Proof Logging for Pseudo-Boolean Optimization

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Based on joint work with Daniel Le Berre, Magnus O. Myreen, Jakob Nordström, Andy Oertel, Yong Kiam Tan, and Marc Vinyals

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  - Simplicity: Keep the proof system simple
  - Trustworthiness:
    - ⋆ Proofs are fully complete, so each step easy to check
    - ⋆ Checker has a formally verified backend

Proof Logging: Existing Work Beyond SAT

- Constraint Programming:
  - Early work [VS10]: no full coverage, not trustworthy
  - *VeriPB* Proof Logging [EGMN20, GMN22, MM23, MMN24, MM25]: efficiency is a problem (e.g. logging overhead $\times 10$, checking overhead $\times 1000$)

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- SMT solving: *Alethe* [SFBF21], *Carcara* [ALB23]: no full coverage, efficiency problem

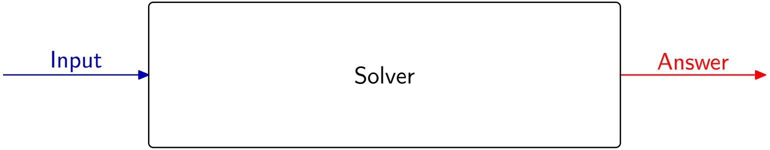
This Talk

- Efficient *VeriPB* proof logging and checking for pseudo-Boolean optimization [KLM+25]
- Covers all techniques in state-of-the-art solvers *RoundingSat* [EN18] and *Sat4j* [LP10]

This Talk

- Efficient *VeriPB* proof logging and checking for pseudo-Boolean optimization [KLM+25]
- Covers all techniques in state-of-the-art solvers *RoundingSat* [EN18] and *Sat4j* [LP10]
- Performance close to our goals:
  - Proof logging overhead usually $\leqslant 10\%$ (worst-case 50%)
  - Checking overhead usually $\leqslant \times 6$ (worst-case $\times 20$)
- First time practically feasible proof logging beyond SAT

Proof Logging with Certifying Solvers: Workflow



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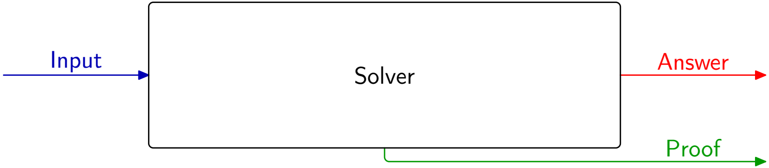

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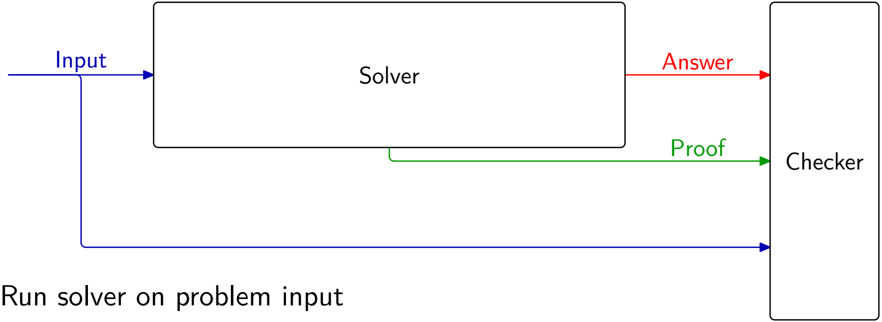

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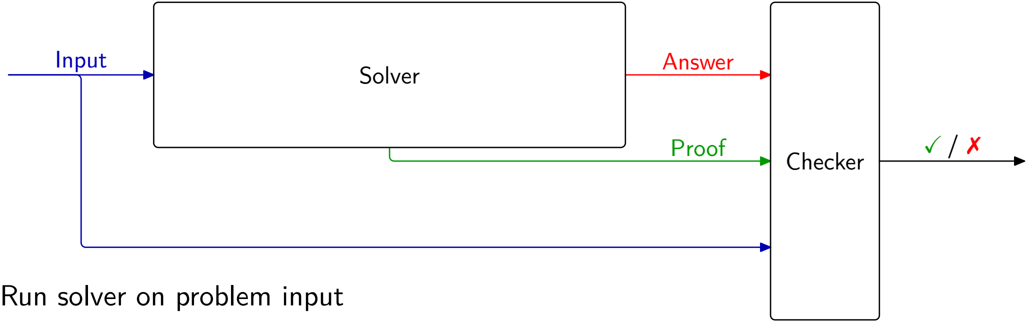

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4. Verify that proof checker says answer is correct

Overview of This Talk

1. Pseudo-Boolean Solving and Optimization
2. The *VeriPB* Proof System
3. Optimization Techniques
4. Core-Guided Optimization
5. LP Integration
6. Empirical Results
7. Concluding Remarks

Pseudo-Boolean Optimization

- Operates on 0-1 integer linear inequalities or pseudo-Boolean constraints:

$$\sum_i a_i \ell_i \geq A$$

  - $a_i, A \in \mathbb{Z}$
  - literals ℓ_i: x_i or $\overline{x}_i$ (where $x_i + \overline{x}_i = 1$)
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  - Cardinality constraints: $x_1 + x_2 + x_3 \geq 2$
  - General constraints: $3x_1 + 2x_2 + x_3 + x_4 \geq 3$

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  Example: After deciding $x_1 = 0$, constraint $3x_1 + 2x_2 + x_3 + x_4 \geq 3$ propagates $x_2 = 1$
- Conflict-driven search:
  - Try to build satisfying assignment literal by literal using decisions and propagations
  - When falsifying constraint, derive constraint explaining the conflict and add to formula

Conflict Analysis Example

- Let

$$C_1 \doteq \overline{z} + \overline{w} \geq 1 \qquad C_2 \doteq \overline{y} + w \geq 1 \qquad C_3 \doteq 2x + y + z \geq 2$$

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Approaches for Pseudo-Boolean Solving and Optimization

- Two main approaches for pseudo-Boolean solving:
  - CNF-based: Translate to CNF and run conflict-driven clause learning (CDCL)
  - Native PB: Generalize conflict-driven search to pseudo-Boolean constraints (focus of this talk)

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  - CNF-based: Translate to CNF and run conflict-driven clause learning (CDCL)
  - Native PB: Generalize conflict-driven search to pseudo-Boolean constraints (focus of this talk)
- New challenges and techniques for native PB solving compared to SAT:
  - Efficient propagation [Dev20, NORZ24]
  - Linear programming (LP) integration [DGN21]
  - Optimization techniques, e.g. solution-improving search, core-guided search [DGD+21]

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  - In PB, explicit description of conflict analysis steps required
- Other techniques pose further challenges:
  - Objective rewriting in core-guided search
  - Linear programming (LP) integration (Farkas certificates, cut generation, ...)
- Low-level challenges for truly efficient proof logging and checking:
  - Logging unit constraints (saying that a variable must take some fixed value, e.g. $x_2 \geq 1$)
  - Logging constraint simplifications (e.g. simplifying away variables with fixed values)
  - Logging and checking solutions
  - Optimizing formally verified proof checking

Pseudo-Boolean Proof Logging Basics

Pseudo-Boolean proof logging based on cutting planes proof system [CCT87]

Input axioms From the input

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Proof of soundness:

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Division is crucial for Boolean (as opposed to real-valued) reasoning:

- Addition and multiplication valid over the reals
- Literal axioms $\ell_i \geq 0$ and $\overline{\ell_i} = 1 - \ell_i \geq 0$ valid for all reals in $[0, 1]$
- Division only valid over the integers: e.g. $2x_1 \geq 1$ implies $x_1 \geq 1$

Conflict Analysis Example: *VeriPB* Derivation

$$
\text{Add}\ \dfrac{\text{Add}\ \dfrac{\overline{z} + \overline{w} \geq 1 \qquad \overline{y} + w \geq 1}{\overline{y} + \overline{z} \geq 1} \qquad 2x + y + z \geq 2}{\text{Divide by } 2\ \dfrac{2x \geq 1}{x \geq 1}}
$$

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By naming constraints by labels as

$$\text{Constraint @C1} \doteq \overline{z} + \overline{w} \geq 1$$
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$$\dfrac{\text{Divide by 2}\quad\dfrac{\text{Add}\quad\dfrac{\text{Add}\quad\dfrac{\overline{z}+\overline{w}\geq 1 \qquad \overline{y}+w\geq 1}{\overline{y}+\overline{z}\geq 1} \qquad 2x+y+z\geq 2}{2x\geq 1}}{}}{x\geq 1}$$

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such a calculation is written in the proof log in reverse Polish notation as

```
pol  @C1  @C2  +  @C3  +  2  d ;
```

Advanced Pseudo-Boolean Proof Logging

We need a rule for deriving non-implied constraints (e.g. introducing new variables)

Redundance-based strengthening ([BT19, GN21], inspired by [JHB12], simplified)

F and $F \cup \{C\}$ are equisatisfiable if there is a substitution ω (mapping variables to truth values or literals), called a witness, for which

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When using rule in a proof, the implication needs to be efficiently verifiable — every $D \in (F \cup \{C\})\restriction_\omega$ should follow from $F \cup \{\neg C\}$ either "obviously" or by explicit derivation

Redundance-Based Strengthening: Example

Suppose we know $D \doteq x_1 + x_2 + x_3 \geq 2$.

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VeriPB:

```
red +1 x1 +1 x2 +1 x3 -1 y3 <= 2 : y3 -> 1;
red +1 x1 +1 x2 +1 x3 -1 y3 >= 2 : y3 -> 0;
```

Proof by Contradiction

- F and $F \cup \{C\}$ are equisatisfiable if $F \cup \{\neg C\} \models \bot$
- Can be seen as a special case of the redundance rule (empty witness ω)

Proof by Contradiction: Example

- From

$$C_1 \doteq 2t + x_1 + x_2 \geq 2 \qquad C_2 \doteq 2\bar{t} + x_1 + x_2 \geq 2$$

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VeriPB:

```
pbc +1 x1 +1 x2 >= 2 :  subproof
    pol @C1 -1 + 2 d @C2 -1 + 2 d + ;
qed pbc :  -1;
```

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- Decision problems:
  - Satisfiable instances: just provide a solution
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- Optimization problems: provide:
  (i) a solution with value UB, and
  (ii) a derivation of the inequality $Obj \geq LB$

  (Optimality proven if $UB = LB$)

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- Find solutions with better and better objective values
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- Example: Let $Obj = x_1 + 2x_2 + x_3$

  We find the solution $x_1 = x_3 = 1$, $x_2 = 0$ with objective value 2

  Then `soli x1 ~x2 x3` introduces constraint $Obj \leq 1$, i.e. $x_1 + 2x_2 + x_3 \leq 1$

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- Can also do this starting from pre-chosen literal values
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- These pre-chosen values are called assumptions

- Possible outcomes:
  - Consistent $\rightarrow$ find solution to formula
  - Inconsistent $\rightarrow$ learn constraint (called core) why assumptions are inconsistent

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- Introduce fresh variables y_k such that

$$\sum_{i=1}^{k} \ell_i = A + \sum_{i=A+1}^{k} y_i$$

(y_j is true iff $\sum_{i=1}^{k} \ell_i \geq j$ for $A + 1 \leq j \leq k$)

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- Repeat with rewritten objective

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- Rewrite objective:

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- Shows that $Obj \geq 4$
- Next assume $x_1 = x_3 = x_4 = y_3 = 0 \ldots$

Proof Logging for Core-Guided Optimization: Example

We know $D \doteq x_2 + x_3 + x_4 \geq 2$. Want to introduce a variable y_3 such that

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using condition $F \cup \{\neg C\} \models (F \cup \{C\})\restriction_\omega$.

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- $F \cup \{C_1\} \cup \{\neg C_2\} \models (F \cup \{C_1\} \cup \{C_2\})\restriction_\omega$
  Choose $\omega = \{y_3 \mapsto 0\}$ — F untouched; new constraint $C_2\restriction_\omega$ follows from D;
  $\neg C_2 \doteq x_2 + x_3 + x_4 \leq 1 + y_3$ implies $C_1\restriction_\omega \doteq x_2 + x_3 + x_4 \leq 2$

VeriPB:

```
red +1 x2 +1 x3 +1 x4 -1 y3 <= 2 : y3 -> 1;
red +1 x2 +1 x3 +1 x4 -1 y3 >= 2 : y3 -> 0;
```

Proof Logging for Core-Guided Optimization: Some Further Details

$$Obj_{\text{orig}} = x_1 + 2(x_2 + x_3 + x_4) + x_3 + 2x_4$$
$$Obj_{\text{rewritten}} = x_1 + 2(2 + y_3) + x_3 + 2x_4$$

- Multiplying $x_2 + x_3 + x_4 \geq 2 + y_3$ by 2 yields inequality $Obj_{\text{orig}} \geq Obj_{\text{rewritten}}$ (after canceling rest of objective from both sides)
- Used to show, e.g., that $Obj_{\text{rewritten}} \geq LB$ implies $Obj_{\text{orig}} \geq LB$
- Other inequality needed in solver

LP Relaxation

- Linear programming (LP) relaxation: allow variables to take any real value in $[0, 1]$
- In practice usually solved quickly using simplex algorithm
- Relaxation has a better/lower optimal objective value

Pseudo-Boolean Solving: LP Integration

- Recall: conflict-driven search tries to build satisfying assignment
- Partial assignments may yield unsatisfiable subproblem even over the reals
- Propagation does not necessarily detect this, but LP solving can

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- Possible outcomes when solving LP relaxation on formula + partial assignment:
  - infeasibility → generate Farkas certificate
  - found integral solution → this solution is optimal
  - found fractional solution → add constraints ‘cutting away’ fractional solution: cut generation

Farkas Certificates

If solver decides $y = 0$, then constraints

$$\begin{aligned}
C_1 &\doteq y + x_1 + x_2 + x_3 \geq 2 \\
C_2 &\doteq y + 3x_1 + 2x_2 + x_3 + x_4 \geq 3 \\
C_3 &\doteq - 2x_1 - 2x_2 - x_3 \geq -1
\end{aligned}$$

are infeasible over the reals, so $y \geq 1$ must hold

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Round multipliers provided by LP solver to integers and check in exact arithmetic

Farkas Certificates: Proof Logging

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a Farkas certificate is

$$C_1 + C_2 + 2 \cdot C_3 + (\overline{x_4} \geq 0) + (x_2 \geq 0) \doteq 2y \geq 2$$

Divide by 2 to get $y \geq 1$

Farkas Certificates: Proof Logging

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Divide by 2 to get $y \geq 1$

VeriPB: `pol @C1 @C2 + @C3 2 * + ~x4 + x2 + 2 d;`

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- Cut generation:
  - Add constraint (*cut*) implied by input formula
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- Example: Minimize $x_1 + x_2 + x_3$ subject to

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  - Rational optimum $x_1 = x_2 = x_3 = \frac{1}{2}$

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  - Cutting planes division by 2 yields $x_1 + x_2 + x_3 \geq 2$
  - *VeriPB*: `pol @C1 @C2 + @C3 + 2 d;`

Advanced Cut Generation

- Cut generation with mixed integer rounding (MIR) rule [MW01, DGN21] more challenging
- MIR rule is stronger than cutting planes division
- Reasoning uses integer slack variables (not supported by *VeriPB*)
- Proof logging instead uses proof by contradiction

Advanced Cut Generation: MIR cut

- MIR cut: given a constraint $\sum_i a_i \ell_i \geq A$ and a divisor $d \in \mathbb{N}^+$, derive

$$\sum_i \left(\min\{a_i \bmod d, A \bmod d\} + \left\lfloor \frac{a_i}{d} \right\rfloor (A \bmod d) \right) \ell_i \geq \left\lceil \frac{A}{d} \right\rceil (A \bmod d)$$

- We call $R = A \bmod d$ the multiplier of the MIR cut
- Example: Applying a MIR cut with divisor $d = 5$ to

$$10x_1 + 5x_2 + 6x_3 + 3x_4 + x_5 \geq 12$$

  yields

$$4x_1 + 2x_2 + 3x_3 + 2x_4 + x_5 \geq 6$$

- Cutting planes division by $d = 5$ and multiplying by $R = 12 \bmod 5 = 2$ yields weaker constraint

$$4x_1 + 2x_2 + 4x_3 + 2x_4 + 2x_5 \geq 6$$

Advanced Cut Generation: Example

- For constraints

$$C_1 \doteq 6x_1 + 5x_2 + 2x_3 + 3x_4 \geq 8, \qquad C_2 \doteq x_1 + x_3 \geq 1$$

introduce integral slack variables $s_1, s_2 \geq 0$ to obtain

$$C_1' \doteq 6x_1 + 5x_2 + 2x_3 + 3x_4 - s_1 = 8, \qquad C_2' \doteq x_1 + x_3 - s_2 = 1$$

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- Compute linear combination $C_1' + 4 \cdot C_2'$, and only keep $\geq$ part:

$$10x_1 + 5x_2 + 6x_3 + 3x_4 - s_1 - 4s_2 \geq 12$$

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$$4x_1 + 2x_2 + 3x_3 + 2x_4 - s_2 \geq 6$$

- Subtract C_2' to obtain

$$3x_1 + 2x_2 + 2x_3 + 2x_4 \geq 5$$

Proof Logging for Advanced Cut Generation: Example

$$C_1 \doteq 6x_1 + 5x_2 + 2x_3 + 3x_4 \geq 8, \qquad C_2 \doteq x_1 + x_3 \geq 1$$

- We prove resulting cut $D \doteq 3x_1 + 2x_2 + 2x_3 + 2x_4 \geq 5$ by contradiction
- Can use negation $\neg D \doteq 3x_1 + 2x_2 + 2x_3 + 2x_4 \leq 4 \doteq -3x_1 - 2x_2 - 2x_3 - 2x_4 \geq -4$

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$$\text{Add} \quad \frac{-3x_1 - 2x_2 - 2x_3 - 2x_4 \geq -4 \qquad 6x_1 + 5x_2 + 2x_3 + 3x_4 \geq 8}{3x_1 + 3x_2 + x_4 \geq 4}$$

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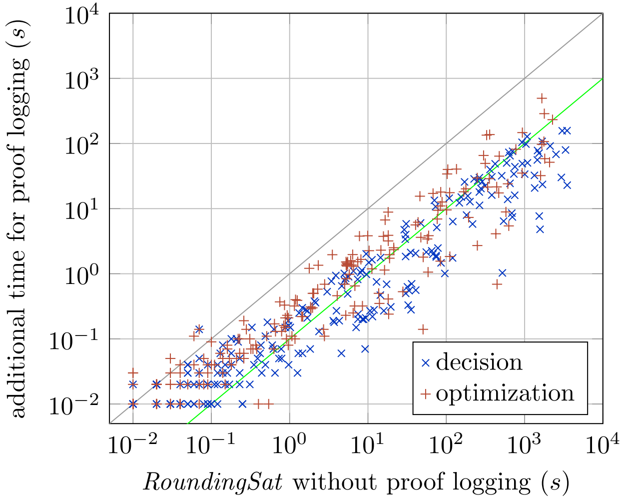
VeriPB:

```
pbc +3 x1 +2 x2 +2 x3 +2 x4 >= 5 :  subproof
    pol -1 @C1 + 3 d 2 * -1 + @C2 + ;
qed pbc :  -1;
```

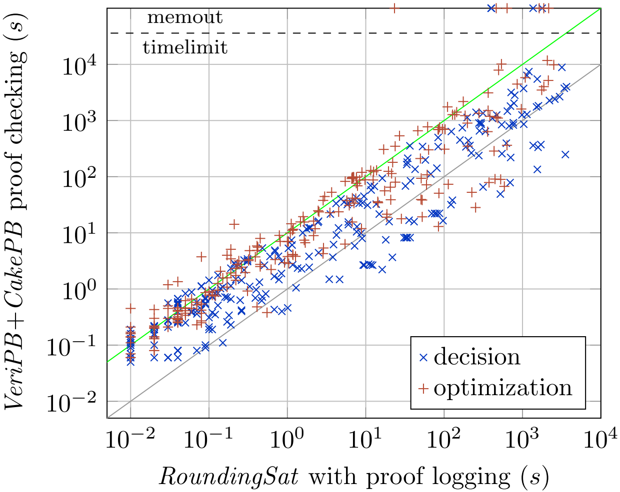
Empirical Results: Proof Logging Overhead for *RoundingSat*

- Usually $\leqslant 10\%$
- Decision instances: worst-case 20%
- Optimization instances: worst-case 50%
- Goal: $\leqslant 10\%$
- Overheads gets smaller for larger solving times



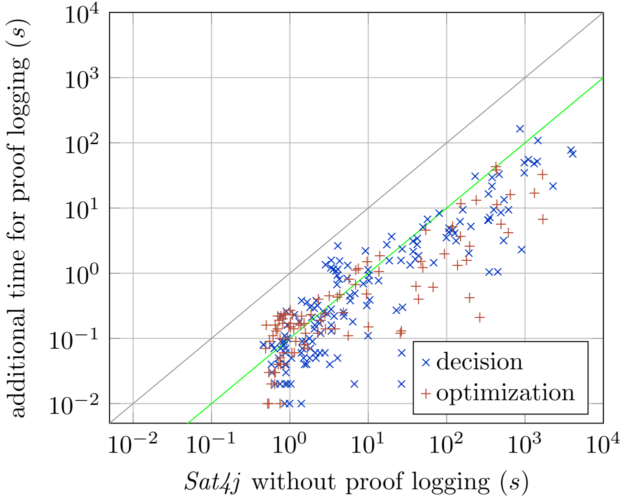
Empirical Results: Proof Checking Overhead for *RoundingSat*

- Usually $\leqslant \times 6$
- Decision instances: worst-case $\times 10$
- Optimization instances: worst-case $\times 20$
- Goal: $\leqslant \times 10$



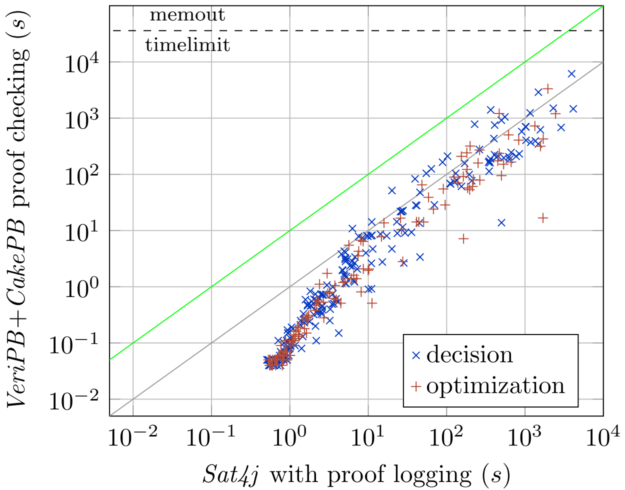
Empirical Results: Proof Logging Overhead for *Sat4j*

- Usually $\leqslant 10\%$
- Worst-case 60%
- Goal: $\leqslant 10\%$



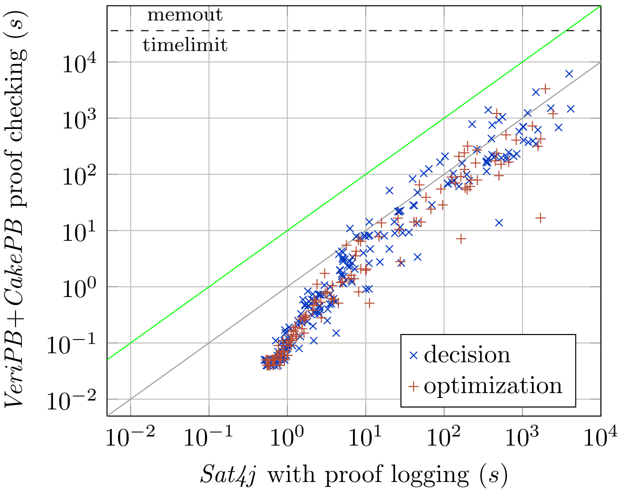
Empirical Results: Proof Checking Overhead for *Sat4j*

- Usually $\leqslant \times 2$
- Worst-case $\times 4$
- Goal: $\leqslant \times 10$



Empirical Results: Proof Checking Overhead for *Sat4j*

- Usually $\leqslant \times 2$
- Worst-case $\times 4$
- Goal: $\leqslant \times 10$
- Lower overheads than *RoundingSat*:
  - Fewer advanced techniques
  - Java is a bit slower than C++



Using Proof Logging to Detect Inefficiency Bugs

- Main purpose of proof logging: detect soundness bugs
- Can also detect bugs leading to inefficiencies (but not unsound reasoning)

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- Two examples:
  - Unnecessarily large coefficient in a constraint
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- Having to specify derivation explicitly (in contrast to SAT) can also be an advantage

Challenges for Efficient Proof Logging and Checking

- Attention to detail
  - Caveat: many low-level details skipped
  - Getting these right requires in-depth understanding of both solver and *VeriPB*
  - So efficient proof logging is not just adding a few simple print statements

Challenges for Efficient Proof Logging and Checking

- Attention to detail
  - Caveat: many low-level details skipped
  - Getting these right requires in-depth understanding of both solver and *VeriPB*
  - So efficient proof logging is not just adding a few simple print statements
- Different perspectives in solver and proof checker
  - *Sat4j* simplifies input constraints but considers them “the same”
  - In the proof these constraints are clearly different
  - Requires painful book-keeping during proof logging
  - New feature of labelling constraints very helpful for this

Future Work

- Even faster proof logging and checking for pseudo-Boolean optimization
  - Branch-and-bound search (checking solutions currently a bottleneck)
  - Native efficient support for simplifications of constraints
  - Low-level optimizations in *VeriPB* and formally verified backend *CakePB*

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  - Branch-and-bound search (checking solutions currently a bottleneck)
  - Native efficient support for simplifications of constraints
  - Low-level optimizations in *VeriPB* and formally verified backend *CakePB*
- Faster proof logging and checking for further paradigms:
  - MaxSAT solving
  - Subgraph solving
  - Constraint programming
  - ...

Take-away Message

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  - Plus explanations how to certify correctness with proof logging
  - First example of practically feasible certified solving beyond SAT

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  - Efficient certified solving in other paradigms
- Is this the start of a new era: practically feasible proof logging beyond SAT?

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  - Survey of some techniques in pseudo-Boolean optimization
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- Is this the start of a new era: practically feasible proof logging beyond SAT?

Thank you! Any questions?

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