Tutorial on Conflict-Driven Pseudo-Boolean Solving

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Pseudo-Boolean?

Pseudo-Boolean (PB) function: $f : \{0,1\}^n \to \mathbb{R}$

Studied since 1960s in operations research and 0–1 integer linear programming [BH02]

Such function f can always be represented as multivariate polynomial of total degree $\leq n$

Restriction for these lectures: f represented as linear form

Many problems expressible as optimizing value of linear pseudo-Boolean function under linear pseudo-Boolean constraints

Pseudo-Boolean vs. SAT

- PB format richer than conjunctive normal form (CNF)

Compare

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \geq 3$$

and

$$\begin{aligned}
& (x_1 \vee x_2 \vee x_3 \vee x_4) \wedge (x_1 \vee x_2 \vee x_3 \vee x_5) \wedge (x_1 \vee x_2 \vee x_3 \vee x_6) \\
\wedge\ & (x_1 \vee x_2 \vee x_4 \vee x_5) \wedge (x_1 \vee x_2 \vee x_4 \vee x_6) \wedge (x_1 \vee x_2 \vee x_5 \vee x_6) \\
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- And pseudo-Boolean reasoning exponentially stronger than conflict-driven clause learning (CDCL)
- Yet close enough to SAT to benefit from SAT solving advances
- Also possible synergies with 0–1 integer linear programming (ILP)

Outline of Tutorial on Pseudo-Boolean Solving

1. Preliminaries
   - Pseudo-Boolean Constraints
   - Pseudo-Boolean Solving and Optimization
2. Conflict-Driven Pseudo-Boolean Solving
   - The Conflict-Driven Paradigm
   - Pseudo-Boolean Conflict Analysis Using Saturation
   - Pseudo-Boolean Conflict Analysis Using Division
3. More About Pseudo-Boolean Reasoning
   - Other Pseudo-Boolean Reasoning Rules
   - Challenges for Efficient PB Solving
   - Some Further References

Linear Pseudo-Boolean Constraints and Normalized Form

For us, pseudo-Boolean constraints are always 0–1 integer linear constraints

$$\sum_i a_i \ell_i \bowtie A$$

- $\bowtie \in \{\geq, \leq, =, >, <\}$
- $a_i, A \in \mathbb{Z}$
- literals ℓ_i: x_i or $\overline{x}_i$ (where $x_i + \overline{x}_i = 1$)
- variables x_i take values $0 = \mathit{false}$ or $1 = \mathit{true}$

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Convenient to use normalized form [Bar95] (without loss of generality)

$$\sum_i a_i \ell_i \geq A$$

- constraint always greater-than-or-equal
- $a_i, A \in \mathbb{N}$ non-negative
- $A = \mathit{deg}(\sum_i a_i \ell_i \geq A)$ referred to as degree (of falsity)

Some Types of Pseudo-Boolean Constraints

1. Clauses are pseudo-Boolean constraints

$$x \vee \overline{y} \vee z \quad \Leftrightarrow \quad x + \overline{y} + z \geq 1$$

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3. General constraints

$$x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$$

Conversion to Normalized Form: Example

Normalized form used for convenience and without loss of generality

$$-x_1 + 2x_2 - 3x_3 + 4x_4 - 5x_5 < 0$$

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1. Make inequality non-strict

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$$x_1 - 2x_2 + 3x_3 - 4x_4 + 5x_5 \geq 1$$

3. Replace $-\ell$ by $-(1-\overline{\ell})$ [where we define $\overline{\overline{x}} \doteq x$]

$$\begin{aligned} x_1 - 2(1-\overline{x}_2) + 3x_3 - 4(1-\overline{x}_4) + 5x_5 &\geq 1 \\ x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 &\geq 7 \end{aligned}$$

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4. Replace "$=$" by two inequalities "$\geq$" and "$\leq$"

Conversion to Normalized Form: Formal Details

Given linear form $\sum_i a_i \ell_i$ with $\sum_i a_i = W$

Syntactic sugar	**Meaning**
$\sum_i a_i \ell_i > A$	$\sum_i a_i \ell_i \geq A + 1$
$\sum_i a_i \ell_i \leq A$	$\sum_i a_i \overline{\ell}_i \geq W - A$
$\sum_i a_i \ell_i < A$	$\sum_i a_i \overline{\ell}_i \geq W - A + 1$
$\sum_i a_i \ell_i = A$	$\sum_i a_i \ell_i \geq A$ and $\sum_i a_i \overline{\ell}_i \geq W - A$

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$\sum_i a_i \ell_i = A$	$\sum_i a_i \ell_i \geq A$ and $\sum_i a_i \overline{\ell}_i \geq W - A$

In what follows:

- Use syntactic sugar when convenient
- Assume (implicit) normalization whenever it matters
- Write $\doteq$ for syntactic equality

Some Notation for Operations on Constraints (1/2)

Given

- constraints $C_1 \doteq \sum_i a_i \ell_i \geq A$ and $C_2 \doteq \sum_i b_i \ell_i \geq B$
- linear form $L \doteq \sum_i c \ell_i$
- positive integer $k \in \mathbb{N}^+$

we will use notation:

$$C_1 + C_2 \doteq \sum_i (a_i + b_i) \cdot \ell_i \geq A + B$$

$$C_1 + L \doteq \sum_i (a_i + c_i) \cdot \ell_i \geq A$$

$$k \cdot C_1 \doteq \sum_i k a_i \cdot \ell_i \geq kA$$

(assuming appropriate normalization whenever needed)

Some Notation for Operations on Constraints (2/2)

Given constraint $C \doteq \sum_i a_i \ell_i \geq A$ with $\sum_i a_i = W$

Negation

$$\neg C \doteq \sum_i a_i \overline{\ell}_i \geq W - A + 1$$

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Reification

$$z \Rightarrow C \doteq A \cdot \overline{z} + \sum_i a_i \ell_i \geq A$$

$$z \Leftarrow C \doteq (W - A + 1) \cdot z + \sum_i a_i \overline{\ell}_i \geq W - A + 1$$

$$z \Leftrightarrow C \doteq z \Rightarrow C \text{ and } z \Leftarrow C$$

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Some calculations

$$C + \neg C \doteq 0 \geq 1$$
$$z \Leftarrow C \doteq \overline{z} \Rightarrow \neg C$$
$$deg(C) \cdot (z \geq 1) + (z \Rightarrow C) \doteq C$$
$$C + (z \Leftarrow C) \doteq deg(\neg C) \cdot z \geq 1$$

Linearization

Possible to linearize nonlinear pseudo-Boolean constraints

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with

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For instance, using fresh variables y_i we can write:

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We won't go further into this in this presentation, though...

Formulas, Decision Problems, and Optimization Problems

Pseudo-Boolean (PB) formula

Conjunction of pseudo-Boolean constraints

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(Maximization: minimize $-\sum_i w_i \ell_i$)

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This lecture:

- Focus on pseudo-Boolean solving
- But not hard to extend to (simple) optimization algorithm

Some Problems Expressed as PBO (1/2)

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- undirected graph $G = (V, E)$
- weight function $w : V \to \mathbb{N}^+$

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Weighted minimum vertex cover

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Find $H \subseteq \mathcal{U}$ such that

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Note: In all of these examples, the problem is to

- optimize a linear function
- subject to a CNF formula (all constraints are clausal)

Already expressive framework!

Approaches for Pseudo-Boolean Problems

What we will discuss in the coming lectures:

1. Pseudo-Boolean (PB) solving and optimization
2. MaxSAT solving
3. Integer linear programming (ILP) — or, more generally, mixed integer linear programming (MIP)

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Rough conceptual difference:

- **PB/SAT:** Focus on integral solutions, try to find optimal one
- **ILP/MIP:** Find optimal non-integer solution; search for integral solutions nearby

Basic trade-off: Inference power vs. inference speed

A Quick Recap of Modern SAT Solving

DPLL method [DP60, DLL62]

- Assign values to variables (in some smart way)
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- More efficient backtracking
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CDCL Main Loop Pseudocode

CDCL(F)

$\mathcal{D} \leftarrow F$; `// initialize clause database to contain formula`
$\rho \leftarrow \emptyset$; `// initialize assignment trail to empty`
forever do
    if *ρ falsifies some clause $C \in \mathcal{D}$* **then**
        $A \leftarrow$ analyzeConflict$(\mathcal{D}, \rho, C)$;
        if $A = \perp$ **then** output `UNSATISFIABLE` and exit ;
        else add learned clause A to $\mathcal{D}$ and backjump by shrinking ρ ;
    else if *exists clause $C \in \mathcal{D}$ unit propagating x to $b \in \{0, 1\}$ under ρ* **then**
        add propagated assignment $x \stackrel{C}{=} b$ to ρ ;
    else if *time to restart* **then** $\rho \leftarrow \emptyset$;
    else if *time for clause database reduction* **then**
        erase (roughly) half of learned clauses in $\mathcal{D} \setminus F$ from $\mathcal{D}$
    else if *all variables assigned* **then** output `SATISFIABLE` and exit ;
    else
        use decision scheme to choose x and b and add assignment $x \stackrel{\mathsf{d}}{=} b$ to ρ ;

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Conflict Analysis Pseudocode

analyzeConflict$(\mathcal{D}, \rho, C_{\mathrm{confl}})$

$C_{\mathrm{learn}} \leftarrow C_{\mathrm{confl}}$;
while C_{learn} *not UIP clause* **and** $C_{\mathrm{learn}} \neq \bot$ **do**
    $\ell \leftarrow$ literal assigned last on trail ρ ;
    if ℓ *propagated* **and** $\overline{\ell}$ *occurs in* C_{learn} **then**
        $C_{\mathrm{reason}} \leftarrow \mathrm{reason}(\ell, \rho, \mathcal{D})$;
        $C_{\mathrm{learn}} \leftarrow \mathrm{resolve}(C_{\mathrm{learn}}, C_{\mathrm{reason}})$;
    $\rho \leftarrow \rho \setminus \{\ell\}$;
return C_{learn} ;

SAT-Based Approaches to Pseudo-Boolean Solving

Conversion to disjunctive clauses

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Conversion to disjunctive clauses

- Lazy approach: learn clauses from PB constraints
  - SAT4J [LP10] (one of versions in library)
- Eager approach: re-encode to clauses and run CDCL
  - MINISAT+ [ES06]
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Want to do “same thing” as in conflict-driven clause learning (CDCL) SAT solving but with pseudo-Boolean constraints without re-encoding

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- At conflict
  1. Do conflict analysis to derive new constraint
  2. Add new constraint to constraint database
  3. Backjump by rolling back decisions so that learned constraint propagates asserting literal (flipping it to opposite value)

Propagation, Conflict, and Slack

Let ρ current assignment of solver (a.k.a. trail)

Represent as $\rho = \{$(ordered) set of literals assigned true$\}$

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Consider $C \doteq x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$

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Note: constraint can be conflicting though not all variables assigned!

Conflict Analysis Invariant

Consider example CDCL conflict analysis from SAT solving lecture

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$p \stackrel{\mathsf{d}}{=} 0$

$u \stackrel{p \lor \overline{u}}{=} 0$

$q \stackrel{\mathsf{d}}{=} 0$

$r \stackrel{q \lor r}{=} 1$

$w \stackrel{\overline{r} \lor w}{=} 1$

$x \stackrel{\mathsf{d}}{=} 0$

$y \stackrel{u \lor x \lor y}{=} 1$

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Assignment "left on trail" always falsifies derived clause

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$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

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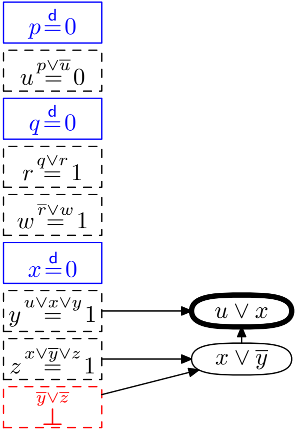

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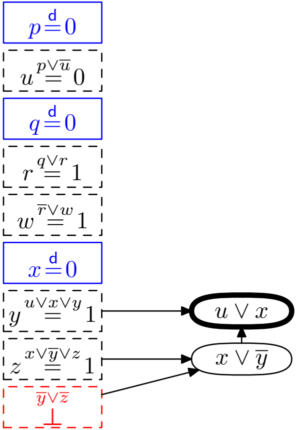

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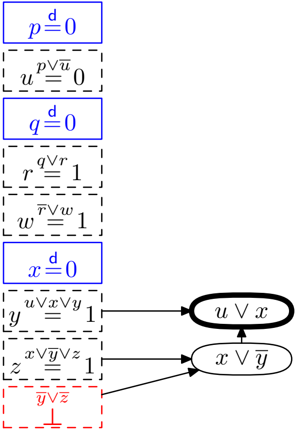

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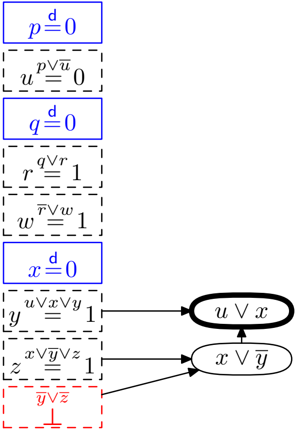

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"Nice" means asserting: after backjump, some variable guaranteed to flip

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Can mimic resolution step

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by adding clauses as pseudo-Boolean constraints

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(Recall $z + \overline{z} = 1$)

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(Recall $z + \overline{z} = 1$)

Generalized resolution rule (from [Hoo88, Hoo92])
Positive linear combination so that some variable cancels

$$\frac{a_1 x_1 + \sum_{i \geq 2} a_i \ell_i \geq A \qquad b_1 \overline{x}_1 + \sum_{i \geq 2} b_i \ell_i \geq B}{\sum_{i \geq 2} \left(\frac{c}{a_1} a_i + \frac{c}{b_1} b_i\right) \ell_i \geq \frac{c}{a_1} A + \frac{c}{b_1} B - c} \quad [c = \mathrm{lcm}(a_1, b_1)]$$

Saturation

Actually, not quite the right constraint in mimicking of resolution

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Sound over integers, not over reals (need such rules for SAT solving)

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[Generalized resolution as defined in [Hoo88, Hoo92] includes fix above, but convenient here to make the two separate steps explicit]

Analyze Conflict with Generalized Resolution + Saturation!

$$C_1 \doteq 2x_1 + 2x_2 + 2x_3 + x_4 \geq 4$$
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Trail $\rho = \{x_1 \stackrel{\mathsf{d}}{=} 0, x_2 \stackrel{C_1}{=} 1, x_3 \stackrel{C_1}{=} 1\}$ $\Rightarrow$ Conflict with C_2

(Note: same constraint can propagate several times!)

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- Resolve reason$(x_3, \rho) = C_1$ with C_2 over x_3 to get resolve(C_1, C_2, x_3)

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- Non-negative slack w.r.t. $\rho' = \{x_1 \stackrel{\mathsf{d}}{=} 0, x_2 \stackrel{C_1}{=} 1\}$
  Not conflicting! Does not explain mistake in assignment

What Went Wrong? And What to Do About It?

Accident report

- Generalized resolution sound over the reals
- Given $\rho' = \{x_1 = 0, x_2 = 1\}$, over the reals have
  - $C_1 \doteq 2x_1 + 2x_2 + 2x_3 + x_4 \geq 4$ propagates $x_3 \geq \frac{1}{2}$
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Remedial action

- Strengthen propagation to $x_3 \geq 1$ also over the reals
- I.e., want reason C with $\mathit{slack}(C; \rho') = 0$
- **Fix** (non-obvious): Apply weakening

$$\text{weaken}\left(\textstyle\sum_i a_i \ell_i \geq A, \ell_j\right) \doteq \textstyle\sum_{i \neq j} a_i \ell_i \geq A - a_j$$

  to reason constraint and then saturate
- Approach in [CK05] (goes back to observations in [Wil76])

Try to Reduce the Reason Constraint

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2. Saturate weakened constraint
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$$\dfrac{\dfrac{\dfrac{2x_1 + 2x_2 + 2x_3 + x_4 \geq 4}{2x_1 + 2x_3 + x_4 \geq 2}\ \text{weaken } x_2}{2x_1 + 2x_3 + x_4 \geq 2}\ \text{saturate} \qquad 2\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3 \geq 3}{2\overline{x}_2 + x_4 \geq 1}\ \text{resolve } x_3$$

Try to Reduce the Reason Constraint

$$C_1 \doteq 2x_1 + 2x_2 + 2x_3 + x_4 \geq 4$$
$$C_2 \doteq 2\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3 \geq 3$$

Trail $\rho = \{x_1 \stackrel{\mathsf{d}}{=} 0, x_2 \stackrel{C_1}{=} 1, x_3 \stackrel{C_1}{=} 1\}$ $\Rightarrow$ Conflict with C_2

Let's try to

1. Weaken reason on non-falsified literal (but not last propagated)
2. Saturate weakened constraint
3. Resolve with conflicting constraint over propagated literal

$$\text{weaken } x_2 \; \dfrac{2x_1 + 2x_2 + 2x_3 + x_4 \geq 4}{2x_1 + 2x_3 + x_4 \geq 2}$$
$$\text{saturate} \; \dfrac{2x_1 + 2x_3 + x_4 \geq 2}{2x_1 + 2x_3 + x_4 \geq 2}$$
$$\text{resolve } x_3 \; \dfrac{2x_1 + 2x_3 + x_4 \geq 2 \qquad 2\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3 \geq 3}{2\overline{x}_2 + x_4 \geq 1}$$

Bummer! Still non-negative slack — not conflicting

Try Again to Reduce the Reason Constraint...

$$C_1 \doteq 2x_1 + 2x_2 + 2x_3 + x_4 \geq 4$$

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$$\text{resolve } x_3 \dfrac{\text{saturate} \dfrac{\text{weaken } \{x_2, x_4\} \dfrac{2x_1 + 2x_2 + 2x_3 + x_4 \geq 4}{2x_1 + 2x_3 \geq 1}}{x_1 + x_3 \geq 1} \qquad 2\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3 \geq 3}{2\overline{x}_2 \geq 1}$$

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Negative slack — conflicting! Derived constraint shows setting x_2 true was a mistake

Try Again to Reduce the Reason Constraint. . .

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Negative slack — conflicting! Derived constraint shows setting x_2 true was a mistake

Backjump propagates to conflict without solver making any decisions
Done! Next conflict analysis will derive contradiction
(Or, in practice, solver terminates immediately at conflict without decisions)

Reason Reduction Using Saturation [CK05]

reduceSat($C_{\mathrm{reason}}, C_{\mathrm{learn}}, \ell, \rho$)

while $\mathit{slack}(\text{resolve}(C_{\mathrm{learn}}, C_{\mathrm{reason}}, \ell); \rho) \geq 0$ **do**
  $\ell' \leftarrow$ literal in $C_{\mathrm{reason}} \setminus \{\ell\}$ not falsified by ρ ;
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$$\mathit{slack}(c \cdot C + d \cdot D; \rho) \leq c \cdot \mathit{slack}(C; \rho) + d \cdot \mathit{slack}(D; \rho)$$

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- By invariant have $\mathit{slack}(C_{\mathrm{learn}}; \rho) < 0$
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- Saturation decreases slack — hit 0 when max #literals weakened

Pseudo-Boolean Conflict Analysis Pseudocode

analyzePBconflict($\mathcal{D}, \rho, C_{\mathrm{confl}}$)

$C_{\mathrm{learn}} \leftarrow C_{\mathrm{confl}}$;
while C_{learn} *not asserting* **and** $C_{\mathrm{learn}} \neq \bot$ **do**
  $\ell \leftarrow$ literal assigned last on trail ρ ;
  if ℓ *propagated* **and** $\overline{\ell}$ *occurs in* C_{learn} **then**
    $C_{\mathrm{reason}} \leftarrow \mathrm{reason}(\ell, \rho, \mathcal{D})$;
    $C_{\mathrm{reduced}} \leftarrow \mathrm{reduceSat}(C_{\mathrm{reason}}, C_{\mathrm{learn}}, \ell, \rho)$;
    $C_{\mathrm{learn}} \leftarrow \mathrm{resolve}(C_{\mathrm{learn}}, C_{\mathrm{reduced}}, \ell)$;
    $C_{\mathrm{learn}} \leftarrow \mathrm{saturate}(C_{\mathrm{learn}})$;
  $\rho \leftarrow \rho \setminus \{\ell\}$;
return C_{learn} ;

Reduction of reason new compared to CDCL — otherwise same conflict analysis algorithm

Essentially conflict analysis used in SAT4J [LP10]

Some Problems Compared to CDCL

- Compared to clauses harder to detect propagation for constraints like

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  $\Rightarrow$ coefficient sizes can explode (expensive arithmetic)

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- Generalized resolution for general pseudo-Boolean constraints
  $\Rightarrow$ lots of lcm computations
  $\Rightarrow$ coefficient sizes can explode (expensive arithmetic)

- For CNF inputs, degenerates to resolution!
  $\Rightarrow$ CDCL but with super-expensive data structures

The Cutting Planes Proof System

Cutting planes from the theory literature [CCT87] doesn't use saturation but instead division (a.k.a. Chvátal-Gomory cut) and can be defined as having rules

Literal axioms $$\frac{}{\ell_i \geq 0}$$

Linear combination $$\frac{\sum_i a_i \ell_i \geq A \qquad \sum_i b_i \ell_i \geq B}{\sum_i (c_A a_i + c_B b_i) \ell_i \geq c_A A + c_B B}$$

Division $$\frac{\sum_i a_i \ell_i \geq A}{\sum_i \lceil a_i / c \rceil \ell_i \geq \lceil A/c \rceil}$$

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- Cutting planes with saturation is **not** [VEG$^+$18]
- Can division yield stronger conflict analysis?

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- Cutting planes with division implicationally complete
- Cutting planes with saturation is **not** [VEG+18]
- Can division yield stronger conflict analysis?
  (Explored for integer linear programming in CutSat [JdM13])

Using Division to Reduce the Reason

$$C_1 \doteq 2x_1 + 2x_2 + 2x_3 + x_4 \geq 4$$

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Terminate immediately!

Reason Reduction Using Division [EN18]

reduceDiv$(C_{\text{reason}}, C_{\text{learn}}, \ell, \rho)$

$c \leftarrow \mathit{coeff}(C_{\text{reason}}, \ell)$;
while $\mathit{slack}(\text{resolve}(C_{\text{learn}}, \text{divide}(C_{\text{reason}}, c), \ell); \rho) \geq 0$ **do**
  $\ell_j \leftarrow$ literal in $C_{\text{reason}} \setminus \{\ell\}$ such that $\overline{\ell}_j \notin \rho$ and $c \nmid \mathit{coeff}(C, \ell_j)$;
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return divide(C_{reason}, c) ;

So now why does **this** work?

- Sufficient to get reason with slack 0 since
    1. $\mathit{slack}(C_{\mathrm{learn}}; \rho) < 0$
    2. slack is subadditive

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- Slack same after weakening $\Rightarrow$ always $0 \leq \mathit{slack}(C_{\text{reason}}; \rho) < c$
- After max #weakenings have $0 \leq \mathit{slack}(\text{divide}(C_{\text{reason}}, c); \rho) < 1$

Round-to-1 Reduction used in RoundingSat

Reduction method in RoundingSat [EN18] does maximal weakening right away

roundToOne(C, ℓ, ρ)

$c \leftarrow \mathit{coeff}(C, \ell)$;
foreach *literal* ℓ_j *in* C **do**
  if $\overline{\ell}_j \notin \rho$ *and* $c \nmid \mathit{coeff}(C, \ell_j)$ **then**
    $C \leftarrow \mathrm{weaken}(C, \ell_j)$;
return divide(C, c) ;

Guaranteed to work by same proof as before

And roundToOne also used more aggressively in conflict analysis
(though modifications of this explored in more recent versions of RoundingSat. . .)

RoundingSat Conflict Analysis [EN18]

analyzePBconflictRS$(\mathcal{D}, \rho, C_{\mathrm{confl}})$

$C_{\mathrm{learn}} \leftarrow C_{\mathrm{confl}}$;
while C_{learn} *contains no or multiple falsified literals on last level* **do**
    if *no decisions in* ρ **then** output `UNSATISFIABLE` and terminate ;
    $\ell \leftarrow$ literal assigned last on trail ρ ;
    if ℓ *propagated* **and** $\overline{\ell}$ *occurs in* C_{learn} **then**
        $C_{\mathrm{learn}} \leftarrow \mathsf{roundToOne}(C_{\mathrm{learn}}, \overline{\ell}, \rho)$;
        $C_{\mathrm{reduced}} \leftarrow \mathsf{roundToOne}(\mathsf{reason}(\ell, \rho, \mathcal{D}), \ell, \rho)$;
        $C_{\mathrm{learn}} \leftarrow \mathsf{resolve}(C_{\mathrm{learn}}, C_{\mathrm{reduced}}, \ell)$;
    $\rho \leftarrow \rho \setminus \{\ell\}$;
$\ell \leftarrow$ literal in C_{learn} last falsified by ρ ;
return $\mathsf{roundToOne}(C_{\mathrm{learn}}, \ell, \rho)$;

Division vs. Saturation

- Higher conflict speed when PB reasoning doesn't help [EN18]
- Seems to perform better when PB reasoning crucial [EGNV18]
- Keeps coefficients small — can (often) do fixed-precision arithmetic
- But SAT4J still better for some circuit verification problems [LBD$^+$20]
- And it is still equally hard to detect propagation
- Also, conflict analysis still degenerates to resolution for CNF inputs
- Sometimes very poor performance even on infeasible 0–1 LPs!

Other PB Rules I: Cardinality Constraint Reduction

Given PB constraint

$$3x_1 + 2x_2 + x_3 + x_4 \geq 4$$

can compute least #literals that have to be true

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Cardinality constraint reduction rule

$$\frac{\sum_i a_i \ell_i \geq A}{\sum_{i\,:\,a_i>0} \ell_i \geq T} \quad T = \min\{|I| \,:\, I \subseteq [n], \sum_{i \in I} a_i \geq A\}$$

Can be simulated with weakening + division

Other PB Rules II: Strengthening

Strengthening by example:

- Set $x = 0$ and propagate on constraints

$$x + y \geq 1 \qquad x + z \geq 1 \qquad y + z \geq 1$$

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Strengthening rule (imported by [DG02] from operations research)

- Suppose $\ell = 0 \Rightarrow \sum_i a_i \ell_i \geq A$ oversatisfied by amount K
- Then can deduce $K\ell + \sum_i a_i \ell_i \geq A + K$

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- Then can deduce $K\ell + \sum_i a_i \ell_i \geq A + K$

In theory, can recover from bad encodings (e.g., CNF)
In practice, seems inefficient and hard to get to work...

Other PB Rules III: "Fusion Resolution"

Suppose have constraints

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But only get from resolution

$$6y + 4z + 2w \geq 4$$

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But only get from resolution + saturation

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$$2y + 2z + \quad w \geq 2$$

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But only get from resolution + saturation + division

$$2y + 2z + w \geq 2$$

"Fusion resolution" [Goc17]

$$\frac{a\ell + \sum_i b_i \ell_i \geq B \qquad a\overline{\ell} + \sum_i b_i \ell_i \geq B'}{\sum_i b_i \ell_i \geq \min\{B, B'\}}$$

No obvious way for cutting planes to immediately derive this

Shows up in some tricky benchmarks in [EGNV18]

Some PB Solving Challenges I: Input Format

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4. Robustness: Make PB solvers less sensitive to presence of extra constraints (anecdotally, CDCL solvers seem more stable)

Some PB Solving Challenges II: Conflict Analysis

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3. Do constraint minimization à la [SB09, HS09]?
4. How to assess quality of learned constraints?
5. Theoretical potential & limitations poorly understood [VEG+18]
   - Separations in power between different methods of PB reasoning?
   - In particular, is reasoning with division stronger than with saturation [GNY19]?

Some PB Solving Challenges III: Solver Heuristics

Many heuristics copied from CDCL — maybe tailor more carefully to PB setting?

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4. Different “modes” for SAT-focused and UNSAT-focused search?
5. Local search for more efficient finding of solutions?

See [Wal20] for a first in-depth investigation of some of these questions

Some PB Solving Challenges IV: Efficiency and Correctness

1. Efficient unit propagation for PB constraints is a major challenge — latest news in [Dev20, NORZ24], but still much left to do

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2. Efficient detection of assertiveness during conflict analysis
3. Efficient and concise proof logging for pseudo-Boolean solving (shameless self-plug: ongoing work on pseudo-Boolean proof checker VeriPB [Ver, BMN22] in [EGMN20, GMN20, GMM+20, GN21, GMN22, GMNO22, BBN+23, BGMN23, BBN+24, DMM+24, GMM+24, HOGN24, IOT+24, MMN24])

Some References for Further Reading (and Watching)

Handbook of Satisfiability [BHvMW21]

- Chapter 7: Proof Complexity and SAT Solving
- Chapter 23: MaxSAT, Hard and Soft Constraints
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- Chapter 28: Pseudo-Boolean and Cardinality Constraints

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Video tutorials on pseudo-Boolean solving

Presentations from today will be available at the MIAO YouTube channel `youtube.com/@MIAOresearch`

Summing up

- Pseudo-Boolean framework expressive and powerful
- Can be approached using successful conflict-driven paradigm from SAT solving
- In theory, potential for exponential increase in performance
- In practice, some highly nontrivial challenges regarding
  - Algorithm design
  - Efficient implementation
  - Theoretical understanding
- But maybe also quite a bit of low-hanging fruit?
  (And clause-based SAT solving took 50+ years to get right)
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Thank you for your attention!

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