Tutorial on Conflict-Driven Pseudo-Boolean Optimization

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From Decision to Optimization Problems

So far in this series of tutorials we have talked about:

- deciding satisfiability of formulas in conjunctive normal form (CNF)
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Many ways of extending this to (linear) optimization problems:

- Hard and soft constraints
  Maximum satisfiability (MaxSAT) and Weighted Boolean optimization (WBO)
- Constraints with preferences among solutions
  Weighted constraint satisfaction problems (WSCP)
- Optimization of linear objective function subject to CNF/pseudo-Boolean formula
  Pseudo-Boolean optimization (PBO) or 0–1 integer linear programming (0–1 ILP)

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  Pseudo-Boolean optimization (PBO) or 0–1 integer linear programming (0–1 ILP)

We will focus on MaxSAT and PBO viewed as 0–1 ILP minimization problems

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- undirected graph $G = (V, E)$
- weight function $w : V \to \mathbb{N}^+$

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Note: In all of these examples, the problem is to

- optimize a linear function
- subject to a CNF formula (all constraints are clausal)

Without loss of generality, this is what a MaxSAT problem is (as we will soon see)

Outline of Tutorial on Pseudo-Boolean Optimization

1. Basics of MaxSAT and Pseudo-Boolean Optimization
   - Problem Definition
   - Maximum Satisfiability (MaxSAT) Solving
   - Solution-Improving SAT-UNSAT Search
2. Core-Directed UNSAT-SAT Search
   - Basic Core-Guided Search
   - Advanced Core-Guided Search Techniques
   - Implicit Hitting Set (IHS) and Abstract Cores
3. Some Open Problems

Classic Definition of MaxSAT Problem

Pseudo-Boolean optimization and MaxSAT solving intimately connected, so let's start with traditional description of MaxSAT problem

Weighted partial MaxSAT problem

Input: Soft clauses $C_1, \ldots, C_m$ with weights $w_i \in \mathbb{N}^+$, $i \in [m]$
Hard clauses $C_{m+1}, \ldots, C_M$

Goal: Find assignment ρ such that

- for all hard clauses $C_{m+1}, \ldots, C_M$ it holds that $\rho(C_j) = 1$
- ρ maximizes $\sum_{\rho(C_i)=1, i \in [m]} w_i$

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- All hard clauses must be satisfied
- Maximize weight of satisfied soft clauses $=$ minimize penalty of falsified soft clauses
  From now on, MaxSAT is a minimization problem
- Write $(C)_w$ for clause C with weight w ($w = \infty$ for hard clause)

From MaxSAT to Pseudo-Boolean Optimization

MaxSAT instance

$$(\overline{x})_5$$
$$(y \lor \overline{z})_4$$
$$(\overline{y} \lor z)_3$$
$$(x \lor y \lor z)_\infty$$
$$(x \lor \overline{y} \lor \overline{z})_\infty$$

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PBO instance

$$\min \; 5b_1 + 4b_2 + 3b_3$$
$$b_1 + \overline{x} \geq 1$$
$$b_2 + y + \overline{z} \geq 1$$
$$b_3 + \overline{y} + z \geq 1$$
$$x + y + z \geq 1$$
$$x + \overline{y} + \overline{z} \geq 1$$

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Add fresh variable b_i to each soft clause C_i and minimize $\sum_i w_i b_i$
So-called blocking variable transformation
Variables b_i are blocking or relaxation variables

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MaxSAT instance	**PBO instance**
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$(y \lor \overline{z})_4$	$b_2 + y + \overline{z} \geq 1$
$(\overline{y} \lor z)_3$	$b_3 + \overline{y} + z \geq 1$
$(x \lor y \lor z)_\infty$	$x + y + z \geq 1$
$(x \lor \overline{y} \lor \overline{z})_\infty$	$x + \overline{y} + \overline{z} \geq 1$

Add fresh variable b_i to each soft clause C_i and minimize $\sum_i w_i b_i$

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Optimal solution $\rho = \{x = 0, y = 1, z = 0\}$ with penalty 3

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\begin{array}{ll}
\min & \sum_{i=1}^{n} w_i \ell_i \\
& C_1 \\
& C_2 \\
& \vdots \\
& C_M
\end{array}
$$

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PBO instance

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MaxSAT/WBO instance

$$\begin{aligned} & (\overline{\ell}_1)_{w_1} \\ & \vdots \\ & (\overline{\ell}_n)_{w_n} \\ & (C_1)_\infty \\ & \vdots \\ & (C_M)_\infty \end{aligned}$$

Flavours of MaxSAT

- Partial MaxSAT: Hard and soft clauses
- MaxSAT: Only soft clauses
- Unweighted MaxSAT: Same weight for soft clauses (w.l.o.g. 1)
- Weighted MaxSAT: Different weights for soft clauses

4 different subproblems
But most current solvers deal with the most general problem

Main Approaches for MaxSAT (and Pseudo-Boolean Optimization)

1. Linear search SAT-UNSAT (LSU) (or solution-improving search)
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List above not exhaustive — omits, e.g., branch-and-bound MaxSAT [LXC+21]

Linear Search SAT-UNSAT (LSU) Algorithm

- Minimize $\sum_{i=1}^{n} w_i \ell_i$
- Subject to collection of PB constraints $F = C_1 \wedge \cdots \wedge C_m$

Set $\rho_{\mathrm{best}} = \emptyset$ and repeat the following:

1. Run SAT/PB solver
2. If solver returns `UNSATISFIABLE`, output ρ_{best} and terminate
3. Otherwise, let $\rho_{\mathrm{best}} :=$ returned solution ρ
4. Add solution-improving constraint $\sum_{i=1}^{n} w_i \ell_i \leq -1 + \sum_{i=1}^{n} w_i \cdot \rho(\ell_i)$
5. Start over from the top

Linear Search Toy Example

1. Given PB formula F and objective function $\min x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6$

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3. Yields objective value $0 + 2 \cdot 0 + 3 \cdot 0 + 4 \cdot 1 + 5 \cdot 1 + 6 \cdot 0 = 9$, so add

$$x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6 \leq 8$$

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5. Yields objective value 6, so add
$$x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6 \leq 5$$

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6. Now solver returns `UNSATISFIABLE`

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6. Now solver returns `UNSATISFIABLE`
7. Hence, minimum value of objective function subject to F is 6

CNF Encoding of Solution-Improving Constraint

For SAT solver, need CNF encoding of solution-improving constraint
$\sum_{i=1}^{n} w_i \ell_i \leq -1 + \sum_{i=1}^{n} w_i \cdot \rho(\ell_i)$

Lots of work on how to do this in smart ways

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For pseudo-Boolean solver, no re-encoding needed — solution-improving constraint can be added as-is

But also no access to auxiliary variables from CNF encodings

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Two possible explanations:

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2. Potentially very different cost for
   - SAT calls (feasible instances where solver will find solution)
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Properties of linear search SAT-UNSAT:

- Can get **some decent** solution quickly, even if not optimal one
- Important for anytime solving (when time is limited and something is better than nothing)
- But get no estimate of how good the solution is

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Solver outputs

- either solution extending σ
- or explanation (clause/pseudo-Boolean inequality referred to as core constraint) why assumptions σ inconsistent with F

Explanation obtained by simple modification of conflict analysis (decision learning scheme)

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decisionLearningPB$(\mathcal{D}, \rho, C_{\mathrm{confl}})$

$C_{\mathrm{learn}} \leftarrow C_{\mathrm{confl}}$;
$\boldsymbol{\rho}_{\mathbf{dec}} \leftarrow$ all decisions in $\boldsymbol{\rho}$;
while $\boldsymbol{\rho} \neq \emptyset$ **and** $C_{\mathrm{learn}} \neq \bot$ **do**
  $\ell \leftarrow$ literal assigned last on trail ρ ;
  if ℓ *propagated* **and** $\overline{\ell}$ *occurs in* C_{learn} **then**
    $C_{\mathrm{reason}} \leftarrow \mathrm{reason}(\ell, \rho, \mathcal{D})$;
    $C_{\mathrm{reduced}} \leftarrow \mathrm{reduce}(C_{\mathrm{reason}}, C_{\mathrm{learn}}, \ell, \boldsymbol{\rho} \cup \boldsymbol{\rho}_{\mathbf{dec}})$;
    $C_{\mathrm{learn}} \leftarrow \mathrm{resolve}(C_{\mathrm{learn}}, C_{\mathrm{reduced}}, \ell)$;
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return C_{learn} ;

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Postcondition: Learned constraint C_{learn} is violated by decisions ρ_{dec}

Solving with Assumptions and Decision Learning Scheme

Given assumptions σ with $S = |\rho|$

Assumptions and decision learning scheme for CDCL

- Run standard CDCL but with first S decisions coming from σ
- When conflict reached at level $\leq S$, switch to decision learning scheme and return learned constraint

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Assumptions and decision learning scheme for pseudo-Boolean solving

- Can reach level $\leq S$ during conflict analysis even for conflict at level $S' > S$
- If this happens, switch to decision learning scheme on-the-fly?
- Or only when conflict reached at level $\leq S$? (Will happen next conflict)
- Or keep going as long as conflict level decreases?

Core-Guided Search

- Minimize $\sum_{i=1}^{n} w_i \ell_i$
- Subject to collection of PB constraints $F = C_1 \wedge \cdots \wedge C_m$

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Core-Guided Search

- Minimize $\sum_{i=1}^{n} w_i \ell_i$
- Subject to collection of PB constraints $F = C_1 \wedge \cdots \wedge C_m$

To see where name comes from, consider MaxSAT instance with ℓ_i as blocking variables

Core-guided search sets $val_{\text{best}} = 0$ and repeats the following:

1. Run SAT solver with assumptions $\ell_i = 0$ for all ℓ_i in objective function
2. If solver returns `SATISFIABLE`, output val_{best} and terminate
3. Otherwise learn clause over assumption variables, say $\ell_1 \vee \cdots \vee \ell_k$
4. Means that soft clauses $K = \{C_1, \ldots, C_k\}$ form a core — can't satisfy K and all hard constraints
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7. Start over from top with updated objective function

Core-Guided Search for Pseudo-Boolean Optimization

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- And rewriting very convenient — just use PB constraints without re-encoding
- Core-guided pseudo-Boolean search: assume optimistically that objective can reach best imaginable value; derive contradiction if not possible
- Let us try to explain by concrete example

Core-Guided Search Toy Example (1/5)

1. Given same PB formula F and objective function

$$\min x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6 \tag{1}$$

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5. Introduce new, fresh counter variables y_3 and y_4 and constraints

$$x_2 + x_3 + x_4 + x_5 = 2 + y_3 + y_4 \tag{4a}$$

$$y_3 \geq y_4 \tag{4b}$$

to enforce that y_j means "$x_2 + x_3 + x_4 + x_5 \geq j$"

Core-Guided Search Toy Example (2/5)

6. Multiply (4a) by 2 to get

$$4 + 2y_3 + 2y_4 - 2x_2 - 2x_3 - 2x_4 - 2x_5 = 0$$

and add to objective function

$$\min x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6$$

in (1) to cancel x_2 and get updated, equivalent objective function

$$\min x_1 + x_3 + 2x_4 + 3x_5 + 6x_6 + 2y_3 + 2y_4 + 4 \tag{5}$$

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$$\min x_1 + x_3 + 2x_4 + 3x_5 + 6x_6 + 2y_3 + 2y_4 + 4 \tag{5}$$

7. Update $val_{\text{best}} = 4$ and run solver on F assuming $\ell = 0$ for all literals ℓ in rewritten objective (5)

Core-Guided Search Toy Example (3/5)

9. Suppose solver returns the clausal core constraint

$$x_4 + x_5 + x_6 + y_3 \geq 1 \tag{6}$$

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10. Introduce new variables z_2, z_3, z_4 and the constraints

$$x_4 + x_5 + x_6 + y_3 = 1 + z_2 + z_3 + z_4 \tag{7a}$$

$$z_2 \geq z_3 \tag{7b}$$

$$z_3 \geq z_4 \tag{7c}$$

to enforce that z_j means "$x_4 + x_5 + x_6 + y_3 \geq j$"

Core-Guided Search Toy Example (4/5)

10. Multiply (7a) by 2 to get

$$2 + 2z_2 + 2z_3 + 2z_4 - 2x_4 - 2x_5 - 2x_6 - 2y_3 = 0$$

and add to rewritten objective

$$\min x_1 + x_3 + 2x_4 + 3x_5 + 6x_6 + 2y_3 + 2y_4 + 4$$

in (5) to get 3rd equivalent objective

$$\min x_1 + x_3 + x_5 + 4x_6 + 2y_4 + 2z_2 + 2z_3 + 2z_4 + 6 \tag{8}$$

Core-Guided Search Toy Example (4/5)

10. Multiply (7a) by 2 to get

$$2 + 2z_2 + 2z_3 + 2z_4 - 2x_4 - 2x_5 - 2x_6 - 2y_3 = 0$$

and add to rewritten objective

$$\min x_1 + x_3 + 2x_4 + 3x_5 + 6x_6 + 2y_3 + 2y_4 + 4$$

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$$\min x_1 + x_3 + x_5 + 4x_6 + 2y_4 + 2z_2 + 2z_3 + 2z_4 + 6 \tag{8}$$

11. Update $val_{\text{best}} = 6$ and run solver on F assuming $\ell = 0$ for all literals ℓ in rewritten objective (8)

Core-Guided Search Toy Example (5/5)

15. Suppose solver reports it is possible to achieve

$$\rho = \{x_1 = x_3 = x_5 = x_6 = y_4 = z_2 = z_3 = z_4 = 0\} \tag{9}$$

Core-Guided Search Toy Example (5/5)

15. Suppose solver reports it is possible to achieve

$$\rho = \{x_1 = x_3 = x_5 = x_6 = y_4 = z_2 = z_3 = z_4 = 0\} \tag{9}$$

16. Under assignment (9) the equality (4a) simplifies to

$$x_2 + x_4 = 2 + y_3 \tag{10}$$

which can hold only if $y_3 = 0$ and $x_2 = x_4 = 1$, and this also satisfies (7a).

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17. Hence, have recovered optimal solution yielding objective value 6 (as in solution-improving search example before)

Properties of (Pure) Core-Guided Search

- Can get decent lower bounds on solution quickly
- Helps to cut off parts of search space “too good to be true”
- But find no actual solution until the final, optimal one
- Also, no estimate of how good the lower bound is
- Linear search much better at finding solutions — how to get the best of both worlds?

Weight Stratification [ABGL12]

Suppose objective function is

$$x_1 + 2x_2 + 3x_3 + 11x_4 + 12x_5 + 13x_6 + 101x_7 + 102x_8 + 103x_9$$

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Focus on variables with largest weight in objective:

1. First assume $x_7 = x_8 = x_9 = 0$ and try to get core
2. If this fails, assume $x_4 = x_5 = x_6 = x_7 = x_8 = x_9 = 0$ and try to get core
3. Only then assume all of objective function is 0

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What do we gain from this?

- More compact core yielding larger increase of lower bound, or
- Decent solution found early on

Disjoint Cores and Weight-Aware Core Extraction

Consider our core-guided toy example with objective function

$$x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6$$

and assumptions $x_1 = x_2 = x_3 = \ldots = x_6 = 0$ yielding core $3x_2 + 2x_3 + x_4 + x_5 \geq 4$
rounded to cardinality constraint $x_2 + x_3 + x_4 + x_5 \geq 2$

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Disjoint cores [DB11, DB13, Sai15]

- Remove variables in core found
- Call solver with remaining assumptions $x_1 = x_6 = 0$

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Disjoint cores [DB11, DB13, Sai15]

- Remove variables in core found
- Call solver with remaining assumptions $x_1 = x_6 = 0$

Weight-aware core extraction [BJ17]

- Remove only variables that cancel in objective rewriting, in our case x_2
- Call solver with assumptions $x_1 = x_3 = x_4 = x_5 = x_6 = 0$

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Find more independent cores that contribute to larger increase of lower bound

Improving the Core Constraints

Suppose solver returns a core $\sum_i a_i \ell_i \geq A$ (where in MaxSAT $a_i = A = 1$)

Core exhaustion [IMM19]

- Define counter variable
$$y_{A+1} \Leftrightarrow \sum_i a_i \ell_i \geq A + 1$$
- Run solver with assumption $y_{A+1} = 0$
- If solver returns `UNSATISFIABLE`, can strengthen core

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Core trimming/minimization [Mar10, MIM15]

- Choose different literals ℓ_{j^*}
- Check if $\sum_{i \neq j^*} a_i \ell_i \geq A$ still core
- Straightforward for clauses — less obvious how this would work for PB constraints

Lazy Counter Variables for Cores [MJML14, DGD+21]

Consider again core-guided toy example with first core $3x_2 + 2x_3 + x_4 + x_5 \geq 4$ rounded to cardinality constraint $x_2 + x_3 + x_4 + x_5 \geq 2$

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Eager counter variables

- Add all counter variables y_3 and y_4 and constraints

$$x_2 + x_3 + x_4 + x_5 = 2 + y_3 + y_4$$
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$$y_3 \geq y_4$$

Lazy counter variables

- Add y_3 defined by

$$y_3 \Leftrightarrow x_2 + x_3 + x_4 + x_5 \geq 3$$

- Only introduce $y_4 \Leftrightarrow x_2 + x_3 + x_4 + x_5 \geq 4$ when y_3 cancels in rewritten objective
- Rewriting of objective can introduce huge numbers of variables, slowing down solver
- Adding variables lazily only when needed speeds things up

Combining Core-Guided and Solution-Improving Search

Core boosting [BDS19]

- Start with core-guided search to get good lower bound estimate
- Then switch to solution-improving search to find optimal solution

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Hybrid/interleaving search [ADMR15]

- Switch back and forth repeatedly between core-guided and solution-improving search
- Cumbersome in CNF-based solver
- But fairly cheap (and efficient) in native pseudo-Boolean solver [DGD+21]

Some More Parameters to Play with (Not Exhaustive List)

1. Adjust phase saving
2. Detect intrinsic at-most-1 constraints [IMM19]
3. Rewrite the objective with non-cardinality core constraints [JBJ24]
4. Avoid storing solution-improving constraint in solver database [SBJ21, SBJ22]

Theoretical Analysis of Core-Guided Search?

Lower bound computed by core-guided search

- Core-guided search provides lower bound estimate based on cores found
- Can be viewed as optimum of LP relaxation of problem to minimize objective given cores and counter variable definitions [Kat23]

Inference strength of core-guided search?

- Extension variables very strong in theory — no proof complexity lower bounds
- So far hard to leverage this power in practice
- But core-guided search provides principled way of introducing extension variables
- And is very structured — possible to analyze power of this method with proof complexity?

Evaluation of Core-Guided Pseudo-Boolean Solver in [DGD+21]

RoundingSat with core-guided (CG) and linear SAT-UNSAT search (LSU)

#instances solved to optimality; highlighting **1st**, **2nd**, and **3rd** best

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	PB16opt (1600)	MIPopt (291)	KNAP (783)	CRAFT (985)
Hybrid (interleave CG & LSU)	**968**	**78**	306	**639**
HybridCl (w/ clausal cores)	937	75	298	**618**
HybridNL (w/ non-lazy variables)	936	70	186	607
HybridClNL (w/ both)	917	67	203	612
RoundingSat (only LSU)	853	75	341	309
CoreGuided (only CG)	911	61	43	595
CoreBoosted (10% CG, then LSU)	**959**	**80**	**344**	580
Sat4j	773	61	**373**	105
NaPS	896	65	111	345
SCIP	**1057**	**125**	**765**	**642**

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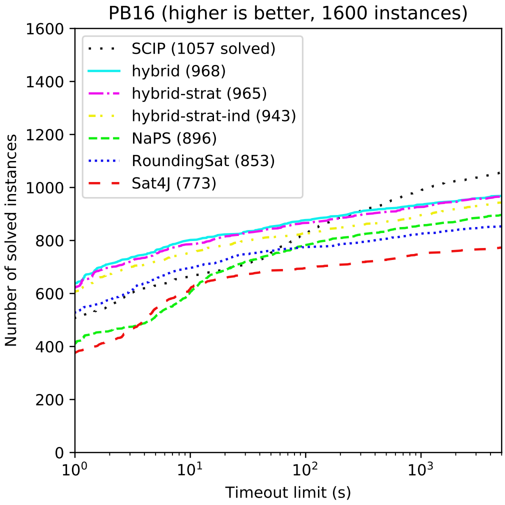
Significant improvement over PB state of the art, but MIP solver SCIP still better

Core-Guided Pseudo-Boolean Solving for PB16 benchmarks [DGD+21]

Cumulative plot for solver performance on PB16 optimization benchmarks

Also including

- stratfication (strat)
- disjoint/independent cores (ind)



Implicit Hitting Set (IHS) Algorithm (1/2)

- Minimize $\sum_{i=1}^{n} w_i \ell_i$
- Subject to collection of PB constraints $F = C_1 \land \cdots \land C_m$ (suppose for now clausal constraints)

As in core-guided search, use solving with assumptions, but maintain collection $\mathcal{K}$ of learned core clauses

$$
\begin{aligned}
C_1 &\doteq \ell_{1,1} \lor \ell_{1,2} \lor \cdots \lor \ell_{1,k_1} \\
C_2 &\doteq \ell_{2,1} \lor \ell_{2,2} \lor \cdots \lor \ell_{2,k_2} \\
&\vdots \\
C_s &\doteq \ell_{s,1} \lor \ell_{s,2} \lor \cdots \lor \ell_{s,k_s}
\end{aligned}
$$

Implicit Hitting Set (IHS) Algorithm (2/2)

Set $\mathcal{K} = \emptyset$ and repeat the following:

1. Run optimization solver to compute minimum hitting set for $\mathcal{K}$, i.e., $H = \{\ell_i\}$ s.t.
   - $H \cap C \neq \emptyset$ for all $C \in \mathcal{K}$ (H is hitting set)
   - $\sum_{\ell_i \in H} w_i$ minimal among H with this property.
2. Run decision solver on F with assumptions $\{\ell_j = 0 \mid \ell_j \notin H\}$
3. If decision solver found solution, it must be optimal (since hitting set is optimal), so return solution with value $\sum_{\ell_i \in H} w_i$
4. Otherwise, decision solver returns new core C_{s+1} — add it to $\mathcal{K}$ and start over

More About the Hitting Sets

- Minimality is actually not needed except in the very final step
- Save time by computing "decent" hitting sets earlier on in the search (good enough to improve currently best solution)
- How to find hitting set?
- This is itself a pseudo-Boolean optimization problem
    - Run integer linear programming (ILP) solver [standard approach]
    - Or pseudo-Boolean solver?
    - Or local search?!

Combine IHS with Pseudo-Boolean Optimization?

IHS and PB Optimization

- In pseudo-Boolean setting, cores will not be subsets of clauses but PB constraints $C_1, \ldots, C_s$ over objective function literals
- "Hitting set" H is partial assignment guaranteed to satisfy all constraints $C_1, \ldots, C_s$
- Want to find minimum-cost set H of literals (w.r.t. objective function) with this property

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- Want to find minimum-cost set H of literals (w.r.t. objective function) with this property

- Explored by CoReO group in Helsinki in [SBJ21, SBJ22]
- Using RoundingSat version in [DGN21] as pseudo-Boolean decision solver

IHS Algorithm for PB Optimization (Simplified)

- Minimize $\sum_{i=1}^{n} w_i \ell_i$
- Subject to collection of PB constraints $F = C_1 \wedge \cdots \wedge C_m$

Set $\mathcal{K} = \emptyset$ and repeat the following:

1. Run optimization solver to minimize $\sum_{i=1}^{n} w_i \ell_i$ under $\mathcal{K}$, yielding solution ρ to objective variables
2. Run decision solver with assumptions ρ on decision problem F
3. If decision solver returns `SATISFIABLE`, we have found optimal solution extending ρ with value $\sum_{i=1}^{n} w_i \cdot \rho(\ell_i)$
4. Otherwise, decision solver returns new core C — add it to $\mathcal{K}$ and start over from top

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$$\min x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6$$

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2. For

$$\mathcal{K}_1 = \emptyset$$

optimization solver returns minimal solution $\rho_1 = \{x_1 = x_2 = \ldots = x_6 = 0\}$

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2. For

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optimization solver returns minimal solution $\rho_1 = \{x_1 = x_2 = \ldots = x_6 = 0\}$

3. Decision solver run with assumptions ρ_1 returns PB core constraint

$$3x_2 + 2x_3 + x_4 + x_5 \geq 4$$

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$$\min x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6$$

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$$\mathcal{K}_1 = \emptyset$$

   optimization solver returns minimal solution $\rho_1 = \{x_1 = x_2 = \ldots = x_6 = 0\}$

3. Decision solver run with assumptions ρ_1 returns PB core constraint

$$3x_2 + 2x_3 + x_4 + x_5 \geq 4$$

4. For

$$\mathcal{K}_2 = \{3x_2 + 2x_3 + x_4 + x_5 \geq 4\}$$

   optimization solver returns minimal solution
   $\rho_2 = \{x_2 = x_3 = 1; x_1 = x_4 = x_5 = x_6 = 0\}$

IHS Toy Example (2/2)

5. Decision solver run with assumptions $\rho'_2 = \{x_1 = x_4 = x_5 = x_6 = 0\}$ returns PB core constraint

$$x_2 + x_4 + x_5 + x_6 \geq 2$$

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8. Hence, we have found an optimal solution with objective value 6 (as for solution-improving search and core-guided search)

Comparison of Core-Guided Search and Implicit Hitting Set

Suppose solver with assumptions returns core

$$C \doteq x_1 + x_2 + x_3 + x_4 \geq 2$$

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Core-guided search

- Introduce new variables by $x_1 + x_2 + x_3 + x_4 = 2 + y_3 + y_4$
- Ignore all x_i with smallest weight in next call (cancelled when objective rewritten)
- Instead assume that "somehow $x_1 + x_2 + x_3 + x_4 \leq 2$ holds" (i.e., assume $y_3 = 0$)
- Lower bound estimate in rewritten objective from LP relaxation of cores [Kat23]

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Implicit Hitting Set

- Add C to collection of cores $\mathcal{K}$
- Find concrete assignment satisfying all of $\mathcal{K}$ as cheaply as possible
- Get lower bound estimate from actual 0–1 solution for cores, not from LP relaxation
- Try that candidate solution as starting point for next call to decision solver

Competitive Advantages of Core-Guided vs. Implicit Hitting Set

- IHS and core-guided approaches for MaxSAT seem orthogonal [Bac21]
- For MaxSAT problems with many interchangeable soft clauses core-guided often better (i.e., when it is not important exactly which of these clauses picked up by the core constraints)
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Theoretical relations between IHS and core-guided search?

Provide a more precise theoretical comparison of IHS and core-guided search with simulations and/or separations

(Some theoretical work on related problems in, e.g., [FMSV20, MIB$^+$19])

Abstract Cores for MaxSAT [BBP20]

Combination of implicit hitting set and core-guided search:

- Run solver with assumption to collect cores $C \doteq \ell_1 + \ell_2 + \cdots + \ell_k \geq 1$

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- How could/should this be generalized to pseudo-Boolean setting?

Challenges and Opportunities for Pseudo-Boolean Optimization (PBO)

1. **Make pseudo-Boolean optimization better known**
    - Personal observation: Most MaxSAT applications seem to be naturally described as pseudo-Boolean optimization problem + PB-to-CNF translation
    - Why not use pseudo-Boolean optimization directly?!

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   - Attempt in [IBJ24], but fixed concrete approach after algorithm instantiation
4. **Leverage ideas from mixed integer programming (MIP)**
   - *Next and final lecture. . .*

Summing up

- MaxSAT problems can be attacked with combination of powerful tools
  - Core-guided solving
  - Implicit hitting set (IHS) solving
  - Integer linear programming
- Approaches with complementary strengths — room for exploiting synergies?
- Lifting core-guided and IHS algorithms to pseudo-Boolean setting presents opportunities and challenges
  - No need for CNF re-encoding
  - More powerful pseudo-Boolean reasoning
  - But also slower than clausal reasoning
  - And more degrees of freedom in algorithm design — more choices needed to get right
- Many interesting questions to explore — and rich pickings of low-hanging fruit?

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Thank you for your attention!

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