Proof Complexity as a Computational Lens

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Three Simple Problems. . .

COLOURING

Does the graph $G = (V, E)$ have a colouring with k colours such that all neighbours have distinct colours?

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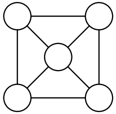

3-colouring?

Three Simple Problems...

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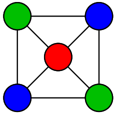

3-colouring? Yes

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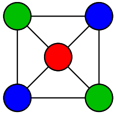

3-colouring? Yes, but no 2-colouring

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CLIQUE

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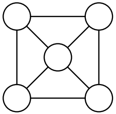

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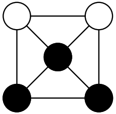

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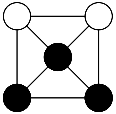

3-clique? Yes, but no 4-clique

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Given propositional logic formula, is there a satisfying assignment?

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Given propositional logic formula, is there a satisfying assignment?

$$(x \lor z) \land (y \lor \neg z) \land (x \lor \neg y \lor u) \land (\neg y \lor \neg u)$$
$$\land\ (u \lor v) \land (\neg x \lor \neg v) \land (\neg u \lor w) \land (\neg x \lor \neg u \lor \neg w)$$

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- Variables should be set to **true** or **false**
- Constraint $(x \lor \neg y \lor z)$: means x or z should be true or y false
- $\land$ means all constraints should hold simultaneously
- Is there a truth value assignment satisfying all constraints?

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COLOURING:	frequency allocation for mobile base stations
CLIQUE:	bioinformatics, computational chemistry
SAT:	easily models these and many other problems

. . . with Huge Practical Implications

- Some more examples of problems that can be encoded as propositional logic formulas:
  - computer hardware verification
  - computer software testing
  - artificial intelligence
  - operations research
  - cryptography
  - bioinformatics
  - et cetera. . .
- Leads to **humongous** formulas (100,000s or even 1,000,000s of variables)
- Can we use computers to solve these problems efficiently?

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- SAT mentioned already in Gödel's famous letter in 1956 to von Neumann
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- SAT problem is NP-complete, so probably very hard [Coo71, Lev73]
- Assuming P $\neq$ NP, even impossible to meaningfully approximate
  - COLOURING [Kho01, Zuc07]
  - CLIQUE [Hås99]
  - SAT [Hås01]

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  - COLOURING [Kho01, Zuc07]
  - CLIQUE [Hås99]
  - SAT [Hås01]
- Except that in practice, there are good algorithms for
  - COLOURING [DLMM08, DLMO09, DLMM11]
  - CLIQUE [Pro12, McC17]

  and amazing conflict-driven clause learning (CDCL) solvers [BS97, MS99, MMZ$^+$01] that solve huge SAT problem instances

How can we understand real-world algorithms for NP-hard problems?

This lecture: Use proof complexity (not only conceivable answer)

Algorithmic View of Proof Complexity

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1. Is there a short proof of the right answer using rules in this proof system?
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Study infeasible problems — proofs of feasibility are trivial

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Question 2: Topic for separate lecture(s) — lots of recent exciting progress; mostly negative (worst-case) results that proof search is hard, e.g., [AM20, GKMP20, dRGN$^+$21]

Applications of Proof Complexity

Three applied reasons for proof complexity:

1. Understand real-world applied algorithmic paradigms [**this lecture**]
2. Get ideas for algorithmic improvements
   [EN18, EN20, LBD+20, DGD+21, DGN21, KBBN22, MBGN23, MSB+25]
   (See, e.g., tutorials youtu.be/VC0CHXoWnS4 and youtu.be/FIJ3k7HWpiQ about RoundingSat)
3. Enhance algorithms to write machine-verifiable certificates of correctness
   [EGMN20, GMN20, GMM+20, GN21, GMN22, GMNO22, BBN+23, BGMN23, MM23, BBN+24, DMM+24, GMM+24, HOGN24, IOT+24, MMN24, DHN+25, KLM+25, MM25]
   (See tutorial youtu.be/s_5BIi4I22w about VeriPB)

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Or just view this as a convenient excuse to study nice computational complexity problems for their own sake. . . ☺

Outline

1. DPLL, CDCL, and Resolution
   - Davis-Putnam-Logemann-Loveland (DPLL) Method
   - Conflict-Driven Clause Learning (CDCL)
   - Resolution Proof System
2. Algebraic and Semi-algebraic Approaches
   - Nullstellensatz
   - Gröbner Bases and Polynomial Calculus
   - Pseudo-Boolean Solving and Cutting Planes
3. Some More Advanced Proof Systems
   - Sherali-Adams and Sums of Squares
   - Stabbing Planes
   - Extended Resolution

Some Preliminaries

- Variable x: takes value **true** $(= 1)$ or **false** $(= 0)$
- Literal ℓ: variable x or its negation $\overline{x}$ (write $\overline{x}$ instead of $\neg x$)
- Clause $C = \ell_1 \lor \cdots \lor \ell_k$: disjunction of literals
  (Consider as sets, so no repetitions and order irrelevant)
- Conjunctive normal form (CNF) formula $F = C_1 \land \cdots \land C_m$: conjunction of clauses

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- Refer to clauses of CNF formula as axioms (as opposed to derived clauses)
- N denotes size of formula (# literals counted with repetitions)
- $\mathcal{O}(f(N))$ grows at most as quickly as $f(N)$ asymptotically
  $\Omega(g(N))$ grows at least as quickly as $g(N)$ asymptotically
  $\Theta(h(N))$ grows equally quickly as $h(N)$ asymptotically

The SAT Problem

The SATISFIABILITY (or just SAT) Problem

Given a formula F in conjunctive normal form (CNF), is it satisfiable?

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For instance, what about our example CNF formula?

$$(x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \lor u) \land (\overline{y} \lor \overline{u})$$
$$\land\ (u \lor v) \land (\overline{x} \lor \overline{v}) \land (\overline{u} \lor w) \land (\overline{x} \lor \overline{u} \lor \overline{w})$$

The Same Problem in Three Different Shapes

$$(x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \lor u) \land (\overline{y} \lor \overline{u})$$
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$$\begin{aligned}
(1-x)(1-z) &= 0 \\
(1-y)z &= 0 \\
(1-x)y(1-u) &= 0 \\
yu &= 0 \\
(1-u)(1-v) &= 0 \\
xv &= 0 \\
u(1-w) &= 0 \\
xuw &= 0
\end{aligned}$$

For **true** $= 1$ and **false** $= 0$, is there a $\{0,1\}$-valued solution?

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$$\begin{aligned}
1 - x - z + xz &= 0 \\
z - yz &= 0 \\
y - xy - yu + xyu &= 0 \\
yu &= 0 \\
1 - u - v + uv &= 0 \\
xv &= 0 \\
u - uw &= 0 \\
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$1 - x - z + xz = 0$	$x + z \geq 1$
$z - yz = 0$	$y + (1 - z) \geq 1$
$y - xy - yu + xyu = 0$	$x + (1 - y) + u \geq 1$
$yu = 0$	$(1 - y) + (1 - u) \geq 1$
$1 - u - v + uv = 0$	$u + v \geq 1$
$xv = 0$	$(1 - x) + (1 - v) \geq 1$
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$1 - x - z + xz = 0$	$x + z \geq 1$
$z - yz = 0$	$y - z \geq 0$
$y - xy - yu + xyu = 0$	$x - y + u \geq 0$
$yu = 0$	$-y - u \geq -1$
$1 - u - v + uv = 0$	$u + v \geq 1$
$xv = 0$	$-x - v \geq -1$
$u - uw = 0$	$-u + w \geq 0$
$xuw = 0$	$-x - u - w \geq -2$

For **true** $= 1$ and **false** $= 0$, is there a $\{0, 1\}$-valued solution?

Clique and Colouring as CNF Formulas

Clique formula
"The graph $G = (V, E)$ has an m-clique"

$q_{k,1} \lor q_{k,2} \lor \cdots \lor q_{k,n}$	$\lvert V \rvert = n;\ 1 \leq k \leq m$	[some vertex is kth member of clique]
$\overline{q}_{k,u} \lor \overline{q}_{k,v}$	$u \neq v \in V;\ 1 \leq k \leq m$	[clique members are uniquely defined]
$\overline{q}_{k,v} \lor \overline{q}_{k',v}$	$v \in V;\ 1 \leq k < k' \leq m$	[no vertex counted as clique member twice]
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"The graph $G = (V, E)$ is m-colourable"

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(Smarter encodings are possible, but these are good enough for our discussion)

DPLL: Attempting Smart Case Analysis

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4. Set $x = 0$, simplify F and make recursive call
5. Set $x = 1$, simplify F and make recursive call
6. If result in both cases "unsatisfiable", then report "unsatisfiable" and return

A DPLL Toy Example

$$\begin{aligned} F = \quad & (x \vee z) \wedge (y \vee \overline{z}) \wedge (x \vee \overline{y} \vee u) \wedge (\overline{y} \vee \overline{u}) \\ \wedge\ & (u \vee v) \wedge (\overline{x} \vee \overline{v}) \wedge (\overline{u} \vee w) \wedge (\overline{x} \vee \overline{u} \vee \overline{w}) \end{aligned}$$

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Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

"Simplify formula" by (mentally) removing

- satisfied clauses
- falsified literals

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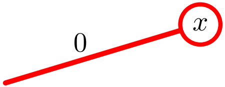

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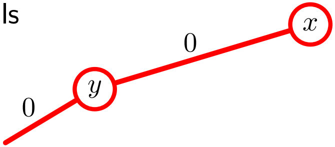
\begin{aligned}
F = \quad & (\quad z) \land (\quad \overline{z}) \land (\quad \overline{y} \lor u) \land (\overline{y} \lor \overline{u}) \\
& \land (u \lor v) \land (\overline{x} \lor \overline{v}) \land (\overline{u} \lor w) \land (\overline{x} \lor \overline{u} \lor \overline{w})
\end{aligned}
$$

Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

"Simplify formula" by (mentally) removing

- satisfied clauses
- falsified literals



A DPLL Toy Example

$$F = (x \lor z) \land (\overline{z}) \land (\overline{y} \lor u) \land (\overline{y} \lor \overline{u}) \land (u \lor v) \land (\overline{x} \lor \overline{v}) \land (\overline{u} \lor w) \land (\overline{x} \lor \overline{u} \lor \overline{w})$$

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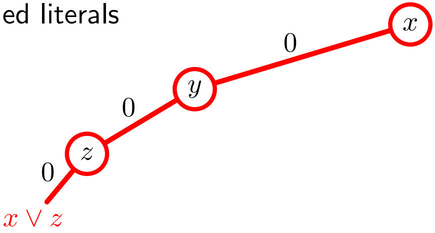

A DPLL Toy Example

$$F = \quad (\qquad z) \land (y \lor \overline{z}) \land (\qquad \overline{y} \lor u) \land (\overline{y} \lor \overline{u})$$
$$\land\ (u \lor v) \land (\overline{x} \lor \overline{v}) \land (\overline{u} \lor w) \land (\overline{x} \lor \overline{u} \lor \overline{w})$$

Visualize execution of DPLL algorithm as search tree

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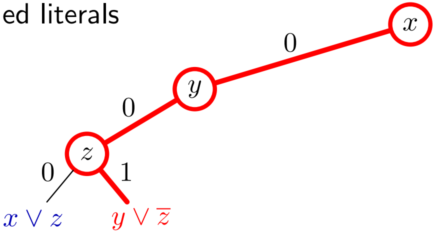

A DPLL Toy Example

$$F = (\quad z) \land (y \lor \overline{z}) \land (\quad u) \land (\quad \overline{u}) \land (u \lor v) \land (\overline{x} \lor \overline{v}) \land (\overline{u} \lor w) \land (\overline{x} \lor \overline{u} \lor \overline{w})$$

Visualize execution of DPLL algorithm as search tree

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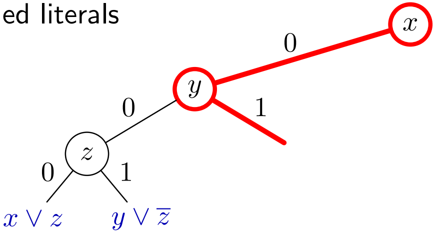

A DPLL Toy Example

$$F = (\quad z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \lor u) \land (\quad \overline{u}) \land (\quad v) \land (\overline{x} \lor \overline{v}) \land (\overline{u} \lor w) \land (\overline{x} \lor \overline{u} \lor \overline{w})$$

Visualize execution of DPLL algorithm as search tree

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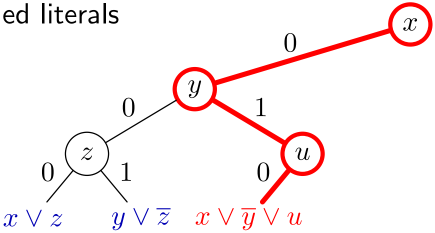

A DPLL Toy Example

$$F = (\quad z) \land (y \lor \overline{z}) \land (\quad u) \land (\overline{y} \lor \overline{u}) \land (u \lor v) \land (\overline{x} \lor \overline{v}) \land (\quad w) \land (\overline{x} \lor \quad \overline{w})$$

Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

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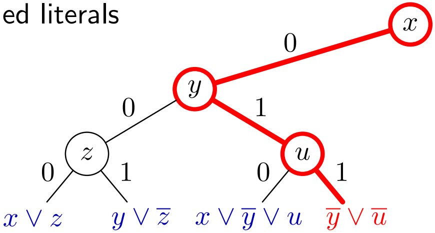


A DPLL Toy Example

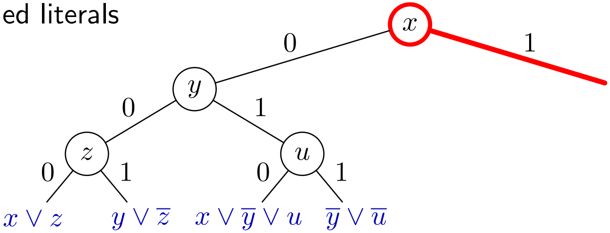
$$F = (x \vee z) \wedge (y \vee \overline{z}) \wedge (x \vee \overline{y} \vee u) \wedge (\overline{y} \vee \overline{u})$$
$$\wedge (u \vee v) \wedge (\overline{v}) \wedge (\overline{u} \vee w) \wedge (\overline{u} \vee \overline{w})$$

Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

"Simplify formula" by (mentally) removing

- satisfied clauses
- falsified literals



A DPLL Toy Example

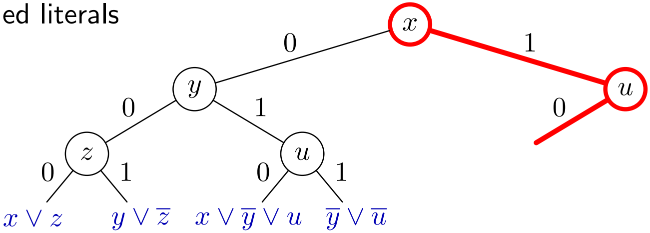
$$F = (x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \quad) \land (\overline{y} \lor \overline{u}) \land (\quad v) \land (\quad \overline{v}) \land (\overline{u} \lor w) \land (\quad \overline{u} \lor \overline{w})$$

Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

"Simplify formula" by (mentally) removing

- satisfied clauses
- falsified literals



A DPLL Toy Example

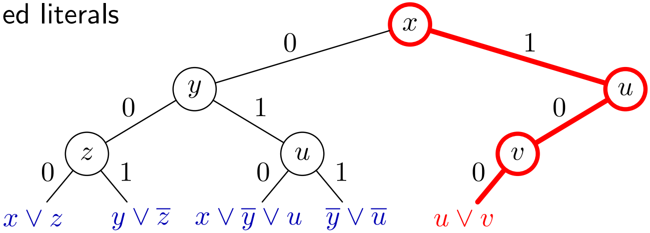
$$F = (x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \quad) \land (\overline{y} \lor \overline{u}) \land (u \lor v) \land (\quad \overline{v}) \land (\overline{u} \lor w) \land (\quad \overline{u} \lor \overline{w})$$

Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

"Simplify formula" by (mentally) removing

- satisfied clauses
- falsified literals



A DPLL Toy Example

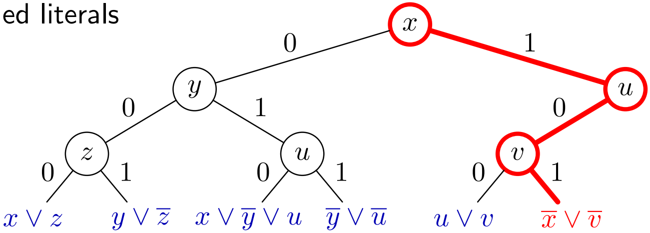
$$F = (x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y}) \land (\overline{y} \lor \overline{u}) \land (v) \land (\overline{x} \lor \overline{v}) \land (\overline{u} \lor w) \land (\overline{u} \lor \overline{w})$$

Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

"Simplify formula" by (mentally) removing

- satisfied clauses
- falsified literals



A DPLL Toy Example

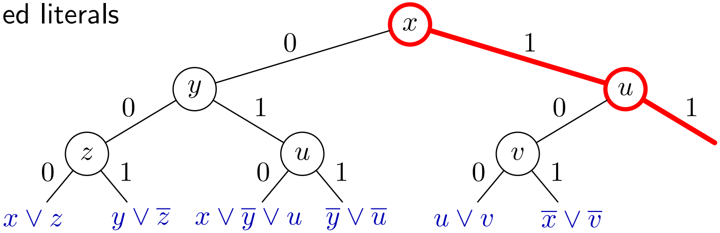
$$\begin{aligned} F = \quad & (x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \lor u) \land (\overline{y} \lor \overline{u}) \\ & \land (u \lor v) \land (\quad \overline{v}) \land (\quad w) \land (\quad \overline{w}) \end{aligned}$$

Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

"Simplify formula" by (mentally) removing

- satisfied clauses
- falsified literals



A DPLL Toy Example

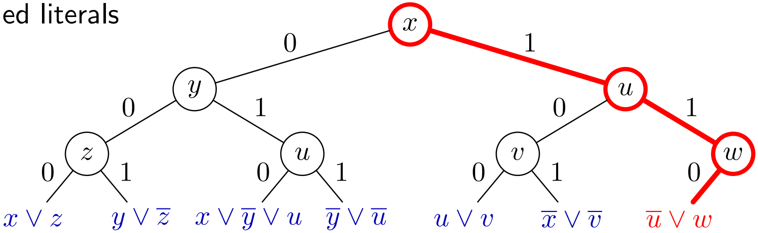
$$\begin{aligned} F = \quad & (x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \lor u) \land (\overline{y} \lor \overline{u}) \\ & \land (u \lor v) \land (\quad \overline{v}) \land (\overline{u} \lor w) \land (\quad \overline{w}) \end{aligned}$$

Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

"Simplify formula" by (mentally) removing

- satisfied clauses
- falsified literals



A DPLL Toy Example

$$F = (x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \lor u) \land (\overline{y} \lor \overline{u}) \land (u \lor v) \land (\overline{v}) \land (w) \land (\overline{x} \lor \overline{u} \lor \overline{w})$$

Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

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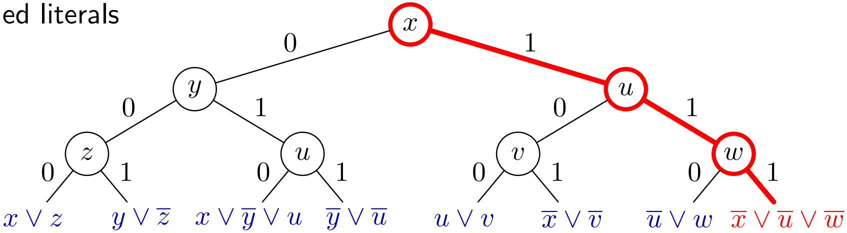

A DPLL Toy Example

$$\begin{aligned} F = \quad & (x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \lor u) \land (\overline{y} \lor \overline{u}) \\ & \land (u \lor v) \land (\overline{x} \lor \overline{v}) \land (\overline{u} \lor w) \land (\overline{x} \lor \overline{u} \lor \overline{w}) \end{aligned}$$

Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

"Simplify formula" by (mentally) removing

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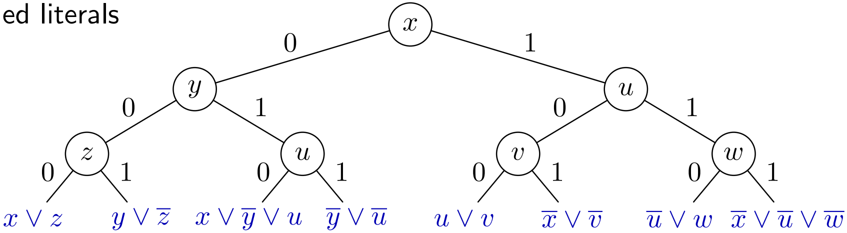

State-of-the-Art SAT Solving in One Slide

High-level description of modern conflict-driven clause learning (CDCL) SAT solving (as pioneered in [BS97, MS99, MMZ^{+}01]):

- Try to build satisfying assignment for formula (branching or decision heuristic crucial)
- When partial assignment violates formula, compute explanation for conflict and add to formula as new clause (clause learning)
- Every once in a while, restart from beginning (but save computed info)

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \vee \overline{u}$ forces $u = 0$

Notation $u \stackrel{p \vee \overline{u}}{=} 0$ ($p \vee \overline{u}$ is reason clause)

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$

$u \stackrel{p \vee \overline{u}}{=} 0$

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \vee \overline{u}$ forces $u = 0$

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Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$

$u \stackrel{p \vee \overline{u}}{=} 0$

$\boxed{q \stackrel{\mathsf{d}}{=} 0}$

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \vee \overline{u}$ forces $u = 0$

Notation $u \stackrel{p \vee \overline{u}}{=} 0$ ($p \vee \overline{u}$ is reason clause)

Always propagate if possible, otherwise decide

Add to assignment trail

Continue until satisfying assignment or conflict

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$

$u \stackrel{p \lor \overline{u}}{=} 0$

$\boxed{q \stackrel{\mathsf{d}}{=} 0}$

$r \stackrel{q \lor r}{=} 1$

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

Notation $u \stackrel{p \lor \overline{u}}{=} 0$ ($p \lor \overline{u}$ is reason clause)

Always propagate if possible, otherwise decide

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Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$\boxed{p \overset{\mathsf{d}}{=} 0}$

$u \overset{p \lor \overline{u}}{=} 0$

$\boxed{q \overset{\mathsf{d}}{=} 0}$

$r \overset{q \lor r}{=} 1$

$w \overset{\overline{r} \lor w}{=} 1$

Decision

Free choice to assign value to variable

Notation $p \overset{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

Notation $u \overset{p \lor \overline{u}}{=} 0$ ($p \lor \overline{u}$ is reason clause)

Always propagate if possible, otherwise decide

Add to assignment trail

Continue until satisfying assignment or conflict

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$

$u \stackrel{p \vee \overline{u}}{=} 0$

$\boxed{q \stackrel{\mathsf{d}}{=} 0}$

$r \stackrel{q \vee r}{=} 1$

$w \stackrel{\overline{r} \vee w}{=} 1$

$\boxed{x \stackrel{\mathsf{d}}{=} 0}$

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \vee \overline{u}$ forces $u = 0$

Notation $u \stackrel{p \vee \overline{u}}{=} 0$ ($p \vee \overline{u}$ is reason clause)

Always propagate if possible, otherwise decide

Add to assignment trail

Continue until satisfying assignment or conflict

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$\boxed{p \overset{\mathsf{d}}{=} 0}$

$u \overset{p \lor \overline{u}}{=} 0$

$\boxed{q \overset{\mathsf{d}}{=} 0}$

$r \overset{q \lor r}{=} 1$

$w \overset{\overline{r} \lor w}{=} 1$

$\boxed{x \overset{\mathsf{d}}{=} 0}$

$y \overset{u \lor x \lor y}{=} 1$

Decision

Free choice to assign value to variable

Notation $p \overset{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

Notation $u \overset{p \lor \overline{u}}{=} 0$ ($p \lor \overline{u}$ is reason clause)

Always propagate if possible, otherwise decide

Add to assignment trail

Continue until satisfying assignment or conflict

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$

$u \stackrel{p \lor \overline{u}}{=} 0$

$\boxed{q \stackrel{\mathsf{d}}{=} 0}$

$r \stackrel{q \lor r}{=} 1$

$w \stackrel{\overline{r} \lor w}{=} 1$

$\boxed{x \stackrel{\mathsf{d}}{=} 0}$

$y \stackrel{u \lor x \lor y}{=} 1$

$z \stackrel{x \lor \overline{y} \lor z}{=} 1$

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

Notation $u \stackrel{p \lor \overline{u}}{=} 0$ ($p \lor \overline{u}$ is reason clause)

Always propagate if possible, otherwise decide

Add to assignment trail

Continue until satisfying assignment or conflict

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

$p \stackrel{\mathsf{d}}{=} 0$

$u \stackrel{p \vee \overline{u}}{=} 0$

$q \stackrel{\mathsf{d}}{=} 0$

$r \stackrel{q \vee r}{=} 1$

$w \stackrel{\overline{r} \vee w}{=} 1$

$x \stackrel{\mathsf{d}}{=} 0$

$y \stackrel{u \vee x \vee y}{=} 1$

$z \stackrel{x \vee \overline{y} \vee z}{=} 1$

$\overset{\overline{y} \vee \overline{z}}{\perp}$

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \vee \overline{u}$ forces $u = 0$

Notation $u \stackrel{p \vee \overline{u}}{=} 0$ ($p \vee \overline{u}$ is reason clause)

Always propagate if possible, otherwise decide

Add to assignment trail

Continue until satisfying assignment or conflict

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$
$u \stackrel{p \lor \overline{u}}{=} 0$
} decision level 1

$\boxed{q \stackrel{\mathsf{d}}{=} 0}$
$r \stackrel{q \lor r}{=} 1$
$w \stackrel{\overline{r} \lor w}{=} 1$
} decision level 2

$\boxed{x \stackrel{\mathsf{d}}{=} 0}$
$y \stackrel{u \lor x \lor y}{=} 1$
$z \stackrel{x \lor \overline{y} \lor z}{=} 1$
$\stackrel{\overline{y} \lor \overline{z}}{\bot}$
} decision level 3

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

Notation $u \stackrel{p \lor \overline{u}}{=} 0$ ($p \lor \overline{u}$ is reason clause)

Always propagate if possible, otherwise decide

Add to assignment trail

Continue until satisfying assignment or conflict

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$

$u \stackrel{p \lor \overline{u}}{=} 0$

decision level 1

$\boxed{q \stackrel{\mathsf{d}}{=} 0}$

$r \stackrel{q \lor r}{=} 1$

$w \stackrel{\overline{r} \lor w}{=} 1$

decision level 2

$\boxed{x \stackrel{\mathsf{d}}{=} 0}$

$y \stackrel{u \lor x \lor y}{=} 1$

$z \stackrel{x \lor \overline{y} \lor z}{=} 1$

$\stackrel{\overline{y} \lor \overline{z}}{\bot}$

decision level 3

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$
$u \stackrel{p \lor \overline{u}}{=} 0$

decision level 1

$\boxed{q \stackrel{\mathsf{d}}{=} 0}$
$r \stackrel{q \lor r}{=} 1$
$w \stackrel{\overline{r} \lor w}{=} 1$

decision level 2

$\boxed{x \stackrel{\mathsf{d}}{=} 0}$
$y \stackrel{u \lor x \lor y}{=} 1$
$z \stackrel{x \lor \overline{y} \lor z}{=} 1$
$\stackrel{\overline{y} \lor \overline{z}}{\bot}$

decision level 3

Could backtrack by erasing conflict level & flipping last decision

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But want to learn from conflict and cut away as much of search space as possible

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$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

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$\stackrel{\overline{y} \vee \overline{z}}{\perp}$

$x \vee \overline{y}$

Could backtrack by erasing conflict level & flipping last decision

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Case analysis over z for last two clauses:

- $x \vee \overline{y} \vee z$ wants $z = 1$
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- Merge clauses & remove z — must satisfy $x \vee \overline{y}$

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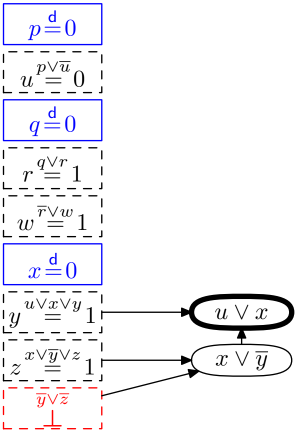

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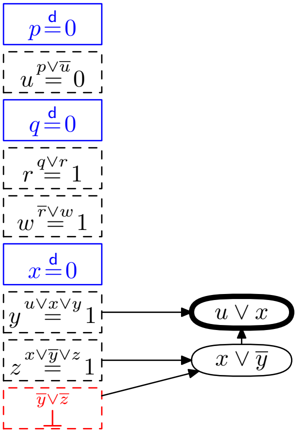
Repeat until UIP clause with only 1 variable at conflict level after last decision — learn and backjump

Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$



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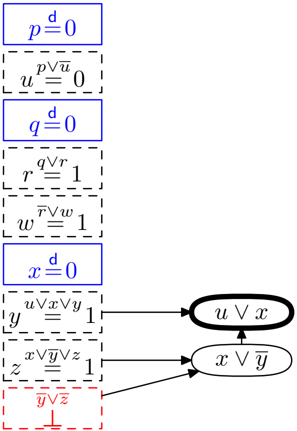

Assertion level 1 (2nd largest level in learned clause) — trim trail to that level

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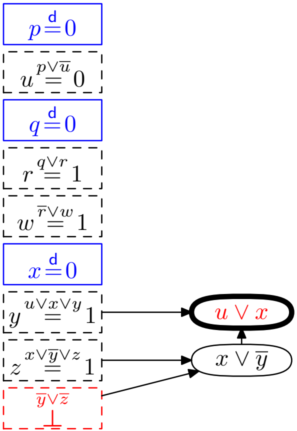

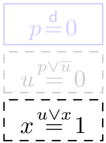

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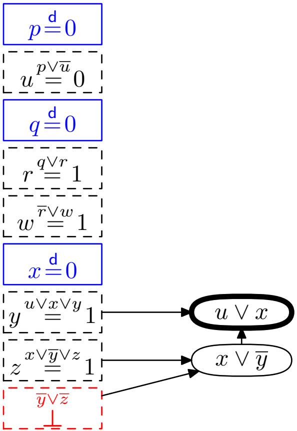

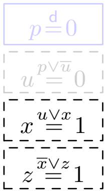

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Then continue as before. . .

Complete Example of CDCL Execution

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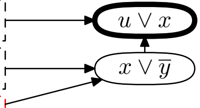

$p \overset{\mathsf{d}}{=} 0$

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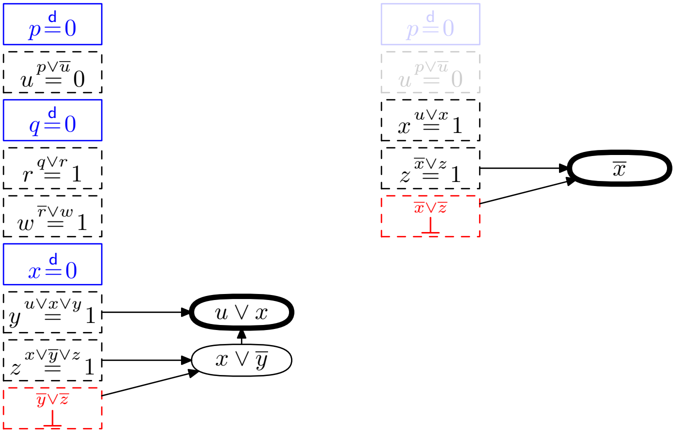

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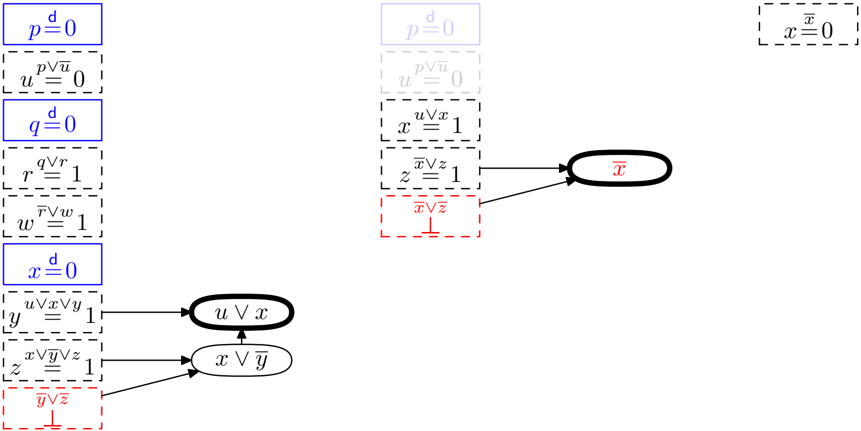

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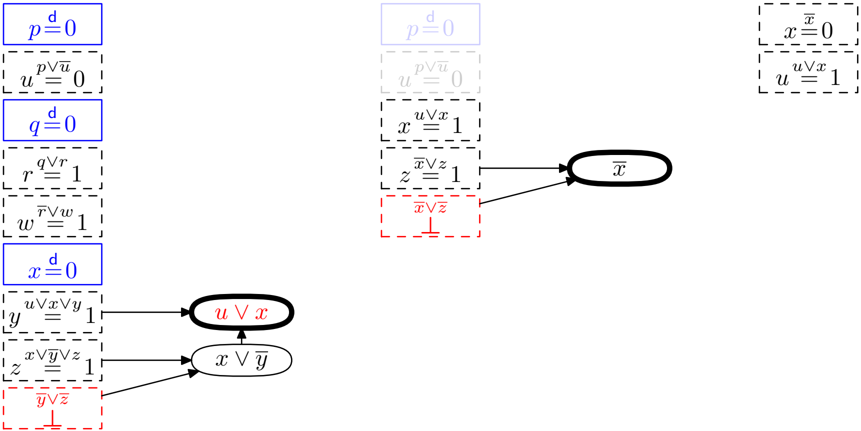

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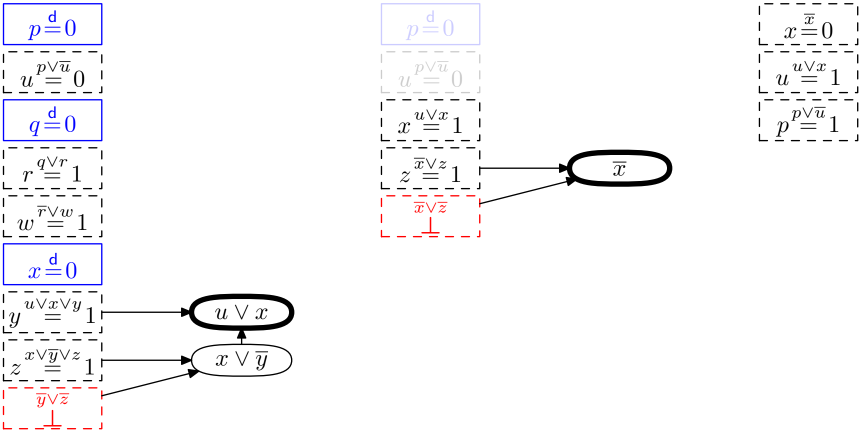

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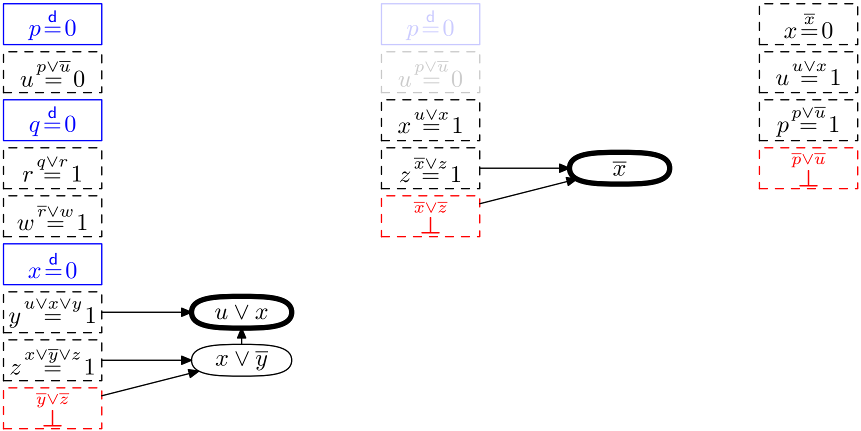

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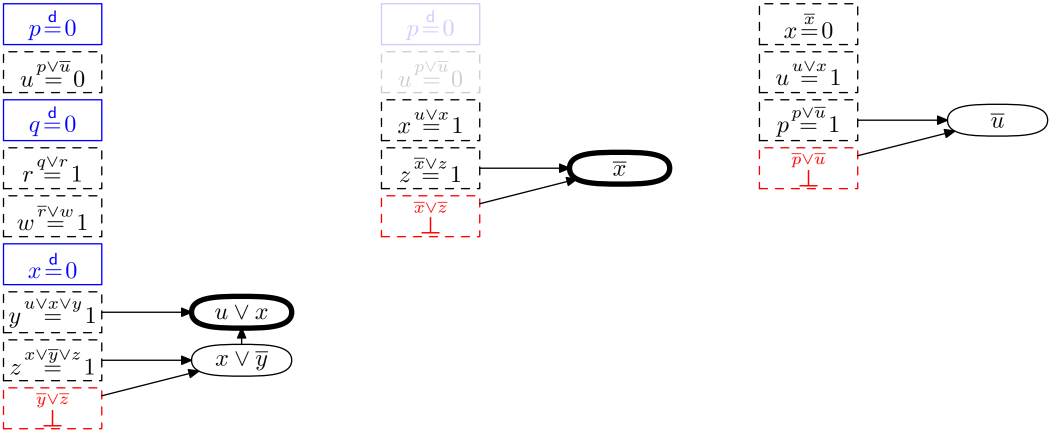

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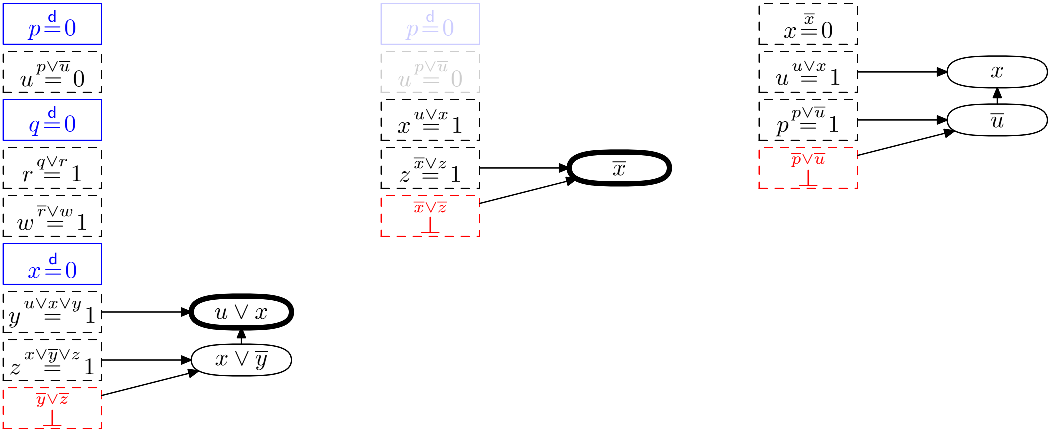

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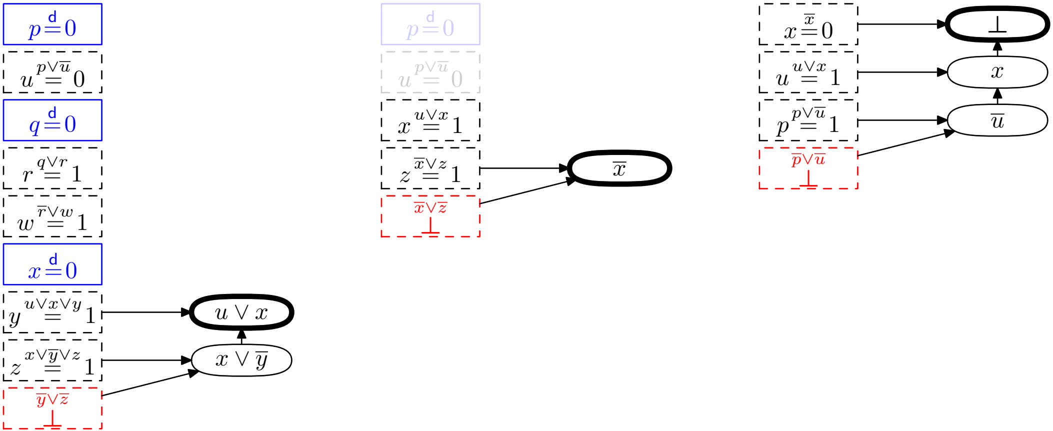

SAT Solver Analysis and the Resolution Proof System

How to make rigorous analysis of SAT solver performance?
Many intricate, hard-to-understand heuristics
So focus instead on underlying method of reasoning

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Resolution proof system [Bla37, Rob65]

- Start with clauses of CNF formula (axioms)
- Derive new clauses by resolution rule

$$\frac{C_1 \lor x \qquad C_2 \lor \overline{x}}{C_1 \lor C_2}$$

Resolution Proofs by Contradiction

Resolution rule:

$$\frac{C_1 \lor x \qquad C_2 \lor \overline{x}}{C_1 \lor C_2}$$

Observation

If F is a satisfiable CNF formula and D is derived from clauses $D_1, D_2 \in F$ by the resolution rule, then $F \land D$ is satisfiable.

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Observation

If F is a satisfiable CNF formula and D is derived from clauses $D_1, D_2 \in F$ by the resolution rule, then $F \land D$ is satisfiable.

So can prove F unsatisfiable by deriving the unsatisfiable empty clause (denoted $\perp$) from F by resolution

Such proof by contradiction also called resolution refutation

DPLL and Resolution Proofs

A DPLL execution is essentially a resolution proof

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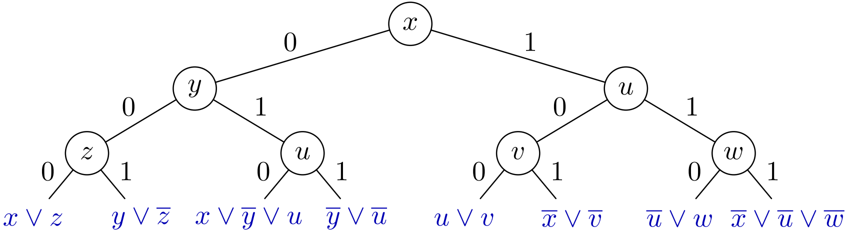
Look at our example again



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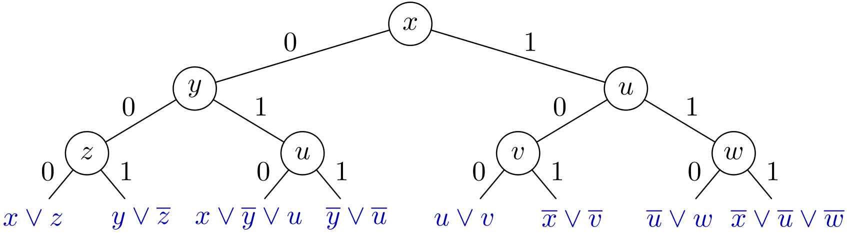

and apply resolution rule $\frac{C_1 \lor x \quad C_2 \lor \overline{x}}{C_1 \lor C_2}$ bottom-up

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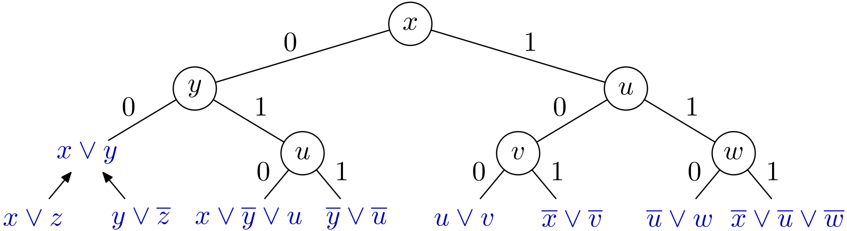

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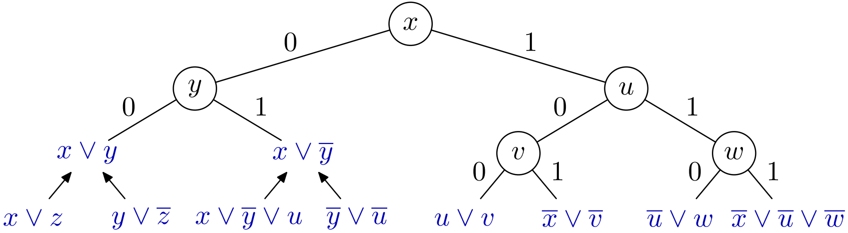

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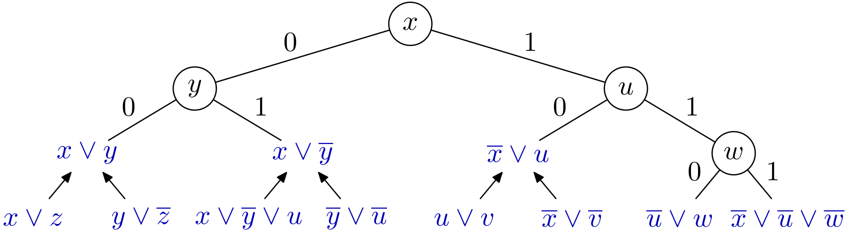

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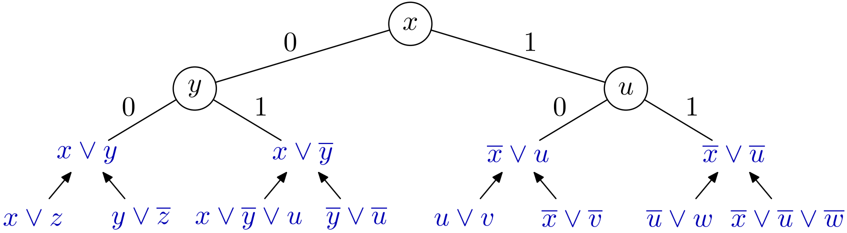

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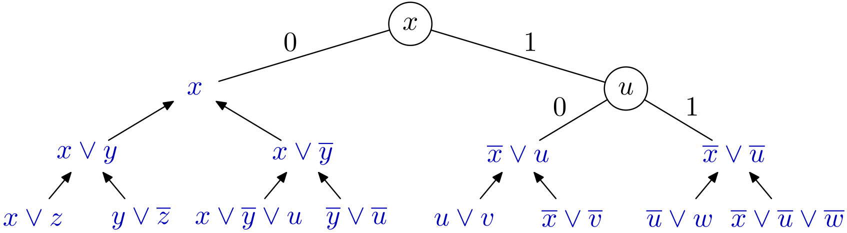

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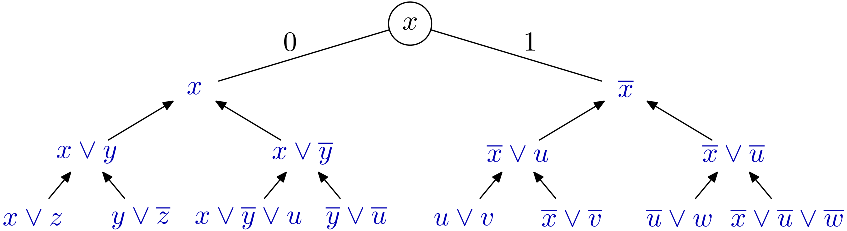

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$$\begin{tikzcd}
 & & & & & & & \bot & & & & & & & \\
 & & & x \arrow[urrrr] & & & & & & & & \overline{x} \arrow[ullll] & & & \\
 & x \lor y \arrow[urr] & & & & x \lor \overline{y} \arrow[ull] & & & & \overline{x} \lor u \arrow[urr] & & & & \overline{x} \lor \overline{u} \arrow[ull] & \\
x \lor z \arrow[ur] & & y \lor \overline{z} \arrow[ul] & & x \lor \overline{y} \lor u \arrow[ur] & & \overline{y} \lor \overline{u} \arrow[ul] & & u \lor v \arrow[ur] & & \overline{x} \lor \overline{v} \arrow[ul] & & \overline{u} \lor w \arrow[ur] & & \overline{x} \lor \overline{u} \lor \overline{w} \arrow[ul]
\end{tikzcd}$$

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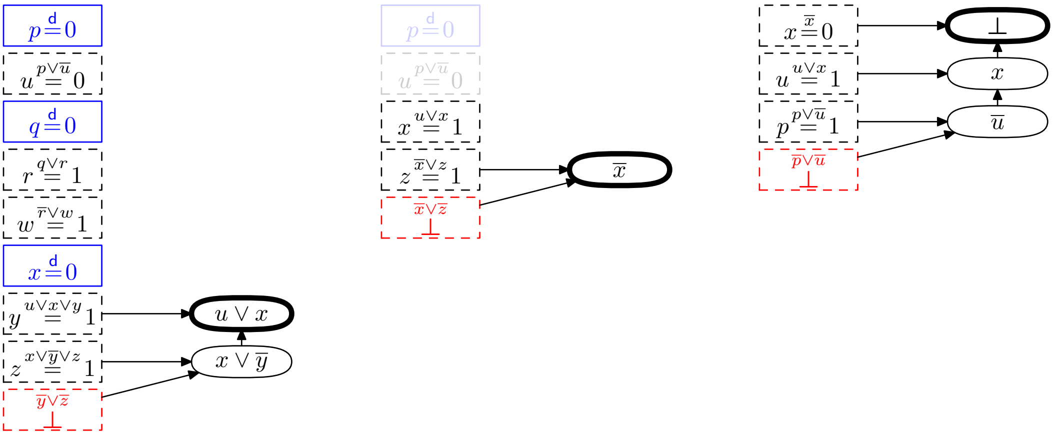
- Conflict-driven clause learning adds "shortcut edges" in tree, but still yields resolution proof

CDCL and Resolution Proofs

Obtain resolution proof...

CDCL and Resolution Proofs

Obtain resolution proof from our example CDCL execution. . .



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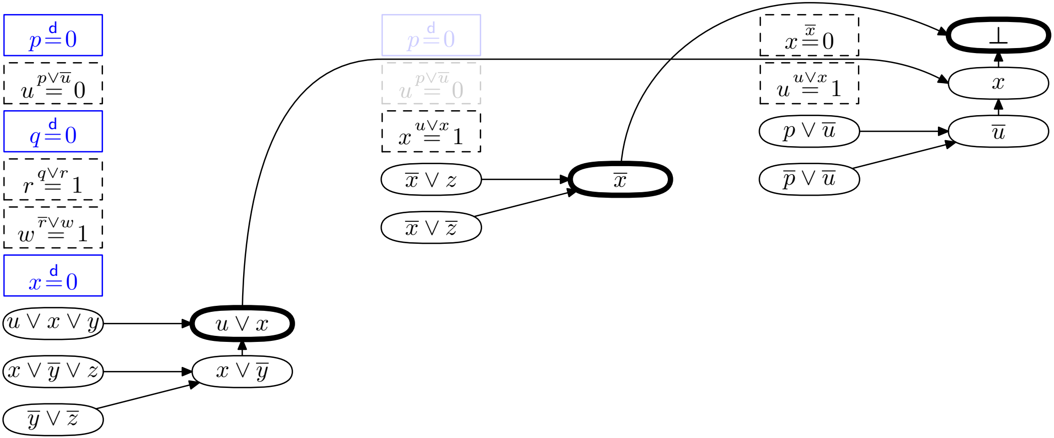
Obtain resolution proof from our example CDCL execution by stringing together conflict analyses:



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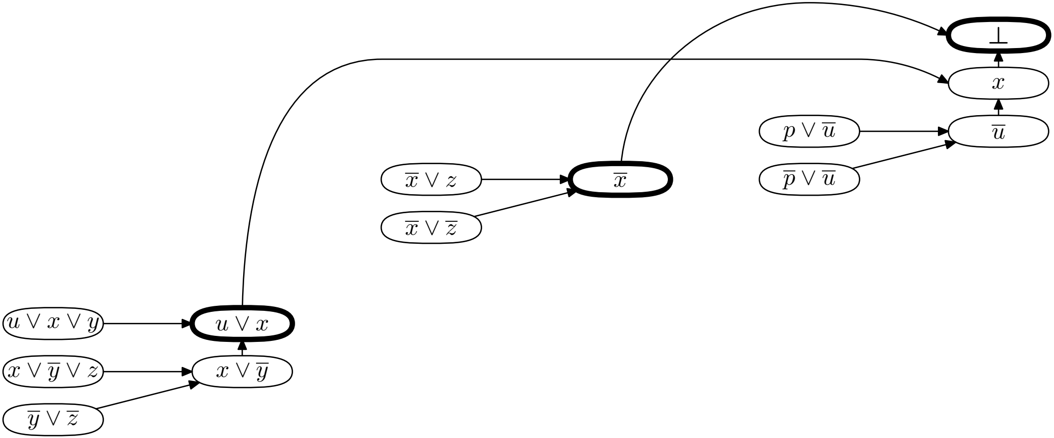

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(*) Except for some preprocessing techniques, which is an important omission, but this gets complicated and we don't have time to go into details...

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- Very poor theoretical understanding:
  - Why do heuristics work?
  - Why are applied instances easy?
- Paradox: resolution quite weak proof system; many strong proof complexity lower bounds for (seemingly) "obvious" formulas

Examples of Hard Formulas for Resolution (1/3)

Pigeonhole principle (PHP) formulas [Hak85]

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Variables $p_{i,j}$ = "pigeon $i \rightarrow$ hole j"; $1 \leq i \leq n+1$; $1 \leq j \leq n$

$p_{i,1} \lor p_{i,2} \lor \cdots \lor p_{i,n}$	[every pigeon i gets a hole]
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Even onto functional PHP hard — **"resolution cannot count"**

Resolution proof requires $\exp(\Omega(n)) = \exp\left(\Omega\left(\sqrt[3]{N}\right)\right)$ clauses
(measured in terms of formula size N, i.e., total number of literals in formula)

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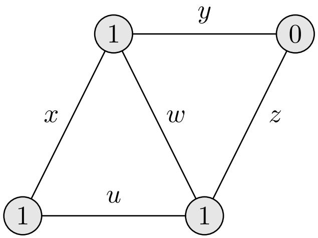

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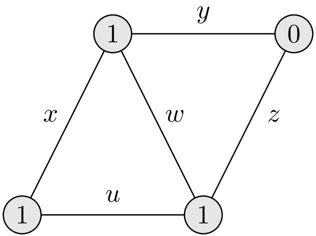

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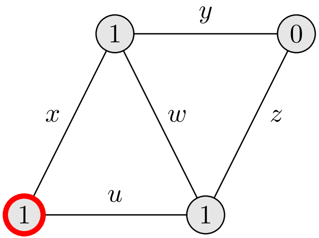

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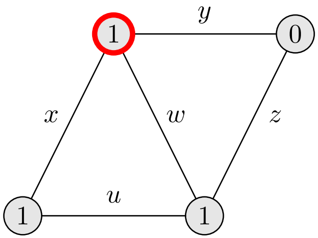

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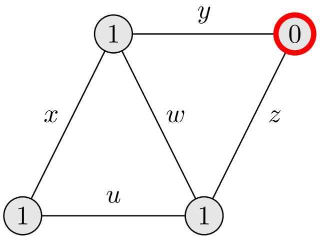

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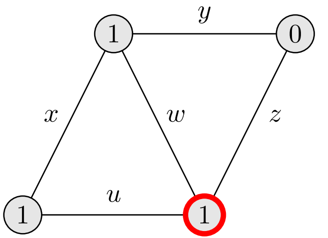

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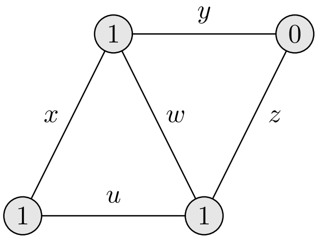

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Requires proof size $\exp(\Omega(N))$ on well-connected so-called expander graphs — **"resolution cannot count** $\mathrm{mod}\ 2$**"**

Examples of Hard Formulas for Resolution (3/3)

Random k-CNF formulas [CS88]

Δn randomly sampled k-clauses over n variables

($\Delta \gtrsim 4.5$ sufficient to get unsatisfiable 3-CNF almost surely)

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And more...

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- Et cetera... (See, e.g., [BN21] for overview)

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But not CLIQUE!

- Refuting existence of k-clique in n-vertex graph should require proof size $n^{\Omega(k)}$
- Only known for restricted so-called regular resolution [ABdR+21]
- For general resolution, best lower bounds are $2^{\Omega(k)}$ for very large k [BIS07]

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The exponential size lower bounds mentioned can all be proven by studying width, i.e., the size of a largest clause in the proof

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There are also other complexity measures of interest such as

- space: memory needed for self-contained presentation of refutation
- depth: longest path in refutation represented as directed acyclic graph (DAG)

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- Add Boolean axioms

$$x_j^2 - x_j = 0$$

  for all variables

Hilbert's Nullstellensatz

Consider any system of polynomial equations

$$\begin{aligned} p_1(x_1,\ldots,x_n) &= 0 \\ p_2(x_1,\ldots,x_n) &= 0 \\ &\vdots \\ p_m(x_1,\ldots,x_n) &= 0 \end{aligned} \qquad\qquad \begin{aligned} x_1^2 - x_1 &= 0 \\ x_2^2 - x_2 &= 0 \\ &\vdots \\ x_n^2 - x_n &= 0 \end{aligned}$$

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Hilbert's Nullstellensatz

System infeasible $\Leftrightarrow$ exist $q_i, r_j \in \mathbb{F}[x_1,\ldots,x_n]$ such that

$$\sum_{i=1}^{m} q_i(x_1,\ldots,x_n) \cdot p_i(x_1,\ldots,x_n) + \sum_{j=1}^{n} r_j(x_1,\ldots,x_n) \cdot (x_j^2 - x_j) = 1$$

Nullstellensatz Proof System [BIK+94]

Nullstellensatz refutation of

$$p_i(x_1, \ldots, x_n) = 0 \qquad i \in [m]$$
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Complexity measures of refutations:

- Size: number of monomials (when all polynomials expanded out)
- Degree: highest total degree of any polynomial

Nullstellensatz Example

$$(x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \lor u) \land (\overline{y} \lor \overline{u})$$
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$$(1-x)(1-z)$$
$$(1-y)z$$
$$(1-x)y(1-u)$$
$$yu$$
$$(1-u)(1-v)$$
$$xv$$
$$u(1-w)$$
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$$\begin{aligned}
& (1-y) \cdot (1-x)(1-z) \\
+ & (1-x) \cdot (1-y)z \\
+ & \quad 1 \cdot (1-x)y(1-u) \\
+ & (1-x) \cdot yu \\
+ & \quad x \cdot (1-u)(1-v) \\
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Size 27
Degree 3
(No use of Boolean axioms)

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$$
\begin{aligned}
&\ 1 - x - y - z + xy + xz + yz - xyz \\
&+ z - xz - yz + xyz \\
&+ y - yu - xy + xyu \\
&+ yu - xyu \\
&+ x - xu - xv + xuv \\
&+ xv - xuv \\
&+ xu - xuw \\
&+ xuw \\
&= 1
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$$

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Nullstellensatz Proof Search

- Solve linear system of equations with coefficients of polynomials q_i, r_j as unknowns
- Used successfully to solve, e.g., graph colouring problems [DLMM08, DLMO09, DLMM11]
- Running time grows exponentially with degree, though high-degree refutations can be very small [BCIP02, dRMNR21]

Dual Variables

- Annoying problem: $x_1 \lor x_2 \lor x_3$ translates to polynomial

$$(1 - x_1)(1 - x_2)(1 - x_3) = 1 - x_1 - x_2 - x_3 + x_1x_2 + x_1x_3 + x_2x_3 - x_1x_2x_3$$

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- More generally, exponential blow-up in # positive literals
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- Translate $C = \bigvee_{i \in \mathcal{P}} x_i \lor \bigvee_{j \in \mathcal{N}} \overline{x}_j$ to monomial equation

$$\prod_{i \in \mathcal{P}} x'_i \cdot \prod_{j \in \mathcal{N}} x_j = 0$$

Dual Variables

- Annoying problem: $x_1 \vee x_2 \vee x_3$ translates to polynomial

$$(1 - x_1)(1 - x_2)(1 - x_3) = 1 - x_1 - x_2 - x_3 + x_1x_2 + x_1x_3 + x_2x_3 - x_1x_2x_3$$

- More generally, exponential blow-up in # positive literals
- Fix: introduce dual variables x'_i and axioms $x_i + x'_i - 1 = 0$
- Translate $C = \bigvee_{i \in \mathcal{P}} x_i \vee \bigvee_{j \in \mathcal{N}} \overline{x}_j$ to monomial equation

$$\prod_{i \in \mathcal{P}} x'_i \cdot \prod_{j \in \mathcal{N}} x_j = 0$$

- Doesn't affect degree (obviously), but can decrease size exponentially [dRLNS21] (also for other algebraic proof systems)

Dynamic Construction of Nullstellensatz Certificates

Nullstellensatz again

Infeasibility of

$$\begin{aligned} p_i(x_1,\ldots,x_n) &= 0 && i \in [m] \\ x_j^2 - x_j &= 0 && j \in [n] \\ x_j + x_j' - 1 &= 0 && j \in [n] \end{aligned}$$

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- Ideal $\mathcal{I}$:
  1. $p, q \in \mathcal{I} \Rightarrow p + q \in \mathcal{I}$
  2. $p \in \mathcal{I} \Rightarrow r \cdot p \in \mathcal{I}$ for any r
- Compute polynomials in this ideal $\mathcal{I}$ step by step
- Use "multivariate division" to check whether 1 lies in ideal or not

Gröbner Bases: Admissible Orderings and Leading Terms

Admissible ordering $\preceq$ on monomials m, m', t:

1. $m \preceq m' \Rightarrow t \cdot m \preceq t \cdot m'$
2. $m \preceq t \cdot m$

Examples:

- Lexicographic
- Degree-lexicographic

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Can write $p = \mathrm{lt}(p) + p'$ for $\mathrm{lt}(p)$ leading term (largest w.r.t. $\preceq$)

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"Multivariate division": Reduce p modulo all q in set of polynomials $\mathcal{G}$ until no further reductions possible

$\mathcal{G}$ is a Gröbner basis if final result uniquely determined

Gröbner Bases: Buchberger's Algorithm

Buchberger's algorithm for computing Gröbner bases (**very** rough)

1. Let $\mathcal{G} :=$ all axioms
2. Pick unprocessed pair $p, q \in \mathcal{G}$ or terminate if none exists
3. Compute $p' = t_p \cdot p$ and $q' = t_q \cdot q$ to make leading terms cancel
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Facts:

- Buchberger's algorithm computes Gröbner basis
- At termination, $1 \in \mathcal{G} \Leftrightarrow$ polynomial equations infeasible

Polynomial Calculus [CEI96, ABRW02]

- Compute polynomials in ideal $\mathcal{I}$ generated by p_i, $x_j^2 - x_j$, and $x_j + x'_j - 1$ step by step:
  - $p_i \in \mathcal{I}$, $x_j^2 - x_j \in \mathcal{I}$, and $x_j + x'_j - 1 \in \mathcal{I}$ (axioms)
  - If $p, q \in \mathcal{I}$, then $\alpha p + \beta q \in \mathcal{I}$ for any $\alpha, \beta \in \mathbb{F}$ (linear combination)
  - If $p \in \mathcal{I}$, then $m \cdot p \in \mathcal{I}$ for any monomial $m = \prod_j x_j$ (multiplication)

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  - If $p \in \mathcal{I}$, then $m \cdot p \in \mathcal{I}$ for any monomial $m = \prod_j x_j$ (multiplication)
- A polynomial calculus refutation is a derivation ending with the polynomial 1
- Complexity measures:
  - Size: total number of monomials in all polynomials in derivation expanded out
  - Degree: highest total degree of any polynomial
- Polynomial calculus (much) stronger than Nullstellensatz w.r.t. both size and degree

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$$\frac{x \lor \overline{y} \lor z \qquad \overline{y} \lor \overline{z}}{x \lor \overline{y}}$$

simulated by polynomial calculus derivation

$$\dfrac{x'yz' \qquad \dfrac{\dfrac{yz}{x'yz} \qquad \dfrac{z + z' - 1}{x'yz + x'yz' - x'y}}{-x'yz' + x'y}}{x'y}$$

Polynomial Calculus is Strictly Stronger than Resolution

Polynomial calculus can be exponentially stronger than resolution

For instance:

- Tseitin formulas on expander graphs if $\mathbb{F} = \mathrm{GF}(2)$
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Other hard formulas:

- Tseitin-like formulas for counting $\bmod\ p$ if $p \neq$ field characteristic [BGIP01]
- Random k-CNF formulas
  - all characteristics except 2 [BI99]
  - all characteristics [AR03]

COLOURING and CLIQUE for Polynomial Calculus

COLOURING

- Exponential worst-case lower bounds in [LN17]
- Exponential average-case lower bounds in [CdRN+23]

CLIQUE

Almost nothing known! (Except lower bounds for very large cliques)

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If any polynomial calculus refutation of k-CNF formula F over n variables requires degree linear in n, then refuting F in polynomial calculus requires size exponential in n

- Other complexity measures analogous to those for resolution are also studied
- Many results analogous to resolution hold, but are much harder to prove
- Some analogous results are *believed* to hold, but remain open

What About Algebraic SAT Solvers?

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- Use dual variables! [KBBN22]

Gröbner bases: Some Problems and Questions

1. Buchberger not a great SAT solving algorithm
   Slow and memory-intensive, and computes too much info (#solutions)
   Possible to use conflict-driven paradigm?!

2. Dual variables increase reasoning power exponentially [dRLNS21]
   But are immediately eliminated by multivariate division
   Possible to design dual-variable-aware Buchberger?!

3. Analysis of polynomial calculus uses degree-lexicographic ordering
   In computational algebra, many other orderings used
   Prove proof complexity separation results for different orderings?

SAT as System of 0–1 Integer Linear Inequalities

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$$C = \bigvee_{i \in \mathcal{P}} x_i \vee \bigvee_{j \in \mathcal{N}} \overline{x}_j$$

to 0-1 integer linear inequalities

$$\sum_{i \in \mathcal{P}} x_i + \sum_{j \in \mathcal{N}} (1 - x_j) \geq 1$$

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$$\sum_{i \in \mathcal{P}} x_i + \sum_{j \in \mathcal{N}} (1 - x_j) \geq 1$$

- Add variable axioms

$$x_j \geq 0$$
$$-x_j \geq -1$$

for all variables

Cutting Planes Proof System [CCT87]

Cutting planes proof system introduced in [CCT87] to model integer linear programming algorithm in [Gom63, Chv73]

Can be applied to any system of 0-1 integer linear inequalities

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Cutting planes derivation rules

Multiplication $$\frac{\sum a_i x_i \geq A}{\sum c a_i x_i \geq cA} \quad c \in \mathbb{N}^+$$

Addition $$\frac{\sum a_i x_i \geq A \qquad \sum b_i x_i \geq B}{\sum (a_i + b_i) x_i \geq A + B}$$

Division $$\frac{\sum a_i x_i \geq A}{\sum \lceil a_i / c \rceil x_i \geq \lceil A/c \rceil} \quad c \in \mathbb{N}^+$$

Cutting Planes Derivations and Refutations

- A cutting planes derivation is a sequence of 0-1 integer linear inequalities derived using
  - Axioms (clauses and variable bounds)
  - Multiplication $\sum a_i x_i \geq A \Rightarrow \sum c a_i x_i \geq cA$
  - Addition $\sum a_i x_i \geq A$, $\sum b_i x_i \geq B \Rightarrow \sum (a_i + b_i) x_i \geq A + B$
  - Division $\sum a_i x_i \geq A \Rightarrow \sum \lceil a_i / c \rceil x_i \geq \lceil A/c \rceil$
- A cutting planes refutation ends with the inequality $0 \geq 1$
- Complexity measures:
  - Length: $\#$ inequalities
  - Size: Count also bit size of representing all coefficients

Cutting Planes vs. Resolution

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- Cutting planes can simulate resolution reasoning efficiently and can be exponentially stronger (e.g., for PHP, just count and argue that #pigeons $>$ #holes)
- And 0-1 linear inequalities are similar to but much more concise than CNF

Compare

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \geq 3$$

and

$$\begin{aligned}
&(x_1 \lor x_2 \lor x_3 \lor x_4) \land (x_1 \lor x_2 \lor x_3 \lor x_5) \land (x_1 \lor x_2 \lor x_3 \lor x_6) \\
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Hard Formulas for Cutting Planes

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Variables:

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$q_{k,1} \lor q_{k,2} \lor \cdots \lor q_{k,n}$	$1 \leq k \leq m$	[some vertex is kth member of clique]
$\overline{q}_{k,v} \lor \overline{q}_{k',v}$	$1 \leq v \leq n;\ k \neq k'$	[no vertex counted as clique member twice]
$p_{u,v} \lor \overline{q}_{k,u} \lor \overline{q}_{k',v}$	$1 \leq u < v \leq n;\ k \neq k'$	[clique members are neighbours]
$r_{v,1} \lor r_{v,2} \lor \cdots \lor r_{v,m-1}$	$1 \leq v \leq n;$	[every vertex has a colour]
$\overline{p}_{u,v} \lor \overline{r}_{u,\ell} \lor \overline{r}_{v,\ell}$	$1 \leq u < v \leq n;\ 1 \leq \ell \leq m-1$	[neighbours have distinct colours]

More Hard Formulas for Cutting Planes?

Lower bound for clique-colouring formulas uses interpolation and circuit complexity

- From small cutting planes proof, build small circuit of special type that can decide whether graph has clique
- Prove separately that no such small circuits can exist
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Some recent developments in [dRMN^{+}20, HP17, FPPR22, GGKS20, Sok24]

- Random $\mathcal{O}(\log n)$-CNF formulas exponentially hard [HP17, FPPR22]
- Lower bound for random k-CNF formulas open
- Surprisingly, Tseitin formulas have refutations of quasi-polynomial size [DT20]!
- Nothing known for COLOURING or CLIQUE

SAT Solvers Based on Cutting Planes?

So-called pseudo-Boolean (PB) solvers using (subset of) cutting planes reasoning developed in, e.g., [CK05, SS06, LP10, EN18]

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Is it truly harder to build good pseudo-Boolean solvers?
Or has just so much more work has been put into CDCL solvers?

Division Versus Saturation

Use negated literals as needed to get all a_i, A positive (normalized form)

Boolean derivation rules for 0–1 integer linear inequalities

$$\text{Division} \quad \frac{\sum a_i \ell_i \geq A}{\sum \lceil a_i / c \rceil \ell_i \geq \lceil A/c \rceil} \quad c \in \mathbb{N}^+$$

$$\text{Saturation} \quad \frac{\sum a_i \ell_i \geq A}{\sum \min\{a_i, A\} \cdot \ell_i \geq A}$$

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- Open how the two variants compare, but clear that division can sometimes be better in theory [GNY19]
- ... And most often also in practice [EN18], though not always [LBD$^+$20]

Separating Division from Saturation?

Even colouring formulas [Mar06]

"$\exists$ $0/1$-colouring of edges so that every vertex has equal number of 0-edges and 1-edges"

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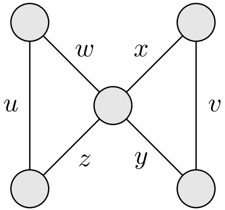

$$
\begin{aligned}
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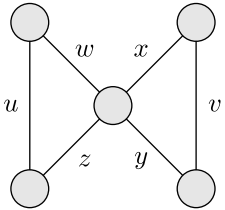

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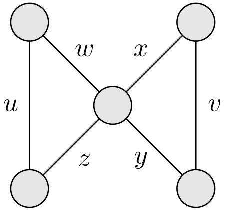

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- Very easy for (tree-like) cutting planes

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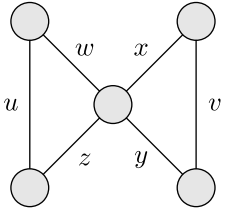

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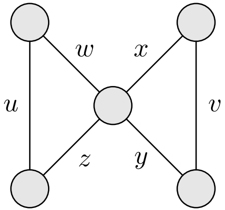
- Very easy for (tree-like) cutting planes
- CNF encoding exponentially hard for resolution (and PB solvers) for expander graphs

Separating Division from Saturation?

Even colouring formulas [Mar06]

"$\exists$ $0/1$-colouring of edges so that every vertex has equal number of 0-edges and 1-edges"



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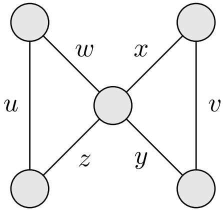

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- Very easy for (tree-like) cutting planes
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- Pseudo-Boolean encoding hard in practice for pseudo-Boolean solvers [EGNV18]
- Possible to prove lower bounds for cutting planes with saturation instead of division?

The Subgraph Isomorphism Problem

Input

- Pattern graph $\mathcal{P}$ with vertices $V(\mathcal{P}) = \{a, b, c, \ldots\}$
- Target graph $\mathcal{T}$ with vertices $V(\mathcal{T}) = \{u, v, w, \ldots\}$
- No loops (for simplicity)

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Task

- Find all subgraph isomorphisms $\varphi : V(\mathcal{P}) \to V(\mathcal{T})$
- I.e., one-to-one mappings φ such that if
  1. $\varphi(a) = u$
  2. $\varphi(b) = v$
  3. $(a, b) \in E(\mathcal{P})$

  then must have $(u, v) \in E(\mathcal{T})$

Cutting Planes Lower Bounds for Subgraph Isomorphism?

Subgraph isomorphism formula

$\sum_{v \in V(\mathcal{T})} x_{a,v} \geq 1$	[every pattern vertex $a \in V(\mathcal{P})$ maps somewhere]
$\sum_{v \in V(\mathcal{T})} -x_{a,v} \geq -1$	[... but only to one target vertex $u \in V(\mathcal{T})$]
$\sum_{b \in V(\mathcal{P})} -x_{b,u} \geq -1$	[mapping is one-to-one]
$-x_{a,u} + \sum_{v \in N(u)} x_{b,v} \geq 0$	[edge $(a,b) \in E(\mathcal{P})$ maps to edge $(u,v) \in E(\mathcal{T})$]

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- All reasoning steps in Glasgow Subgraph Solver [ADH$^+$19, GSS] can be formalized efficiently in cutting planes [GMN20, GMM$^+$24]
- So lower bounds for any graph pairs $(\mathcal{P}, \mathcal{T})$ would establish theoretical limitations on state-of-the-art algorithms

Sherali–Adams (SA) and Sum of Squares (SoS)

Refutation of $p_i \in \mathbb{R}[x_1, \ldots, x_n]$, $i \in [m]$, and $x_j^2 - x_j$, $j \in [n]$

Nullstellensatz

$$\sum_{i=1}^{m} q_i \cdot p_i + \sum_{j=1}^{n} r_j \cdot (x_j^2 - x_j) = 1$$

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Sum of squares (SoS) $(s_k \in \mathbb{R}[x_1, \ldots, x_n])$

$$\sum_{i=1}^{m} q_i \cdot p_i + \sum_{j=1}^{n} r_j \cdot (x_j^2 - x_j) + \sum_{k=1}^{s} s_k^2 = -1$$

Sherali–Adams, Sum of Squares, and Relations to Other Proof Systems

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Strict hierarchy (over $\mathbb{R}$):

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Survey [FKP19] recommended for more reading

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Complexity measures:

- Length: $\#$ branching nodes / sets $\mathcal{S}$
- Size: Count also bit size for representing all coefficients

Stabbing Planes and Cutting Planes

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Recent news: Interpolation and circuit complexity can be used to get similar lower bounds for stabbing planes as for cutting planes! [GP24]

Still possible that stabbing planes is exponentially more powerful than cutting planes, but hard to know what to believe

Extended Resolution [Tse68]

Resolution rule

$$\frac{C_1 \lor x \qquad C_2 \lor \overline{x}}{C_1 \lor C_2}$$

Extension rule introducing clauses

$$a \lor \overline{x} \lor \overline{y} \qquad \overline{a} \lor x \qquad \overline{a} \lor y$$

for fresh variable a (encoding that $a \leftrightarrow (x \land y)$ must hold)

Extended Resolution and SAT Solving

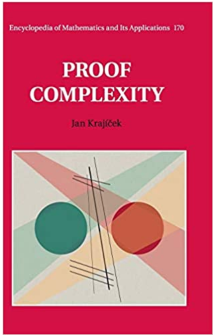
- Closely related (and equivalent) to *DRAT* proof system used to justify correctness of some SAT preprocessing techniques [JHB12]
- *DRAT* also used for SAT solver proof logging [HHW13a, HHW13b, WHH14]
- Attempts to combine extended resolution with CDCL in, e.g., [AKS10, Hua10]
- Without restrictions, as powerful as extremely strong extended Frege system [CR79] — pretty much no lower bounds known
- To analyse solvers using extended resolution, would need to:
  - Describe heuristics/rules actually used
  - See if possible to reason about such restricted proof system

Some More References for Further Reading

Handbook of Satisfiability
(Especially chapter 7 ☺)

[BHvMW21]

Proof Complexity
by Jan Krajíček



[Kra19]

Summing up This Lecture

Overview of some proof systems used in combinatorial solving:

- Resolution $\longleftrightarrow$ conflict-driven clause learning (CDCL)
- Nullstellensatz and polynomial calculus $\longleftrightarrow$ Gröbner bases
- Cutting planes $\longleftrightarrow$ pseudo-Boolean solving

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Thank you for your attention!

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