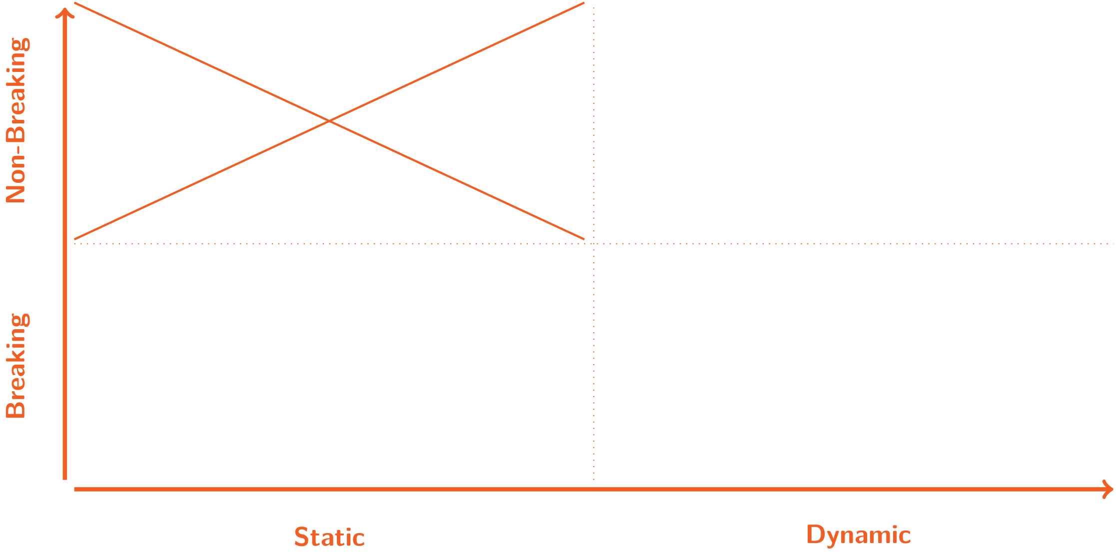

SYMMETRY HANDLING TECHNIQUES

Non-Breaking

Add lex-leader constraint
for symmetries of F:

set of clauses B such that
$\alpha \models B$ iff $\alpha \preceq_{lex} \alpha \circ \sigma$

Breaking

Global symmetry breaking
SHATTER [ASM06]
BREAKID [DBBD16]

Static **Dynamic**

SYMMETRY HANDLING TECHNIQUES

	Static	Dynamic
Non-Breaking	Add lex-leader constraint for symmetries of subformulas of F:	
Breaking	Global symmetry breaking SHATTER [ASM06] BREAKID [DBBD16]	Local symmetry breaking [BS07]

SYMMETRY HANDLING TECHNIQUES

Non-Breaking

For "simple symmetries",
instead of branching on variables,
on the number of variables that are true

Breaking

Global symmetry breaking
SHATTER [ASM06]
BREAKID [DBBD16]

Local symmetry breaking [BS07]
Asymmetric branching SYMCHAFF [Sab09]

Static **Dynamic**

SYMMETRY HANDLING TECHNIQUES

Non-Breaking

Add lex-leader constraint
for symmetries of F

when these clauses would propagate

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Global symmetry breaking
SHATTER [ASM06]
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Local symmetry breaking [BS07]
Asymmetric branching SYMCHAFF [Sab09]
Effective symmetry breaking [MBCK18]

Static | Dynamic

SYMMETRY HANDLING TECHNIQUES

Non-Breaking

Propagator for $\preceq_{lex}$-minimality

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Global symmetry breaking
SHATTER [ASM06]
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SAT modulo symmetries [KS21]
SAT modulo CAS [BKG19]

Static **Dynamic**

SYMMETRY HANDLING TECHNIQUES

Non-Breaking

When SAT solver learns c,
also learn $c{\upharpoonright}_\sigma$ (if this seems "interesting")

Symmetric Learning [HKM+05]
[SHvM09, BNOS10, DBD+12, DBB17]

Breaking

Global symmetry breaking
SHATTER [ASM06]
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Local symmetry breaking [BS07]
Asymmetric branching SYMCHAFF [Sab09]
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Static | **Dynamic**

SYMMETRY HANDLING TECHNIQUES

	Static	Dynamic
Non-Breaking	Hybrid combination of Effective symmetry breaking predicates (first) and symmetric learning (for symmetries not broken completely)	Symmetric Learning [HKM$^+$05] [SHvM09, BNOS10, DBD$^+$12, DBB17]
		ESBP+SP [MBK19]
Breaking	Global symmetry breaking SHATTER [ASM06] BREAKID [DBBD16]	Local symmetry breaking [BS07] Asymmetric branching SYMCHAFF [Sab09] Effective symmetry breaking [MBCK18] SAT modulo symmetries [KS21] SAT modulo CAS [BKG19]

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Symmetric learning

- Recently proposed proof logging [TD20]
  1. Special-purpose, specific approach
  2. Requires adding explicit concept of symmetries
  3. Not compatible with preprocessing techniques

  Better to keep proof system super-simple(?)

THE VERIPB PROOF SYSTEM

A proof system for **pseudo-Boolean optimization problems**

- Reasons with general pseudo-Boolean constraints
- Builds on cutting planes
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Details about the proof checker, see Stephan Gocht's PhD thesis [Goc22]

PSEUDO-BOOLEAN CONSTRAINTS

Pseudo-Boolean constraints are 0-1 integer linear constraints

$$\sum_i a_i \ell_i \geq A$$

- $a_i, A \in \mathbb{Z}$
- literals ℓ_i: x_i or $\overline{x}_i$ (where $x_i + \overline{x}_i = 1$)
- as before, variables x_i take values $0 = \mathit{false}$ or $1 = \mathit{true}$

PSEUDO-BOOLEAN REASONING: CUTTING PLANES [CCT87]

Literal axioms $$\frac{}{\ell_i \geq 0}$$

Linear combination $$\frac{\sum_i a_i \ell_i \geq A \qquad \sum_i b_i \ell_i \geq B}{\sum_i (c_A a_i + c_B b_i)\ell_i \geq c_A A + c_B B} \quad [c_A, c_B \in \mathbb{N}]$$

Division $$\frac{\sum_i c a_i \ell_i \geq A}{\sum_i a_i \ell_i \geq \lceil A/c \rceil} \quad [c \in \mathbb{N}^+]$$

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Sometimes weaker criterion needed — recall that to get variable a encoding

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we introduced pseudo-Boolean constraints

$$3\overline{a} + 3x + 2y + z + w \geq 3 \qquad 5a + 3\overline{x} + 2\overline{y} + \overline{z} + \overline{w} \geq 5$$

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Wish to allow without-loss-of-generality arguments that can derive non-implied constraints

REDUNDANCE-BASED STRENGTHENING

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C is redundant with respect to F iff there is a substitution ω (mapping variables to truth values or literals), called a witness, for which

$$F \wedge \neg C \models (F \wedge C)\restriction_{\omega}$$

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α satisfies $\phi\restriction_\omega$ iff $\alpha \circ \omega$ satisfies ϕ

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Witness ω should be specified, and implication should be efficiently verifiable, which is the case for constraints in $(F \wedge C)\restriction_\omega$ that are, e.g.,

- Reverse unit propagation (RUP) constraints w.r.t. $F \wedge \neg C$
- Obviously implied by a single constraint among $F \wedge \neg C$

DERIVING $A \leftrightarrow (3X + 2Y + Z + W \geq 3)$ USING THE REDUNDANCE RULE

Want to derive

$$3\overline{a} + 3x + 2y + z + w \geq 3 \qquad 5a + 3\overline{x} + 2\overline{y} + \overline{z} + \overline{w} \geq 5$$

using condition $F \land \neg C \models (F \land C)\restriction_\omega$

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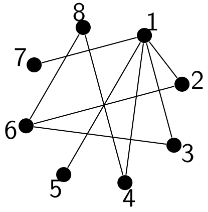
   Choose $\omega = \{a \mapsto 1\}$ — F untouched; new constraint satisfied
   $\neg(5a + 3\overline{x} + 2\overline{y} + \overline{z} + \overline{w} \geq 5)$ forces $3\overline{x} + 2\overline{y} + \overline{z} + \overline{w} \leq 4$
   This is the same constraint as $3x + 2y + z + w \geq 3$
   And VeriPB can automatically detect this

TOY EXAMPLE OF REDUNDANCE RULE

Is the following graph 2-colourable (bipartite)?



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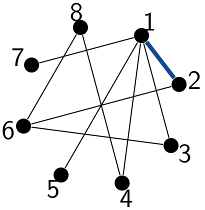

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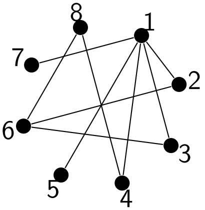

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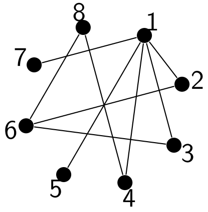

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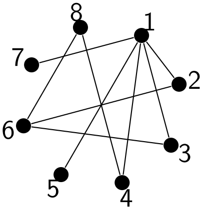

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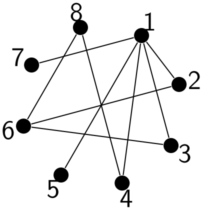

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pseudo-Boolean proof version 2.0
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↪ b6 -> ~b6 b7 -> ~b7 b8 -> ~b8
rup >= 1;
output NONE
conclusion UNSAT
end pseudo-Boolean proof
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- Start the proof

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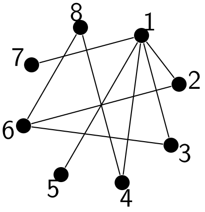

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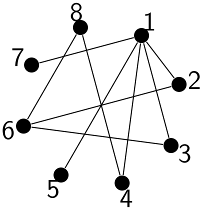

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- Derive contradiction by RUP

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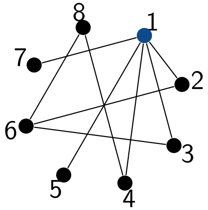

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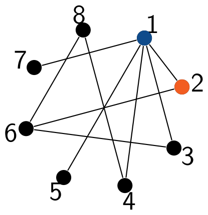

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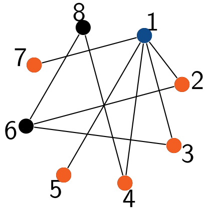

$b_1 + b_2 \geq 1$

$\bar{b}_1 + \bar{b}_2 \geq 1$

$b_1 + b_3 \geq 1$

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$\dots$

Without loss of generality, node 1 is blue (otherwise we can swap the two colours)

Derive $b_1 \geq 1$ with redundance; witness $\omega : b_i \mapsto \bar{b}_i$

$$F = F\restriction_\omega \quad \text{and} \quad \neg(b_1 \geq 1) = (b_1 \geq 1)\restriction_\omega$$

```
pseudo-Boolean proof version 2.0
f 18
red 1 b1  >= 1 ; b1 -> ~b1 b2 -> ~b2
↪ b3 -> ~b3 b4 -> ~b4 b5 -> ~b5
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rup >= 1;
output NONE
conclusion UNSAT
end pseudo-Boolean proof
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- Start the proof
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- Derive contradiction by RUP

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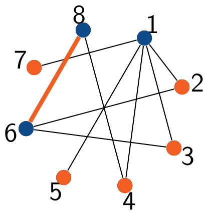

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REDUNDANCE AND DOMINANCE RULES FOR OPTIMISATION

Redundance-based strengthening, optimisation version

Add constraint C to formula F if exists witness substitution ω s.t.

$$F \wedge \neg C \models (F \wedge C)\restriction_\omega \ \wedge \ f\restriction_\omega \leq f$$

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7. ...
8. Can't go on forever, so finally reach α' satisfying $F \wedge C$

STRENGTH OF DOMINANCE RULE

Dominance-based strengthening (stronger, still simplified)

If $C_1, C_2, \ldots, C_{m-1}$ have been derived from F (maybe using dominance), then can derive C_m if exists witness substitution ω s.t.

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Only consider F — no need to show that any $C_i\restriction_\omega$ implied!

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Further extensions:

- Define dominance rule w.r.t. order independent of objective
- Switch between different orders in same proof
- See [BGMN23] for details

STRATEGY FOR SYMMETRY BREAKING IN SAT SOLVING

1. Pretend to solve optimisation problem minimizing $f \doteq \sum_{i=1}^{n} 2^{n-i} \cdot x_i$ (search for lexicographically smallest assignment satisfying formula)

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   (in same spirit as [GMNO22])

$$y_0 \qquad\qquad \overline{y}_j \vee \overline{\sigma(x_j)} \vee x_j$$

$$\overline{y}_{j-1} \vee \overline{x}_j \vee \sigma(x_j) \qquad\qquad y_j \vee \overline{y}_{j-1} \vee \overline{x}_j$$

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$$\overline{y}_j + y_{j-1} \geq 1 \qquad y_j + \overline{y}_{j-1} + \sigma(x_j) \geq 1$$

SYMMETRY BREAKING: EXAMPLE

Example: Pigeonhole principle (PHP) formula

- Variables p_{ij} $(1 \leq i \leq 4, 1 \leq j \leq 3)$ true iff pigeon i in hole j
- Focus on pigeon symmetries — notation:
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Order: "Pick smallest hole for pigeon 1, then smallest for pigeon 2, ..."

$$f \doteq 2^{11} \cdot p_{13} + 2^{10} \cdot p_{12} + 2^{9} \cdot p_{11} + 2^{8} \cdot p_{23} + \cdots + 1 \cdot p_{41}$$

BREAKING A SINGLE SIMPLE SYMMETRY (EXAMPLE)

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Similar to DRAT symmetry breaking [HHW15]

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This idea does not generalize

- Breaking two symmetries

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Intuitively, C_{1234} holds if shifting all the pigeons results in a worse assignment
Can satisfy this constraint by applying $\sigma_{(1234)}$ once, twice, or thrice

BREAKING SYMMETRIES WITH THE DOMINANCE RULE (1/2)

Definition

Given a symmetry σ, the (pseudo-Boolean) breaking constraint of σ is

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Theorem ([BGMN23])

C_σ can be derived from F using dominance with witness σ

$$F \wedge \neg C_\sigma \models F\restriction_\sigma \wedge f\restriction_\sigma < f$$

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Breaking symmetries with the dominance rule

- Surprisingly simple

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  - Works for multiple symmetries (can ignore previously derived symmetry breaking constraints)

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- Generalizes well
  - Works for arbitrary symmetries
  - Works for multiple symmetries (can ignore previously derived symmetry breaking constraints)

$$F \wedge C_{12} \wedge \neg C_{23} \models F\restriction_{\sigma_{(23)}} \wedge f\restriction_{\sigma_{(23)}} < f$$

Why does it work?

- Witness need not satisfy all derived constraints
- Sufficient to just produce “better” assignment

STRATEGY FOR SYMMETRY BREAKING IN SAT SOLVING

1. Pretend to solve optimisation problem minimizing $f \doteq \sum_{i=1}^{n} 2^{n-i} \cdot x_i$ (search for lexicographically smallest assignment satisfying formula)
2. Derive (for proof log only) pseudo-Boolean version of lex-leader constraint

$$C_\sigma \quad \doteq \quad f \leq f\restriction_\sigma \quad \doteq \quad \sum_{i=1}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0$$

3. Derive CNF encoding of lex-leader constraint used by SAT solver from pseudo-Boolean constraint (in same spirit as [GMNO22])

$$\begin{aligned} y_0 &\geq 1 & \overline{y}_j + \overline{\sigma(x_j)} + x_j &\geq 1 \\ \overline{y}_{j-1} + \overline{x}_j + \sigma(x_j) &\geq 1 & y_j + \overline{y}_{j-1} + \overline{x}_j &\geq 1 \\ \overline{y}_j + y_{j-1} &\geq 1 & y_j + \overline{y}_{j-1} + \sigma(x_j) &\geq 1 \end{aligned}$$

SYMMETRY BREAKING IN CNF

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- We use the encoding of BREAKID [DBBD16]:

$$\begin{aligned} y_0 \\ \overline{y}_{j-1} + \overline{x}_j + \sigma(x_j) \\ \overline{y}_j + y_{j-1} \\ \overline{y}_j + \overline{\sigma(x_j)} + x_j \\ y_j + \overline{y}_{j-1} + \overline{x}_j \\ y_j + \overline{y}_{j-1} + \sigma(x_j) \end{aligned}$$

SYMMETRY BREAKING IN CNF

- In SAT symmetry breaking tools, symmetry is broken by adding clausal constraints
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$$y_0$$
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$$y_j + \overline{y}_{j-1} + \overline{x}_j$$
$$y_j + \overline{y}_{j-1} + \sigma(x_j)$$

Define y_j true if x_k equals $\sigma(x_k)$ for all $k \leq j$

$$y_k \Leftrightarrow y_{k-1} \wedge (x_k \Leftrightarrow \sigma(x_k))$$

(derivable with redundance rule)

SYMMETRY BREAKING IN CNF

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Define y_j true if x_k equals $\sigma(x_k)$ for all $k \leq j$

$$y_k \Leftrightarrow y_{k-1} \wedge (x_k \Leftrightarrow \sigma(x_k))$$

(derivable with redundance rule)

If y_{j-1} is true, x_j is at most $\sigma(x_j)$
(derivable from the PB breaking constraint)

BACK TO OUR PIGEONS — PRETEND OPTIMISATION PROBLEM

Start the proof and load input formula

```
pseudo-Boolean proof version 2.0
f 22
pre_order exp
  vars
    left  u1 u2 u3 u4 u5 u6 u7 u8 u9 u10 u11 u12
    right v1 v2 v3 v4 v5 v6 v7 v8 v9 v10 v11 v12
    aux
  end
  def
    -1 u12 1 v12 -2 u11 2 v11 [...] -1024 u2 1024 v2 -2048 u1 2048 v1 >= 0;
  end
  transitivity
    vars
      fresh_right w1 w2 w3 w4 w5 w6 w7 w8 w9 w10 w11 w12
    end
  proof
    proofgoal #1
      pol  1 2 + 3 +
      qed -1
    qed
  end
end
load_order exp p13 p12 p11 p23 p22 p21 p31 p32 p33 p41 p42 p43
```

BACK TO OUR PIGEONS — PRETEND OPTIMISATION PROBLEM

Start the proof and load input formula

1. Pretend to solve optimisation problem minimizing $f \doteq 2^{11} \cdot p_{13} + 2^{10} \cdot p_{12} + 2^9 \cdot p_{11} + 2^8 \cdot p_{23} + \cdots + 2 \cdot p_{42} + 1 \cdot p_{41}$

```
pseudo-Boolean proof version 2.0
f 22
pre_order exp
  vars
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    right v1 v2 v3 v4 v5 v6 v7 v8 v9 v10 v11 v12
    aux
  end
  def
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Start the proof and load input formula

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(Actually defining an order — see [BGMN23] for details)

```
pseudo-Boolean proof version 2.0
f 22
pre_order exp
  vars
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    right v1 v2 v3 v4 v5 v6 v7 v8 v9 v10 v11 v12
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load_order exp p13 p12 p11 p23 p22 p21 p31 p32 p33 p41 p42 p43
```

BACK TO OUR PIGEONS — DERIVING THE CONSTRAINTS

Derived constraints ($\mathcal{D}$):

$$2^{11} \cdot (p_{23} - p_{13}) + 2^{10} \cdot (p_{22} - p_{12}) + \cdots \geq 0$$

```
dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 [...] p23 -> p13 ; begin
  proofgoal #2
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red 1 y0  >= 1 ; y0 -> 1
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red 1 ~p13 1 ~y0  1 y1  >= 1 ; y1 -> 1
pol 26 32 2048 * +
del id 26
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```

Pseudo-Boolean breaking constraint

BACK TO OUR PIGEONS — DERIVING THE CONSTRAINTS

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Pseudo-Boolean breaking constraint

Use dominance with witness $\sigma = (p_{11} p_{21})(p_{12} p_{22})(p_{13} p_{23})$

BACK TO OUR PIGEONS — DERIVING THE CONSTRAINTS

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Pseudo-Boolean breaking constraint

Use dominance with witness $\sigma = (p_{11}p_{21})(p_{12}p_{22})(p_{13}p_{23})$

$$F \wedge \neg C_{12} \models F\restriction_\omega \wedge \left(f\restriction_\omega < f\right)$$

VeriPB fills in all missing subproofs except for $\neg C_{12} \wedge C_{12} \models \bot$

BACK TO OUR PIGEONS — DERIVING THE CONSTRAINTS

Derived constraints ($\mathcal{D}$):

$$2^{11} \cdot (p_{23} - p_{13}) + 2^{10} \cdot (p_{22} - p_{12}) + \cdots \geq 0$$

y_0

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dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 [...] p23 -> p13 ; begin
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rup 1 ~y1  1 ~p12 1 p22 >= 1 ;
```

Derivable by redundance with witness $\omega = \{y_0 \mapsto 1\}$

$$F \wedge \mathcal{D} \wedge \neg(y_0) \models (F \wedge \mathcal{D})\restriction_\omega \wedge (y_0)\restriction_\omega$$

BACK TO OUR PIGEONS — DERIVING THE CONSTRAINTS

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$$F \wedge \mathcal{D} \wedge \neg(y_0) \models (F \wedge \mathcal{D})\restriction_\omega \wedge (y_0)\restriction_\omega$$

$$F \wedge \mathcal{D} \wedge (\overline{y}_0) \models (F \wedge \mathcal{D}) \quad \wedge (1)$$

BACK TO OUR PIGEONS — DERIVING THE CONSTRAINTS

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$\overline{y}_0 + \overline{p}_{13} + \sigma(p_{13})$

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Derivable by RUP

$$F \wedge \mathcal{D} \wedge \neg(\overline{y}_0 + \overline{p}_{13} + \sigma(p_{13}))$$

BACK TO OUR PIGEONS — DERIVING THE CONSTRAINTS

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```

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$$F \wedge \mathcal{D} \wedge \neg(\overline{y}_0 + \overline{p}_{13} + \sigma(p_{13})) \models F \wedge \mathcal{D} \wedge (y_0) \wedge (p_{13}) \wedge (\overline{p}_{23})$$

BACK TO OUR PIGEONS — DERIVING THE CONSTRAINTS

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BACK TO OUR PIGEONS — DERIVING THE CONSTRAINTS

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Derivable by RUP

$$F \wedge \mathcal{D} \wedge \neg(\overline{y}_0 + \overline{p}_{13} + \sigma(p_{13})) \models F \wedge \mathcal{D} \wedge (y_0) \wedge (p_{13}) \wedge (\overline{p}_{23})$$

$$2^{11} \cdot (\quad -1 \quad) + 2^{10} \cdot (p_{22} - p_{12}) + \cdots \geq 0$$

BACK TO OUR PIGEONS — DERIVING THE CONSTRAINTS

Derived constraints ($\mathcal{D}$):

$$2^{11} \cdot (p_{23} - p_{13}) + 2^{10} \cdot (p_{22} - p_{12}) + \cdots \geq 0$$

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pol 26 32 2048 * +
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```

Derivable by RUP

$$F \wedge \mathcal{D} \wedge \neg(\overline{y}_0 + \overline{p}_{13} + \sigma(p_{13})) \models F \wedge \mathcal{D} \wedge (y_0) \wedge (p_{13}) \wedge (\overline{p}_{23})$$

$$2^{11} \cdot (\quad -1 \quad) + 2^{10} \cdot (p_{22} - p_{12}) + \cdots \geq 0$$

where $\sum_{i=1}^{10} 2^i < 2^{11}$

BACK TO OUR PIGEONS — DERIVING THE CONSTRAINTS

Derived constraints ($\mathcal{D}$):

$$2^{11} \cdot (p_{23} - p_{13}) + 2^{10} \cdot (p_{22} - p_{12}) + \cdots \geq 0$$

y_0

$\overline{y}_0 + \overline{p}_{13} + \sigma(p_{13})$

$\overline{y}_1 + y_0$

```
dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 [...] p23 -> p13 ; begin
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red 1 ~p13 1 ~y0  1 y1  >= 1 ; y1 -> 1
pol 26 32 2048 * +
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rup 1 ~y1  1 ~p12 1 p22 >= 1 ;
```

Derivable by redundance with witness $\omega = \{y_1 \mapsto 0\}$

$$F \wedge \mathcal{D} \wedge \neg(\overline{y}_1 + y_0) \models (F \wedge \mathcal{D})\restriction_\omega \wedge (\overline{y}_1 + y_0)\restriction_\omega$$

BACK TO OUR PIGEONS — DERIVING THE CONSTRAINTS

Derived constraints ($\mathcal{D}$):

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pol 26 32 2048 * +
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```

Derivable by redundance with witness $\omega = \{y_1 \mapsto 0\}$

$$F \wedge \mathcal{D} \wedge \neg(\overline{y}_1 + y_0) \models (F \wedge \mathcal{D})\restriction_\omega \wedge (\overline{y}_1 + y_0)\restriction_\omega$$

$$F \wedge \mathcal{D} \wedge (y_1 + \overline{y}_0 \geq 2) \models (F \wedge \mathcal{D}) \quad \wedge (1 + y_0)$$

BACK TO OUR PIGEONS — DERIVING THE CONSTRAINTS

Derived constraints ($\mathcal{D}$):

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y_0

$\overline{y}_0 + \overline{p}_{13} + \sigma(p_{13})$

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```

Derivable by redundance with witness $\omega = \{y_1 \mapsto 0\}$ (essentially same argument)

BACK TO OUR PIGEONS — DERIVING THE CONSTRAINTS

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$\overline{y}_1 + y_0$

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$y_1 + \overline{y}_0 + \overline{p}_{13}$

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rup 1 ~y1  1 ~p12 1 p22 >= 1 ;
```

Derivable by redundance with witness $\omega = \{y_1 \mapsto 1\}$

$$F \wedge \mathcal{D} \wedge \neg(y_1 + \overline{y}_0 + \overline{p}_{13}) \models (F \wedge \mathcal{D})\restriction_\omega \wedge (y_1 + \overline{y}_0 + \overline{p}_{13})\restriction_\omega$$

BACK TO OUR PIGEONS — DERIVING THE CONSTRAINTS

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$y_1 + \overline{y}_0 + \overline{p}_{13}$

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```

Derivable by redundance with witness $\omega = \{y_1 \mapsto 1\}$

$$F \wedge \mathcal{D} \wedge \neg(y_1 + \overline{y}_0 + \overline{p}_{13}) \models (F \wedge \mathcal{D})\restriction_\omega \wedge (y_1 + \overline{y}_0 + \overline{p}_{13})\restriction_\omega$$

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BACK TO OUR PIGEONS — DERIVING THE CONSTRAINTS

Derived constraints ($\mathcal{D}$):

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(same argument)

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Simplify the pseudo-Boolean breaking constraint and delete original constraint

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Continue in the same way for following y_i-variables

...

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  - Isomorphisms might not be symmetries of the SAT encoding
  - How to verify that symmetry breaking was complete?

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- For static (and some dynamic) symmetry breaking, fully general symmetry breaking in VeriPB
  - **Challenge:** get this to work in (some extension of) DRAT
- Makes use heavily of dominance rule
  - **Challenge:** analyze this rule (can extended Frege simulate it?)
- Claim that this generalizes to dynamic symmetry breaking methods
  - **Challenge:** Verify this for other dynamic symmetry breaking methods
  - **Challenge:** What about SAT modulo symmetries?
- For symmetric learning, dedicated proof system has been developed
  - **Challenge:** develop certification in a formalism that doesn't know about symmetries
  - Proofs with lemmas?

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Thank you for your attention!

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