Tutorial on Boolean Satisfiability (SAT) Solving

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University of Copenhagen and Lund University

1st International Workshop on
Solving Linear Optimization Problems
for Pseudo-Booleans and Yonder
Lund, Sweden
November 5, 2024

This Is Me...

Jakob Nordström

Professor

University of Copenhagen
and Lund University

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. . . And This Is What I Do for a Living

$$(x_{1,1} \lor x_{1,2} \lor x_{1,3} \lor x_{1,4} \lor x_{1,5} \lor x_{1,6} \lor x_{1,7}) \land (x_{2,1} \lor x_{2,2} \lor x_{2,3} \lor x_{2,4} \lor x_{2,5} \lor x_{2,6} \lor x_{2,7}) \land (x_{3,1} \lor x_{3,2} \lor x_{3,3} \lor x_{3,4} \lor$$

$$x_{3,5} \lor x_{3,6} \lor x_{3,7}) \land (x_{4,1} \lor x_{4,2} \lor x_{4,3} \lor x_{4,4} \lor x_{4,5} \lor x_{4,6} \lor x_{4,7}) \land (x_{5,1} \lor x_{5,2} \lor x_{5,3} \lor x_{5,4} \lor x_{5,5} \lor x_{5,6} \lor x_{5,7}) \land (x_{6,1} \lor$$

$$x_{6,2} \lor x_{6,3} \lor x_{6,4} \lor x_{6,5} \lor x_{6,6} \lor x_{6,7}) \land (x_{7,1} \lor x_{7,2} \lor x_{7,3} \lor x_{7,4} \lor x_{7,5} \lor x_{7,6} \lor x_{7,7}) \land (x_{8,1} \lor x_{8,2} \lor x_{8,3} \lor x_{8,4} \lor x_{8,5} \lor x_{8,6} \lor$$

$$x_{8,7}) \land (\neg x_{1,1} \lor \neg x_{2,1}) \land (\neg x_{1,1} \lor \neg x_{3,1}) \land (\neg x_{1,1} \lor \neg x_{4,1}) \land (\neg x_{1,1} \lor \neg x_{5,1}) \land (\neg x_{1,1} \lor \neg x_{6,1}) \land (\neg x_{1,1} \lor \neg x_{7,1}) \land (\neg x_{1,1} \lor$$

$$\neg x_{8,1}) \land (\neg x_{2,1} \lor \neg x_{3,1}) \land (\neg x_{2,1} \lor \neg x_{4,1}) \land (\neg x_{2,1} \lor \neg x_{5,1}) \land (\neg x_{2,1} \lor \neg x_{6,1}) \land (\neg x_{2,1} \lor \neg x_{7,1}) \land (\neg x_{2,1} \lor \neg x_{8,1}) \land$$

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Three Simple Problems. . .

Colouring

Does the graph $G = (V, E)$ have a colouring with k colours such that all neighbours have distinct colours?

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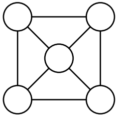

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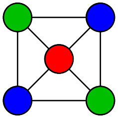

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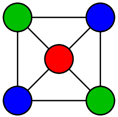

3-colouring? Yes, but no 2-colouring

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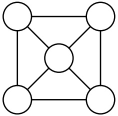

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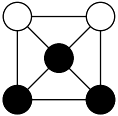

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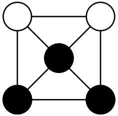

3-clique? Yes, but no 4-clique

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- Variables should be set to **true** or **false**
- Constraint $(x \lor \neg y \lor z)$: means x or z should be true or y false
- $\land$ means all constraints should hold simultaneously
- Is there a truth value assignment satisfying all constraints?

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COLOURING: frequency allocation for mobile base stations
CLIQUE: bioinformatics, computational chemistry
SAT: easily models these and many other problems

. . . with Huge Practical Implications

- Some more examples of problems that can be encoded as propositional logic formulas:
  - computer hardware verification
  - computer software testing
  - artificial intelligence
  - cryptography
  - bioinformatics
  - et cetera. . .
- Leads to **humongous** formulas (100,000s or even 1,000,000s of variables)
- Can we use computers to solve these problems efficiently?
- Question mentioned already in Gödel's famous letter in 1956 to von Neumann (the "father of computer science")
- Topic of intense research in computer science ever since 1960s

Solving Logic Formulas in Practice

- Enormous progress since the turn of the millennium on so-called SAT solvers
  Today routinely used to solve large-scale real-world problems

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  - It's 2024 now — can we go beyond techniques from 1960s?

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  [in the pseudo-Boolean tutorials]

. . . *And in the process also touch on some of the research conducted in the Mathematical Insights into Algorithms for Optimization (MIAO) group* in Copenhagen and Lund

Outline of Tutorial on Boolean Satisfiability (SAT) Solving

1. SAT solving
   - The SATISFIABILITY Problem
   - Davis-Putnam-Logemann-Loveland (DPLL) Method
   - Conflict-Driven Clause Learning (CDCL)
2. Proof Complexity
   - Resolution Proof System
   - Resolution and SAT Solving
   - Lower Bounds for Resolution
3. Future Research Directions
   - Understanding and Improving on the State of the Art
   - Pseudo-Boolean Solving and the Cutting Planes Method
   - Some Research Questions

Formal Description of SAT Problem

- Variable x: takes value 1 (**true**) or 0 (**false**)
- Literal ℓ: variable x or its negation $\overline{x}$ (write $\overline{x}$ instead of $\neg x$)
- Clause $C = \ell_1 \lor \cdots \lor \ell_k$: disjunction of literals
  (Consider as sets, so no repetitions and order irrelevant)
- Conjunctive normal form (CNF) formula $F = C_1 \land \cdots \land C_m$: conjunction of clauses

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The SATISFIABILITY (or just SAT) Problem

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For instance, what about our example formula?

$$(x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \lor u) \land (\overline{y} \lor \overline{u})$$
$$\land\ (u \lor v) \land (\overline{x} \lor \overline{v}) \land (\overline{u} \lor w) \land (\overline{x} \lor \overline{u} \lor \overline{w})$$

How to Solve the SAT Problem?

- Let computer check all possible assignments! Isn't this exactly the kind of monotone routine work at which computers excel?
- But how many cases to check?
- Suppose formula has n variables
- Each variable can be either true or false, so all in all get 2^n different cases
- If formula contains, say, one million variables, we get $2^{1,000,000}$ cases (a number with more than 300,000 digits)

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To understand how large this number is, consider that even if every atom in the known universe was a modern supercomputer that had been running at full speed ever since the beginning of time some 13.7 billion years ago, all of them together would only have covered a completely negligible fraction of these cases by now. So we really would not have time to wait for them to finish...

An Interesting Feature of the Sat Problem

- Deciding whether a satisfying assignment exists may take a long time
- But if you happen to know a satisfying assignment, easy to convince someone else that formula is satisfiable
- How? Just give assignment — can be verified in linear time
- So Sat problem might seem hard to solve, but verifying a solution is easy (not all problems have this property — how do you verify a winning position in chess?)
- The family of problems for which solutions are easy to check have a name: NP

How to Solve the SAT Problem, Take 2

- SAT problem can be used to describe any problem in NP — it is NP-complete [Coo71, Lev73]
- If you can solve SAT efficiently, then you can solve any problem in NP efficiently (this is why SAT is so useful)
- So how hard is it to solve SAT? (Brute force didn't work, but it usually doesn't — maybe can do something smarter?)

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- So how hard is it to solve SAT? (Brute force didn't work, but it usually doesn't — maybe can do something smarter?)
- We don't know
- This one of the million-dollar "Millennium Prize Problems" [Mil00] posed as key challenges for mathematics in the new millennium
- Widely believe to be impossible to solve efficiently on computer in the worst case, but we really don't know

An Attempt at a Smarter Case Analysis: DPLL

Ok, but suppose you're out there in reality and actually have to solve the problem — then what do you do?

Chances are you'll use some variant of the DPLL method developed by Davis, Putnam, Logemann & Loveland [DP60, DLL62]

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- Set $x = 0$, simplify F and make recursive call
- Set $x = 1$, simplify F and make recursive call
- If result in both cases "unsatisfiable", then report "unsatisfiable" and return

A DPLL Toy Example

$$F = (x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \lor u) \land (\overline{y} \lor \overline{u}) \land (u \lor v) \land (\overline{x} \lor \overline{v}) \land (\overline{u} \lor w) \land (\overline{x} \lor \overline{u} \lor \overline{w})$$

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Pick variables in internal nodes; terminate in leaves when conflict reached

"Simplify formula" by (mentally) removing

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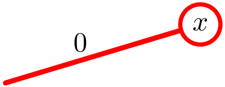

A DPLL Toy Example

$$F = (\quad z) \land (\quad \overline{z}) \land (\quad \overline{y} \lor u) \land (\overline{y} \lor \overline{u}) \land (u \lor v) \land (\overline{x} \lor \overline{v}) \land (\overline{u} \lor w) \land (\overline{x} \lor \overline{u} \lor \overline{w})$$

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Pick variables in internal nodes; terminate in leaves when conflict reached

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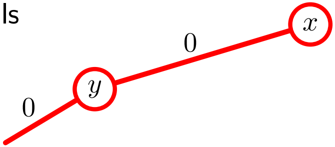

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$$
\begin{aligned}
F = \; & (x \lor z) \land (z) \land (\overline{y} \lor u) \land (\overline{y} \lor \overline{u}) \\
& \land (u \lor v) \land (\overline{x} \lor \overline{v}) \land (\overline{u} \lor w) \land (\overline{x} \lor \overline{u} \lor \overline{w})
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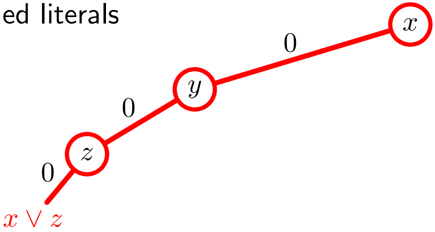

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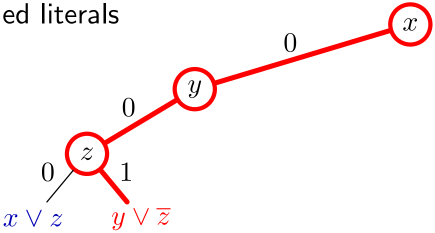

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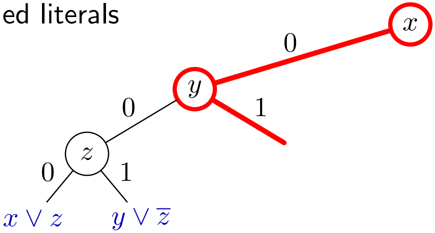

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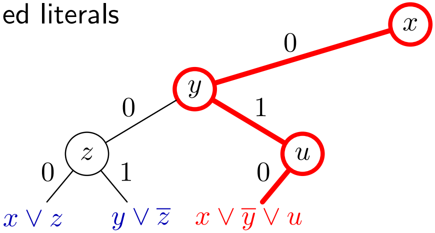

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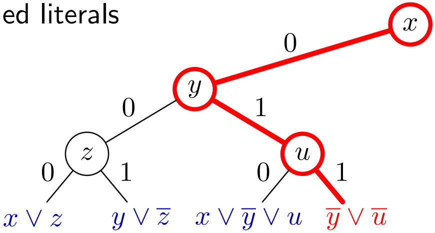

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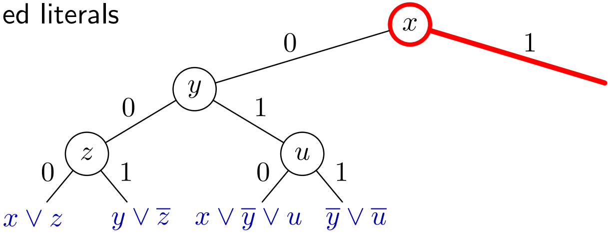

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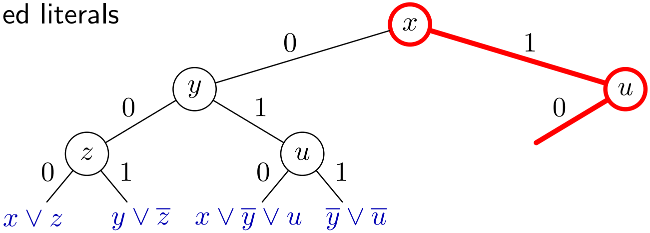

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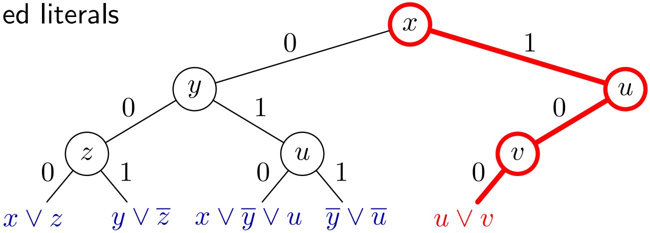

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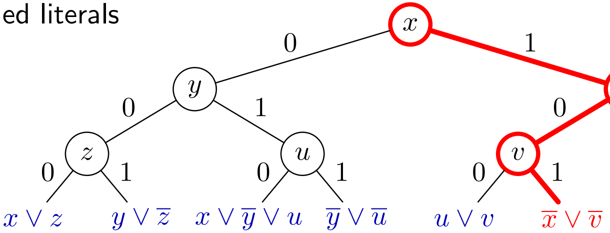

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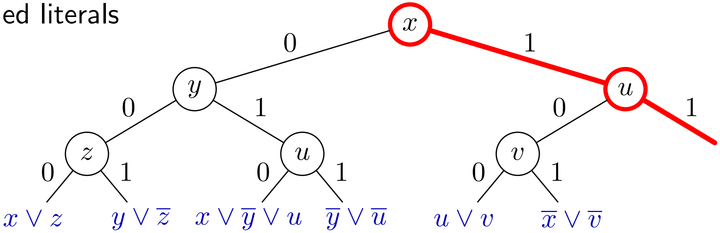

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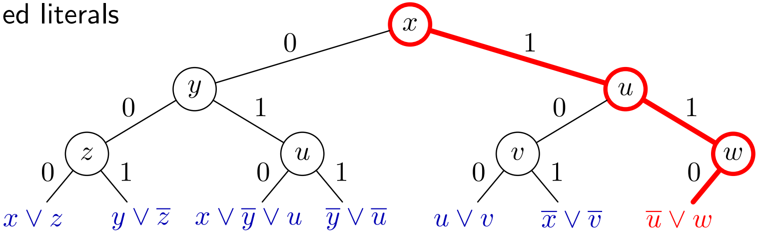

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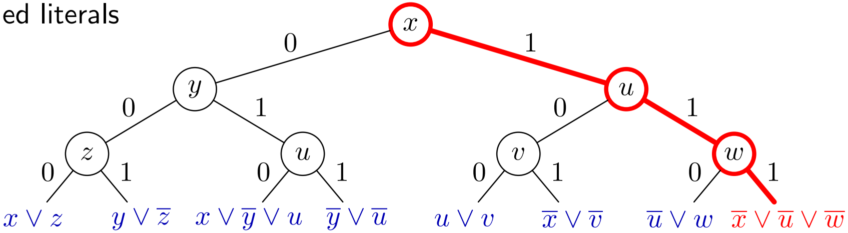

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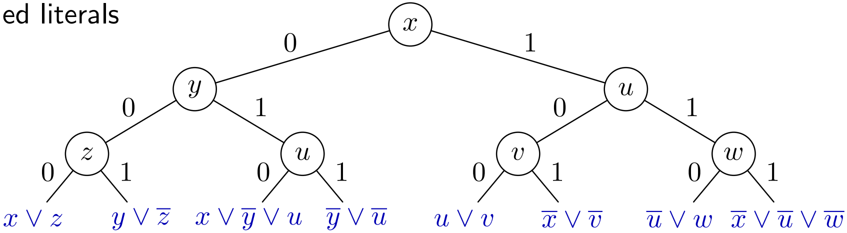

State-of-the-art SAT solvers: Ingredients

Many more ingredients in modern conflict-driven clause learning (CDCL) SAT solvers (as pioneered in [BS97, MS99, MMZ⁺01]), e.g.:

- Branching or decision heuristic (choice of pivot variables crucial)
- When reaching leaf, compute explanation for conflict and add to formula as new clause (clause learning)
- Every once in a while, restart from beginning (but save computed info)
- Preprocessing the formula before the search even starts

Let us discuss some of these ingredients

Variable Assignment Heuristics

Unit propagation

- Suppose current assignment ρ falsifies all literals in $C = \ell_1 \lor \ell_2 \lor \cdots \lor \ell_k$ except one (say ℓ_k) — C is unit under ρ
- Then ℓ_k has to be true, so set it to true
- Known as unit progagation or Boolean constraint progagation
- Always propagate if possible — in modern solvers aim for ≈99% of assignments being unit propagations

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VSIDS (Variable state independent decaying sum)

- When backtracking, score $+1$ for variables "causing conflict"
- Also multiply all scores with factor $\kappa < 1$ — exponential filter rewarding variables involved in recent conflicts
- When no propagations, decide on variable with highest score

Clause Learning

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- Often described in terms of cuts in conflict graph
- More helpful to view conflict analysis as syntactic derivation applied on clauses unit propagating to conflict

Decisions, Unit Propagations, and Conflict

Two kinds of assignments — illustrate on example formula:

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

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Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

Notation $u \stackrel{p \lor \overline{u}}{=} 0$ ($p \lor \overline{u}$ is reason clause)

Always propagate if possible, otherwise decide

Add to assignment trail

Continue until satisfying assignment or conflict

Decisions, Unit Propagations, and Conflict

Two kinds of assignments — illustrate on example formula:

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

Trail	
$p \overset{\mathsf{d}}{=} 0$	decision level 1
$u \overset{p \lor \overline{u}}{=} 0$	
$q \overset{\mathsf{d}}{=} 0$	decision level 2
$r \overset{q \lor r}{=} 1$	
$w \overset{\overline{r} \lor w}{=} 1$	
$x \overset{\mathsf{d}}{=} 0$	decision level 3
$y \overset{u \lor x \lor y}{=} 1$	
$z \overset{x \lor \overline{y} \lor z}{=} 1$	
$\overset{\overline{y} \lor \overline{z}}{\bot}$	

Decision

Free choice to assign value to variable

Notation $p \overset{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

Notation $u \overset{p \lor \overline{u}}{=} 0$ ($p \lor \overline{u}$ is reason clause)

Always propagate if possible, otherwise decide

Add to assignment trail

Continue until satisfying assignment or conflict

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$p \stackrel{\mathsf{d}}{=} 0$

$u \stackrel{p \lor \overline{u}}{=} 0$

decision level 1

$q \stackrel{\mathsf{d}}{=} 0$

$r \stackrel{q \lor r}{=} 1$

$w \stackrel{\overline{r} \lor w}{=} 1$

decision level 2

$x \stackrel{\mathsf{d}}{=} 0$

$y \stackrel{u \lor x \lor y}{=} 1$

$z \stackrel{x \lor \overline{y} \lor z}{=} 1$

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decision level 3

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$p \stackrel{\mathsf{d}}{=} 0$
$u \stackrel{p \lor \overline{u}}{=} 0$
} decision level 1

$q \stackrel{\mathsf{d}}{=} 0$
$r \stackrel{q \lor r}{=} 1$
$w \stackrel{\overline{r} \lor w}{=} 1$
} decision level 2

$x \stackrel{\mathsf{d}}{=} 0$
$y \stackrel{u \lor x \lor y}{=} 1$
$z \stackrel{x \lor \overline{y} \lor z}{=} 1$
$\stackrel{\overline{y} \lor \overline{z}}{\bot}$
} decision level 3

Could backtrack by erasing conflict level & flipping last decision

Conflict Analysis

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- decision level 1:
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  - $u \stackrel{p \lor \overline{u}}{=} 0$
- decision level 2:
  - $q \stackrel{\mathsf{d}}{=} 0$
  - $r \stackrel{q \lor r}{=} 1$
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Could backtrack by erasing conflict level & flipping last decision

But want to learn from conflict and cut away as much of search space as possible

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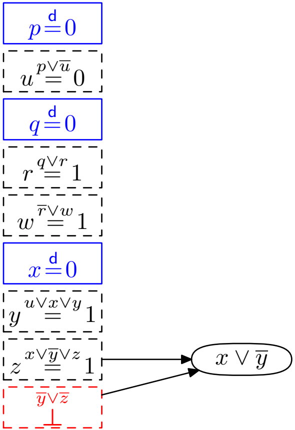

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But want to learn from conflict and cut away as much of search space as possible

Case analysis for last two clauses over propagated variable:

- $x \lor \overline{y} \lor z$ wants $z = 1$
- $\overline{y} \lor \overline{z}$ wants $z = 0$
- Merge clauses & remove z — must satisfy $x \lor \overline{y}$

Conflict Analysis

Time to analyse this conflict and learn from it!

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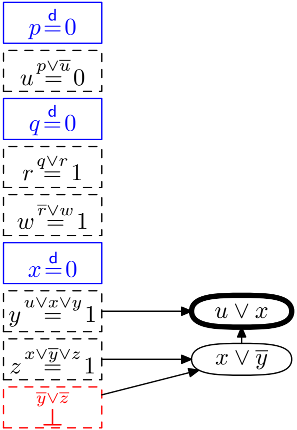

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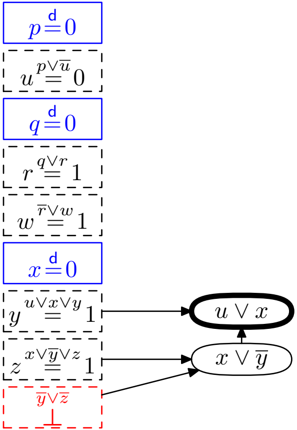
- $\overline{y} \lor \overline{z}$ wants $z = 0$
- Merge clauses & remove z — must satisfy $x \lor \overline{y}$

Repeat until UIP clause with only 1 variable at conflict level after last decision — learn and backjump

Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$



Complete Example of CDCL Execution

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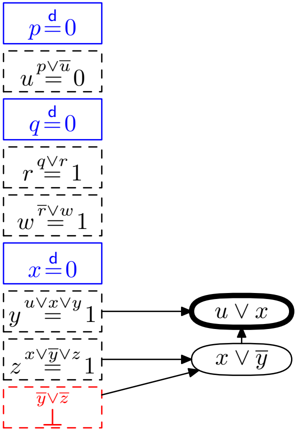
$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

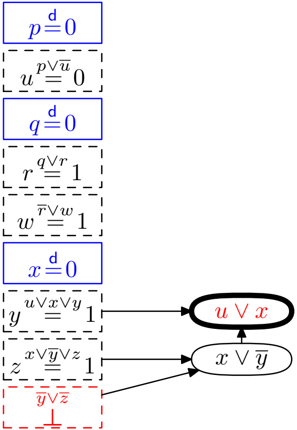


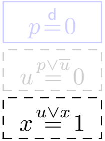
Assertion level 1 (2nd largest level in learned clause) — trim trail to that level

Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$





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Now UIP literal guaranteed to flip (assert) — but this is a propagation, not a decision

Complete Example of CDCL Execution

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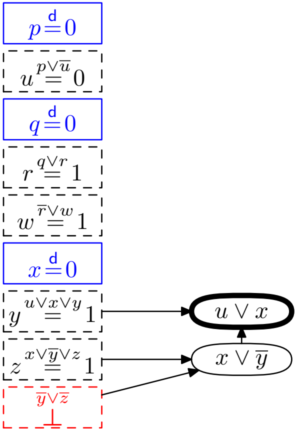
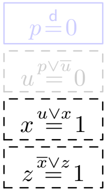
$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$





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Then continue as before. . .

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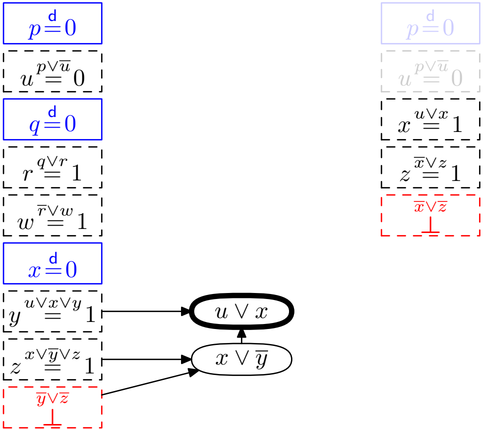

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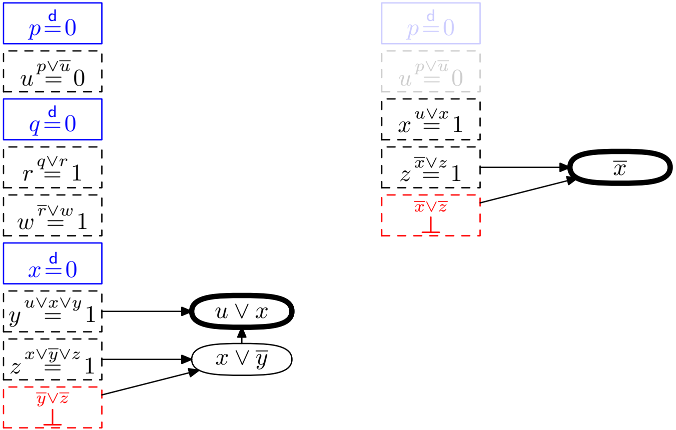

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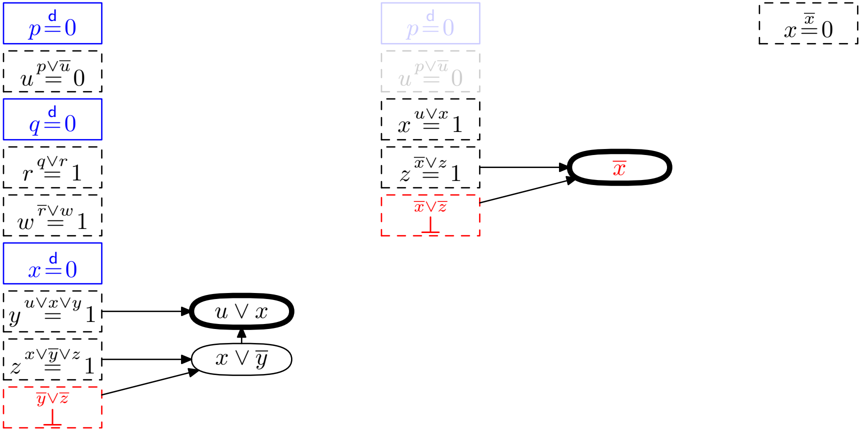

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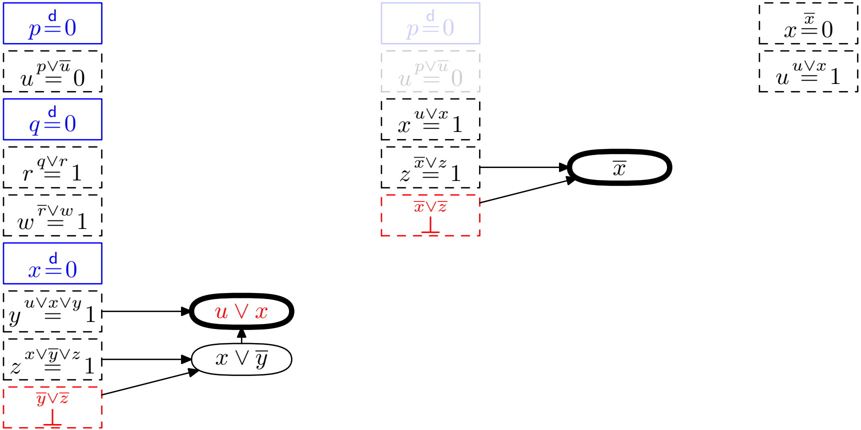

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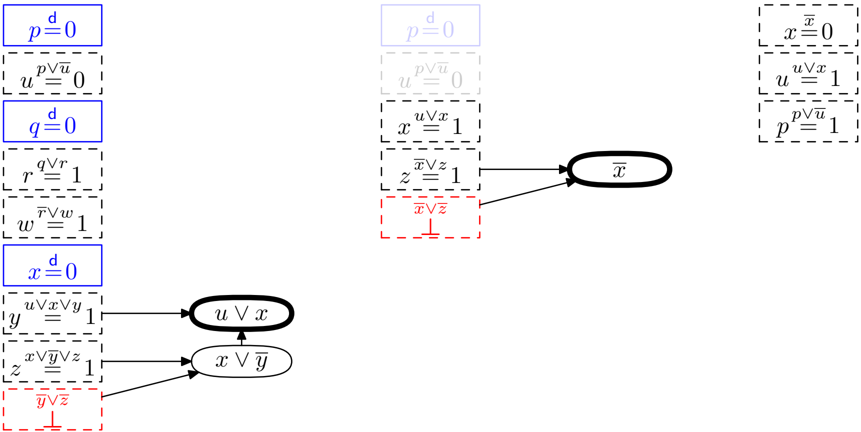

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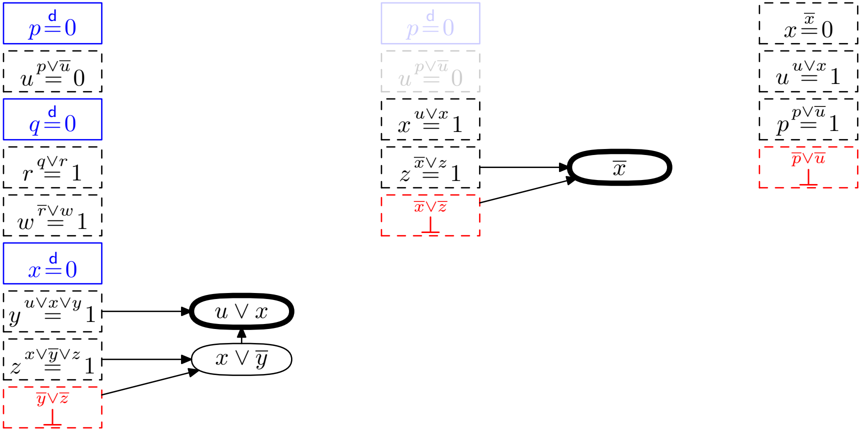

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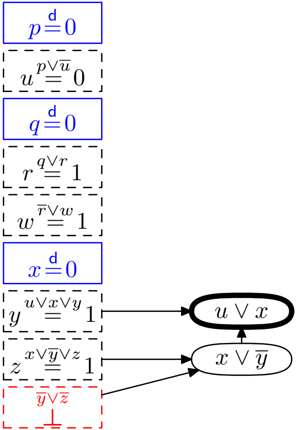

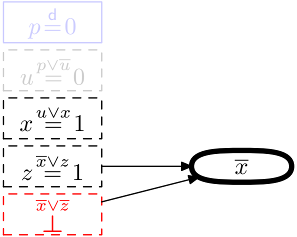

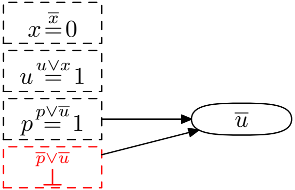

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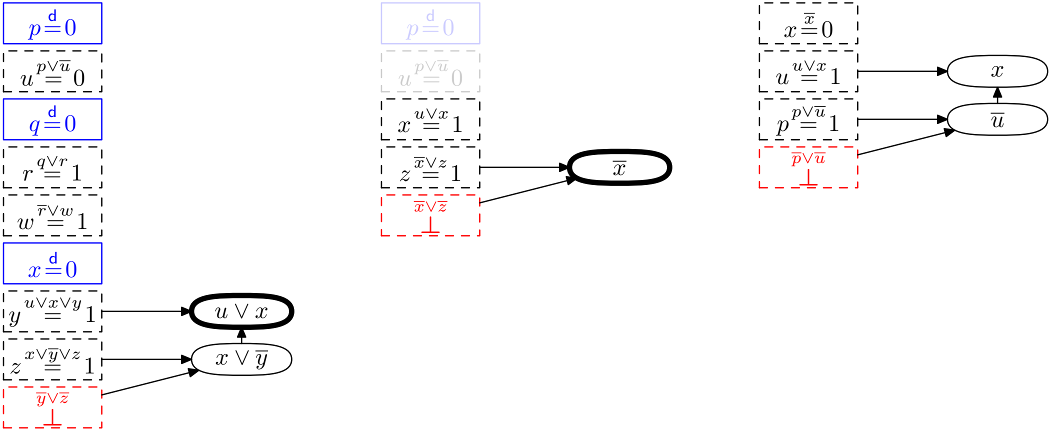

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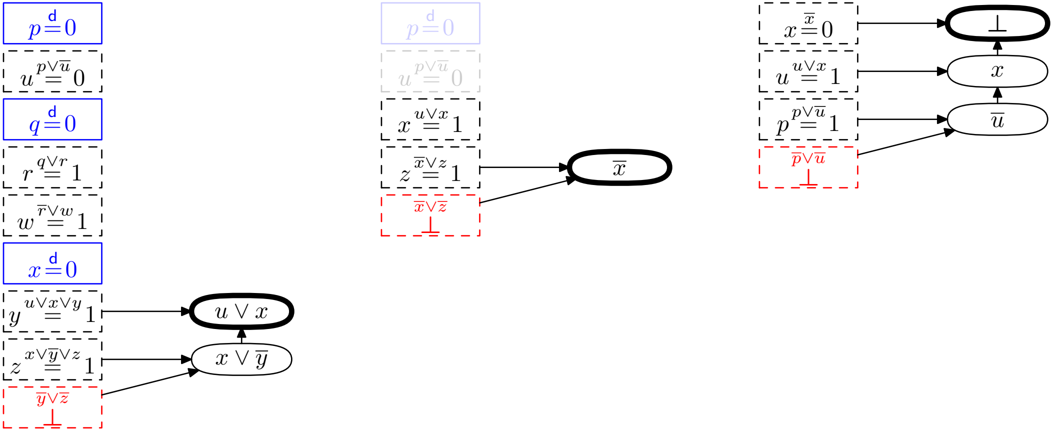

Clause Database Reduction

- In addition to learning clauses, also erase learned clauses that don't seem useful
- Modern solvers do this very aggressively
- Speeds up CDCL search (in particular, unit propagation, which dominates running time)
- But erasing too aggressively can throw away clauses that would have made solver terminate faster [EGG+18]
- So potential trade-off between search speed and search quality
- Except sometimes getting rid of clauses improves search quality too! [KN20]

Restarts

- Fairly frequently, start search all over (but keep learned clauses)
- Original intuition: stuck in bad part of search tree — go somewhere else
- Not the reason this is done now
- Popular variables with high VSIDS scores get set again [MMZ+01]
- Are even set to same values (phase saving) [PD07]
- Current intution: improves the search by focusing on important variables
- Restart at fixed intervals or (better) make adaptive restarts depending on “quality” of learned clauses [AS09, AS12]

Conflict-Driven Clause Learning in Pseudocode (Slightly Simplified)

CDCL(F)

$\mathcal{D} \leftarrow F$; `// initialize clause database to contain formula`
$\rho \leftarrow \emptyset$; `// initialize assignment trail to empty`
forever do
  if *ρ falsifies some clause $C \in \mathcal{D}$* **then**
    $A \leftarrow$ analyzeConflict($\mathcal{D}, \rho, C$) ;
    if $A = \bot$ **then** output `UNSATISFIABLE` and exit ;
    else add learned clause A to $\mathcal{D}$ and backjump by shrinking ρ ;
  else if *exists clause $C \in \mathcal{D}$ unit propagating x to $b \in \{0,1\}$ under ρ* **then**
    add propagated assignment $x \stackrel{C}{=} b$ to ρ ;
  else if *time to restart* **then** $\rho \leftarrow \emptyset$;
  else if *time for clause database reduction* **then**
    erase (roughly) half of learned clauses in $\mathcal{D} \setminus F$ from $\mathcal{D}$
  else if *all variables assigned* **then** output `SATISFIABLE` and exit ;
  else
    use decision scheme to choose x and b and add assignment $x \stackrel{\mathsf{d}}{=} b$ to ρ ;

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Conflict Analysis Pseudocode

analyzeConflict$(\mathcal{D}, \rho, C_{\mathrm{confl}})$

$C_{\mathrm{learn}} \leftarrow C_{\mathrm{confl}}$;
while C_{learn} *not UIP clause* **and** $C_{\mathrm{learn}} \neq \bot$ **do**
    $\ell \leftarrow$ literal assigned last on trail ρ ;
    if ℓ *propagated* **and** $\overline{\ell}$ *occurs in* C_{learn} **then**
        $C_{\mathrm{reason}} \leftarrow \mathrm{reason}(\ell, \rho, \mathcal{D})$;
        $C_{\mathrm{learn}} \leftarrow \mathrm{resolve}(C_{\mathrm{learn}}, C_{\mathrm{reason}})$;
    $\rho \leftarrow \rho \setminus \{\ell\}$;
return C_{learn} ;

State-of-the-art SAT solvers: What About the Recipe?

List of ingredients again (not exhaustive):

- Variable decisions & propagations
- Conflict analysis
- Restarts
- Clause database reduction
- Preprocessing

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- How best to combine these ingredients into a recipe?
- When and why does this recipe work?

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Why SAT solvers actually work so well is poorly understood

Plenty of research has been done to comprehend this better
(Among other places in the MIAO group)

SAT Solver Analysis and the Resolution Proof System

How to make rigorous analysis of CDCL SAT solver performance?
Many intricate, hard-to-understand heuristics
So focus instead on underlying method of reasoning

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Resolution proof system [Bla37, Rob65]

- Start with clauses of CNF formula (axioms)
- Derive new clauses by resolution rule

$$\frac{C_1 \lor x \qquad C_2 \lor \overline{x}}{C_1 \lor C_2}$$

Resolution Proofs by Contradiction

Resolution rule:

$$\frac{C_1 \vee x \qquad C_2 \vee \overline{x}}{C_1 \vee C_2}$$

Observation

If F is a satisfiable CNF formula and D is derived from clauses $D_1, D_2 \in F$ by the resolution rule, then $F \wedge D$ is satisfiable.

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So can prove F unsatisfiable by deriving the unsatisfiable empty clause (denoted $\perp$) from F by resolution

Such proof by contradiction also called resolution refutation

DPLL and Resolution Proofs

A DPLL execution is essentially a resolution proof

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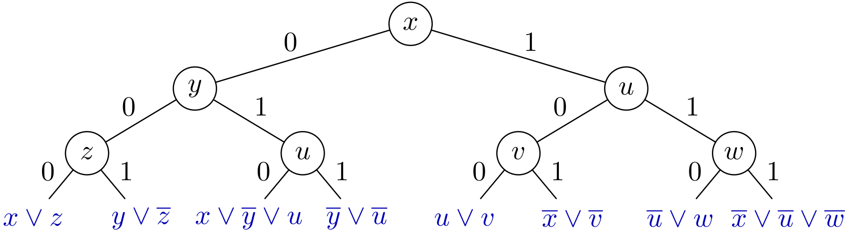
Look at our example again



DPLL and Resolution Proofs

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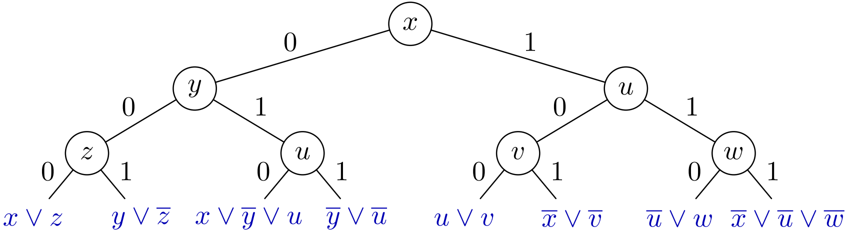

and apply resolution rule $\frac{C_1 \lor x \quad C_2 \lor \overline{x}}{C_1 \lor C_2}$ bottom-up

DPLL and Resolution Proofs

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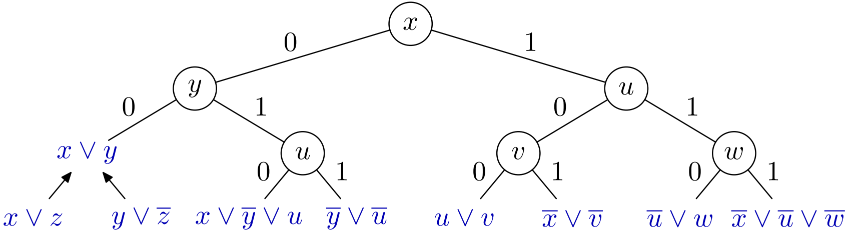

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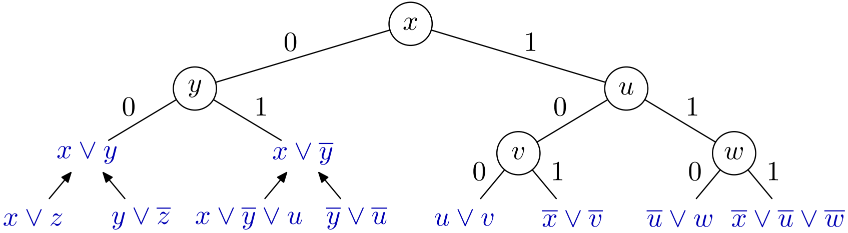

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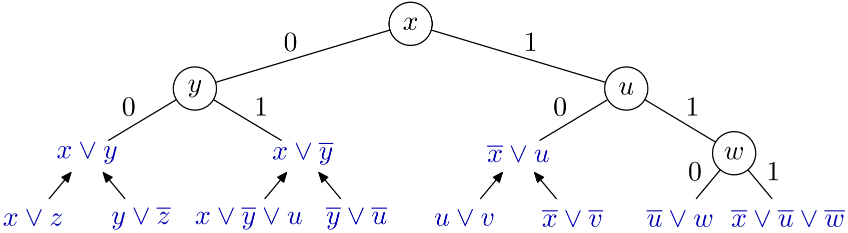

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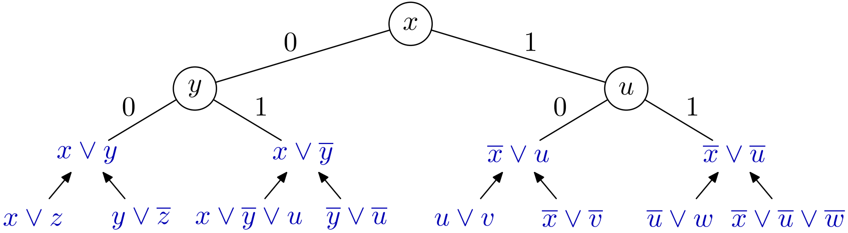

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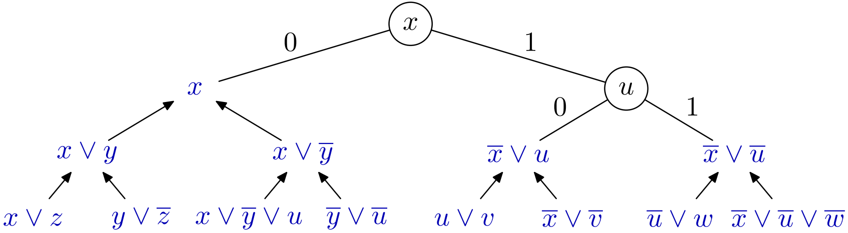

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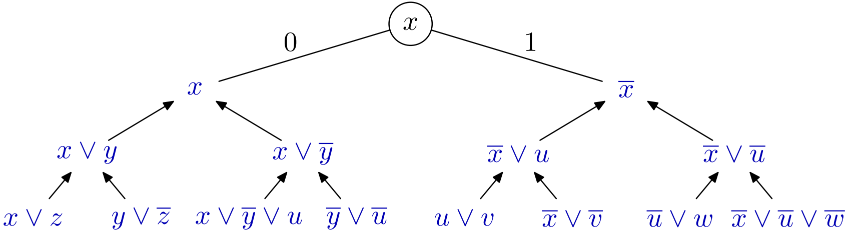

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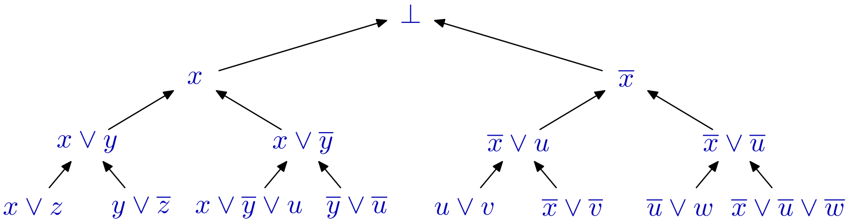

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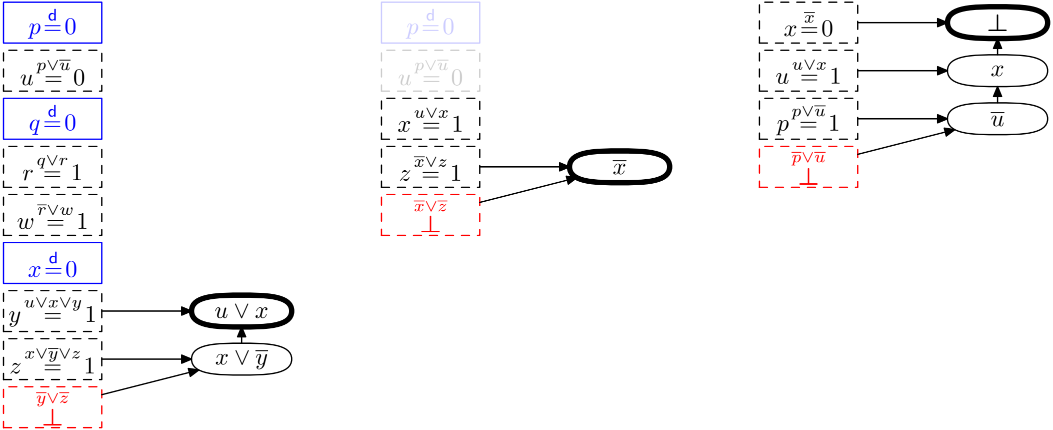

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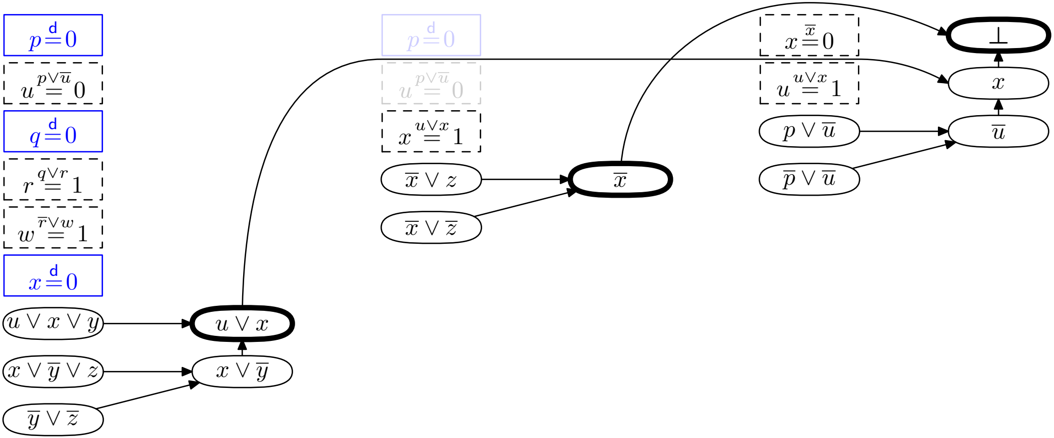

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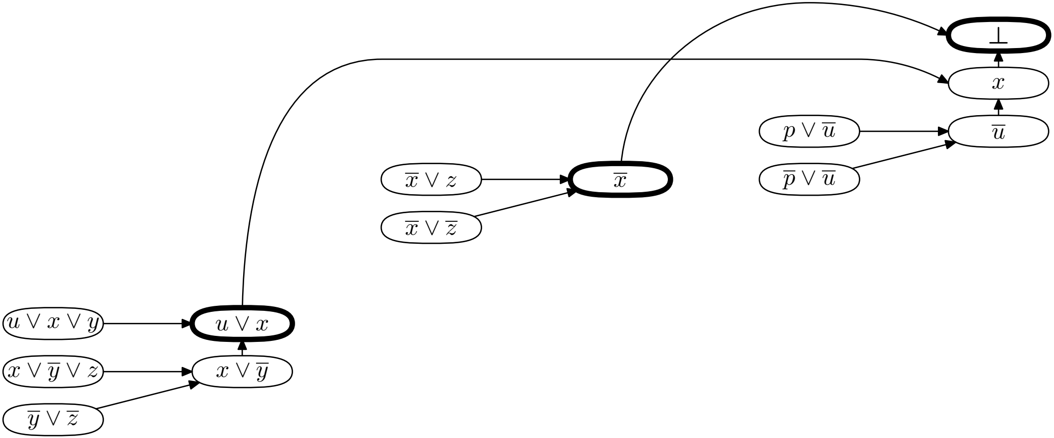

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(*) Except for some preprocessing techniques, which is an important omission, but this gets complicated and we don't have time to go into details...

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  - Why do heuristics work?
  - Why are applied instances easy?
- Paradox: resolution quite weak proof system; many strong proof complexity lower bounds for (seemingly) "obvious" formulas

Examples of Hard Formulas for Resolution (1/3)

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Even onto functional PHP hard — **"resolution cannot count"**

Resolution proof requires $\exp(\Omega(n)) = \exp\left(\Omega\left(\sqrt[3]{N}\right)\right)$ clauses
(measured in terms of formula size N)

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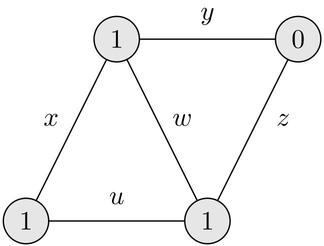

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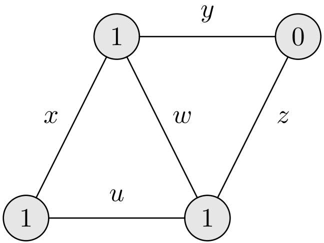

$$
\begin{array}{llll}
 & (u \lor x) & \land & (y \lor \overline{z}) \\
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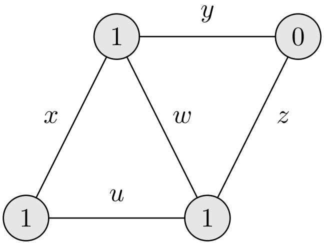

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Requires proof size $\exp(\Omega(N))$ on well-connected so-called expander graphs — **"resolution cannot count** $\bmod 2$**"**

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Δn randomly sampled k-clauses over n variables

($\Delta \gtrsim 4.5$ sufficient to get unsatisfiable 3-CNF formula almost surely)

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And more...

- COLOURING [BCMM05]
- CLIQUE and VERTEXCOVER [BIS07] (though open questions remain [ABdR+21])
- Zero-one designs [Spe10, VS10, MN14]
- ...
- See Chapter 7 on *Proof Complexity and SAT Solving* in the *Handbook of Satisfiability* for more details [BN21]

Theoretical Lower Bounds and Practical Reality

- If resolution so weak, how can CDCL SAT solvers be so good?
- One answer: "tricky" formulas don't show up too often in practice
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- Explore stronger methods of reasoning!
- Algorithms based on such methods could potentially lead to exponential speed-ups *[stay tuned for following lectures. . .]*

Cutting Planes Proof System

Introduced in [CCT87] to model integer linear programming algorithm in [Gom63, Chv73]

Clauses translated to linear inequalities over the reals with integer coefficients

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Example: $x \vee y \vee \overline{z}$ gets translated to $x + y + (1 - z) \geq 1$
or equivalently $x + y - z \geq 0$

Derivation rules

Variable axioms $\dfrac{}{0 \leq x \leq 1}$

Multiplication $\dfrac{\sum a_i x_i \geq A}{\sum c a_i x_i \geq cA} \quad [c \in \mathbb{N}^+]$

Addition $\dfrac{\sum a_i x_i \geq A \quad \sum b_i x_i \geq B}{\sum (a_i + b_i) x_i \geq A + B}$

Division $\dfrac{\sum c a_i x_i \geq A}{\sum a_i x_i \geq \lceil A/c \rceil} \quad [c \in \mathbb{N}^+]$

Cutting Planes Refutations of CNF Formulas

- Translate CNF formula to set of 0-1 linear inequalities
- Apply derivation rules
- Derive $0 \geq 1 \Leftrightarrow$ formula unsatisfiable
- Also makes sense for more general 0-1 linear inequalities (not just translations of CNF formulas)
- Cutting planes can simulate resolution reasoning efficiently and is sometimes exponentially stronger (e.g., for PHP, just count to see #pigeons $>$ #holes)

Building SAT Solvers Based on Cutting Planes Reasoning?

- So-called pseudo-Boolean solvers using cutting planes developed in [CK05, LP10, EN18]
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Is it truly harder to build good pseudo-Boolean solvers?
Or has just so much more work has been put into CDCL solvers?

So... Is There a Smarter Way Than Brute-Force to Solve SAT?

In theory, probably no...

- COLOURING, CLIQUE, SAT, and 1000s other problems are "all the same" — efficient algorithm for one can solve all (the problems are all NP-complete)
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Stark disconnect between theory and practice...

Research Goals in the MIAO Group (1/2)

Strengthen the mathematical analysis of algorithmic methods

- Study methods of reasoning powerful enough to capture state-of-the-art algorithms used in practice
- Prove theorems about their power and limitations
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Construct stronger algorithms for combinatorial problems

- Use insights into stronger mathematical methods of reasoning to build algorithms for SAT and other combinatorial problems
- Aiming for exponential speed-ups over state of the art
- E.g., use cutting planes to build pseudo-Boolean solvers

Research Goals in the MIAO Group (2/2)

Improve understanding of efficient computation in practice

- Use computational complexity theory to study "real-world" (not worst-case) problems
- Combine theoretical study and empirical experiments
- E.g., take "crafted formulas" with provable theoretical properties and investigate correlation with practical solver performance

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Certify correctness for modern combinatorial solvers

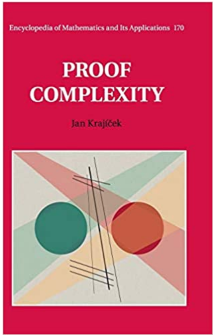
- In many combinatorial optimization paradigms, state-of-the-art solvers are known to be buggy
- Develop methods to make solvers output not just answer but machine-verifiable proof of correctness of this answer

Some References for Further Reading

Handbook of Satisfiability
(Especially chapter 7 ☺)

[BHvMW21]

Proof Complexity
by Jan Krajíček



[Kra19]

And survey papers, slides, and videos at small `www.jakobnordstrom.se`

Take-Home Message

- Modern SAT solvers, although based on old and simple DPLL method, can be enormously efficient in practice
- SAT solving more of an art form than a science — theoretical understanding lagging far behind
- Can use proof complexity to analyze potential and limitations of SAT solvers
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Thanks for listening!

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