Certified MaxSAT Preprocessing

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Our work on one slide

- Success-story of SAT solving:
  - Solvers are fast
  - **And they produce proofs**
- Certifying SAT-based optimization has remained a challenge
  - Many proposals, e.g., [BLM07, LNOR11, MM11, MIB+19, FMSV20, PCH20, PCH21]
  - Proof logging for state-of-the-art MaxSAT only very recently [VDB22, BBN+23, BBN+24]
  - And only for main solver algorithm after preprocessing
- **Contribution of this work:**
  - Proof logging for standalone MaxSAT preprocessor
  - Proofs for equioptimality (and equisatisfiability) with VeriPB
  - Formally verified checker CakePB for the proofs

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- Proof logging for standalone MaxSAT preprocessor MaxPre
  - 15+ different preprocessing techniques certified with VeriPB
- Updated VeriPB proof format
  - Support for equioptimality (and equisatisfiability) proofs
- Formal verification with CakePB
  - HOL4 proof assistant
  - CakeML tools
- Experimental evaluation

Bounded Variable Elimination

Unit Propagation

Structure-Based Labeling

Subsumed Label Elimination

Self-Subsuming Resolution

Hidden Literal Elimination

Intrinsic At-most-ones

Subsumed Literal Elimination

TrimMaxSAT

Failed Literals

Equivalent Literal Detection

Binary Core Removal

Label Matching

Hardening

Subsumption Elimination

Blocked Clause Elimination

Hidden Tautology Elimination

Variable Instantiation

Bounded Variable Addition

Backbones

Generalized Subsumed Label Elimination

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Traditional view of maximum satisfiability (MaxSAT)

Weighted CNF (WCNF) representation

- Hard clauses
- Weighted soft clauses
  - $\langle(\bar{x}_2 \vee \bar{x}_3), 2\rangle$: "incur cost 2 if $(\bar{x}_2 \vee \bar{x}_3)$ is falsified"

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Example of WCNF instance

$\mathcal{F} = (F_H, F_S)$
$F_H = \{(x_1 \vee \bar{x}_2), (x_2 \vee \bar{x}_3), (x_3 \vee \bar{x}_1)\}$
$F_S = \{\langle(x_1), 1\rangle, \langle(\bar{x}_2 \vee \bar{x}_3), 2\rangle\}$

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Conversion using blocking variables

$\mathcal{F}^b = (F_H^b, F_S^b)$
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Same as satisfying F_H^b while minimizing objective $O = \bar{x}_1 + 2x_4$

Modern objective-centric view of MaxSAT

- Optimization variant of SAT
  - CNF formula F
  - Linear objective function O
- Minimize value of O subject to F,

Example:

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- Applications in:
  - Planning
  - Scheduling
  - Configuration
  - Artificial intelligence
  - Combinatorial problems
  - Verification and security
  - Bioinformatics
  - $\ldots$

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There are three solutions:

$\tau_1 = \{x_1 \to 1, x_2 \to 1, x_3 \to 1, x_4 \to 1\}$

$O(\tau_1) = 2$,

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MaxSAT preprocessing

- Simplify instance before solving
  - Remove clauses
  - Introduce new clauses
  - Change the objective function
- In MaxSAT solvers or by a **standalone preprocessor**

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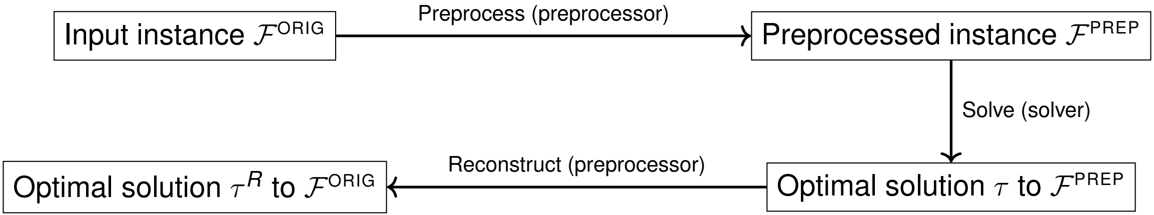

- Certified preprocessing: Verify equioptimality of $\mathcal{F}^{\text{ORIG}}$ and $\mathcal{F}^{\text{PREP}}$

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$$\begin{tikzcd}
\boxed{\text{Input instance } \mathcal{F}^{\mathrm{ORIG}}} \arrow[r, "\text{Preprocess (preprocessor)}"] & \boxed{\text{Preprocessed instance } \mathcal{F}^{\mathrm{PREP}}} \arrow[d, "\text{Solve (solver)}"] \\
\boxed{\text{Optimal solution } \tau^{R} \text{ to } \mathcal{F}^{\mathrm{ORIG}}} & \boxed{\text{Optimal solution } \tau \text{ to } \mathcal{F}^{\mathrm{PREP}}} \arrow[l, "\text{Reconstruct (preprocessor)}"']
\end{tikzcd}$$

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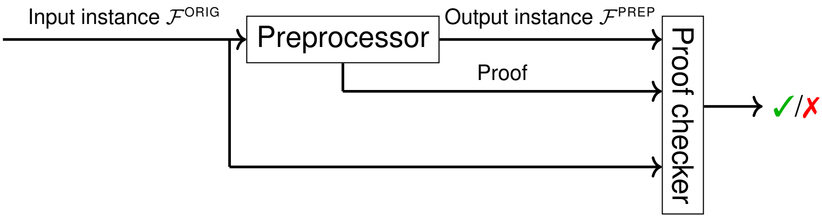
Concrete workflow of MaxPre MaxSAT preprocessor

1. Reading of input
2. Preprocessing on WCNF
3. Conversion to objective-centric
4. Preprocessing on objective-centric
5. Removing constant from the objective function + renaming variables
6. Writing of output

How to verify equioptimality



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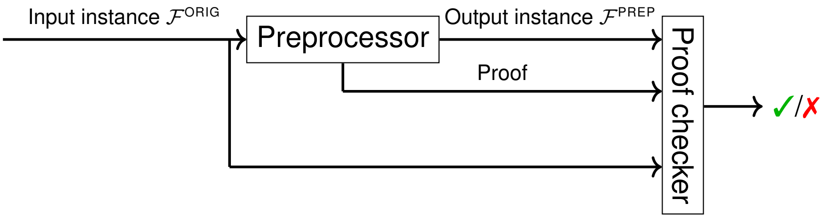

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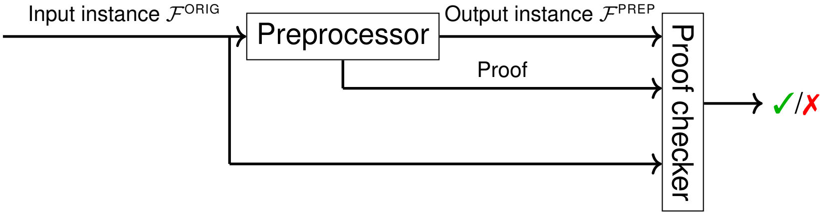

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Which proof system to use for MaxSAT?

- MaxSAT: Formula in CNF
  - Use SAT proof logging techniques (DRAT)?
    - We also have the objective function
  - Extend SAT proof systems to MaxSAT?
    - Difficult for actual solvers to produce proofs
- Objective-function essentially pseudo-Boolean
  - Use pseudo-Boolean proof system
  - PB not too far from MaxSAT
- VeriPB simple but powerful, recent successes:
  - Advanced SAT techniques [GN21, GMNO22, BGMN23]
  - **MaxSAT solving** [VDB22, BBN+23, BBN+24]
  - Constraint programming [EGMN20, GMN22, MM23, MMN24]
  - Subgraph solving [GMM+20, GMN20, GMM+24]
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VeriPB proof system [BGMN23, GN21]

Cutting planes method [CCT87]

Addition

$$\frac{3x_1 + 2\bar{x}_2 + x_3 \geq 3 \qquad x_3 + \bar{x}_4 \geq 1}{3x_1 + 2\bar{x}_2 + 2x_3 + \bar{x}_4 \geq 4}$$

Divide (here by 2)

$$\frac{3x_1 + 2\bar{x}_2 + x_3 \geq 3}{\left\lceil \frac{3}{2} \right\rceil x_1 + \bar{x}_2 + \left\lceil \frac{1}{2} \right\rceil x_3 \geq \left\lceil \frac{3}{2} \right\rceil}$$

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$$F = \{x + y \geq 1, x + \bar{y} \geq 1\}$$

Introduce $x \geq 1$,
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Redundance-based strengthening

$F = \{\bar{x}_1 + \bar{x}_2 \geq 1\}$

Introduce $x_1 + x_2 + x_3 \geq 2$
witness $\omega = \{x_1 \to \bar{x}_2, x_3 \to 1\}$

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Update the objective function

If we can prove $O^{\text{OLD}} = O^{\text{NEW}}$

Practical example:

Input instance (MaxSAT):

$(x_1 \vee x_2 \vee x_3)$
$(\bar{x}_2 \vee \bar{x}_4 \vee \bar{x}_5)$
$(x_2 \vee x_4)$
$(x_3 \vee \bar{x}_5)$
$(x_1 \vee \bar{x}_5)$
$(x_3 \vee \bar{x}_4)$

$O = x_1 + 2x_3 + \bar{x}_5$

In proof (pseudo-Boolean optimization):

$x_1 + x_2 + x_3 \geq 1$
$\bar{x}_2 + \bar{x}_4 + \bar{x}_5 \geq 1$
$x_2 + x_4 \geq 1$
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Practical example: Autarky detection / Subsumed literal elimination

- Consider the MaxSAT instance:
  - $F = \{(x_1 \lor x_2 \lor x_3)^1, (\bar{x}_2 \lor \bar{x}_4 \lor \bar{x}_5)^2, (x_2 \lor x_4)^3, (x_3 \lor \bar{x}_5)^4, (x_1 \lor \bar{x}_5)^5, (x_3 \lor \bar{x}_4)^6\}$
  - $O = x_1 + 2x_3 + \bar{x}_5$
- Consider literals x_2 and $\bar{x}_4$
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  - $F = \{(x_1 \lor x_2 \lor x_3)^1, (\bar{x}_2 \lor \bar{x}_4 \lor \bar{x}_5)^2, (x_2 \lor x_4)^3, (x_3 \lor \bar{x}_5)^4, (x_1 \lor \bar{x}_5)^5, (x_3 \lor \bar{x}_4)^6\}$
  - $O = x_1 + 2x_3 + \bar{x}_5$
- Consider literals x_2 and $\bar{x}_4$
- Remove all clauses containing x_2 or x_4 (by fixing $x_2 = 1$, $x_4 = 0$)

Proof

- Introduce $x_2 \geq 1, \omega = \{x_4 \to 0, x_2 \to 1\}$
- Introduce $\bar{x}_4 \geq 1, \omega = \{x_4 \to 0, x_2 \to 1\}$
- Delete clauses where x_4 or x_2 appear (RUP)
- Delete $x_2 \geq 1, \omega = \{x_2 \to 1\}$
- Delete $\bar{x}_4 \geq 1, \omega = \{x_4 \to 0\}$

```
red +1 x2 >= 1 ; x4 -> 0 x2 -> 1
core id 7
red +1 ~x4 >= 1 ; x4 -> 0 x2 -> 1
core id 8
```

Practical example: Autarky detection / Subsumed literal elimination

- Consider the MaxSAT instance:
  - $F = \{ \quad (x_3 \vee \bar{x}_5)^4, (x_1 \vee \bar{x}_5)^5 \quad \}$
  - $O = x_1 + 2x_3 + \bar{x}_5$
- Consider literals x_2 and $\bar{x}_4$
- Remove all clauses containing x_2 or x_4 (by fixing $x_2 = 1$, $x_4 = 0$)

Proof

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- Delete $x_2 \geq 1, \omega = \{x_2 \to 1\}$
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```
red +1 x2 >= 1 ; x4 -> 0 x2 -> 1
core id 7
red +1 ~x4 >= 1 ; x4 -> 0 x2 -> 1
core id 8


del id 1
del id 2
del id 3
del id 6
```

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red +1 ~x4 >= 1 ; x4 -> 0 x2 -> 1
core id 8


del id 1
del id 2
del id 3
del id 6


del id 7 ; x2 -> 1
del id 8 ; x4 -> 0
```

Practical example: Hardening

- Continue with the formula
  - $F = \{(x_3 \vee \bar{x}_5)^4, (x_1 \vee \bar{x}_5)^5\}$
  - $O = x_1 + 2x_3 + \bar{x}_5$
- There is a solution $\tau = \{x_1 \to 0, x_3 \to 0, x_5 \to 0\}$, $O(\tau) = 1$
  - Definitely no optimal solution sets $x_3 = 1$
  - We can fix $x_3 = 0$

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Proof

- Introduce $\bar{x}_3 \geq 1$, $\omega = \{x_1 \to 0, x_3 \to 0, x_5 \to 0\}$
- Remove x_3 from the objective function
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  - Delete $(x_3 \lor \bar{x}_5)$
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```
red +1 ~x3 >= 1 ; x1 -> 0 x3 -> 0 x5 -> 0
core id 9
```

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```
red +1 ~x3 >= 1 ; x1 -> 0 x3 -> 0 x5 -> 0
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obju diff -2 x3
```

Practical example: Hardening

- Continue with the formula
  - $F = \{(x_3 \lor \bar{x}_5)^4, (x_1 \lor \bar{x}_5)^5, (\bar{x}_5)^{10}\}$
  - $O = x_1 \qquad\qquad + \bar{x}_5$
- There is a solution $\tau = \{x_1 \to 0, x_3 \to 0, x_5 \to 0\}$, $O(\tau) = 1$
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```
red +1 ~x3 >= 1 ; x1 -> 0 x3 -> 0 x5 -> 0
core id 9

obju diff -2 x3

rup 1 ~x5 >= 1 ;
core id 10
del id 4
```

Practical example: Hardening

- Continue with the formula
  - $F = \{ \quad (x_1 \vee \bar{x}_5)^5, (\bar{x}_5)^{10}\}$
  - $O = x_1 \quad + \bar{x}_5$
- There is a solution $\tau = \{x_1 \to 0, x_3 \to 0, x_5 \to 0\}$, $O(\tau) = 1$
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```

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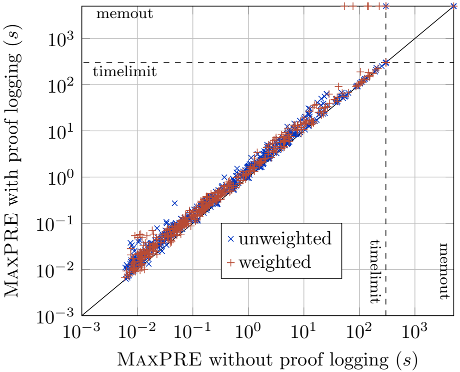
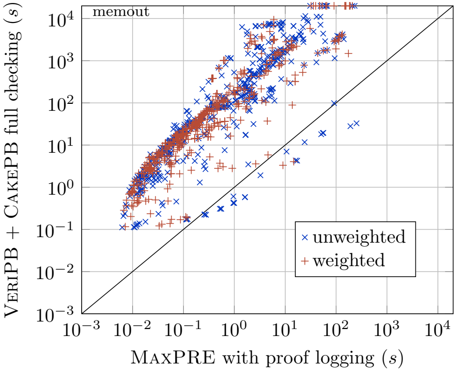
Overhead of proof logging



Overhead of proof logging 46% (geometric mean)

Overhead of formally verified proof checking



Formally verified checking 113 times slower than preprocessing (geometric mean)

Discussion of performance

Putting things in perspective

- Preprocessing only small part of solving time
- So even pretty bad overhead can be negligible in the grand scheme of things
- Still, it is an important challenge to make proof checking faster

Potential for improvements

- Lots of software engineering-level things to improve
- Faster unit propagation
- Binary proof format
- But this is not the full story

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Difference between SAT and MaxSAT preprocessing?

SAT preprocessing techniques

- Are fast for the preprocessor
- But generate lots of proofs, which take time to write
- These proofs then take lots of time to check

MaxSAT preprocessing techniques

- Generate proofs at much slower rate
- And so cause less overhead for proof checking

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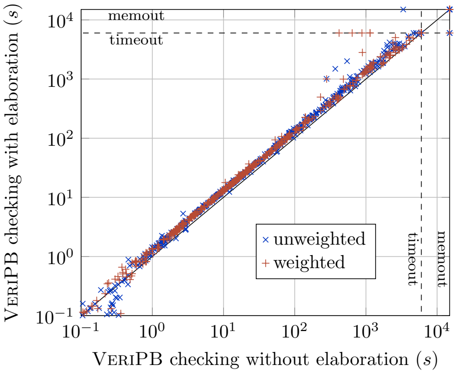
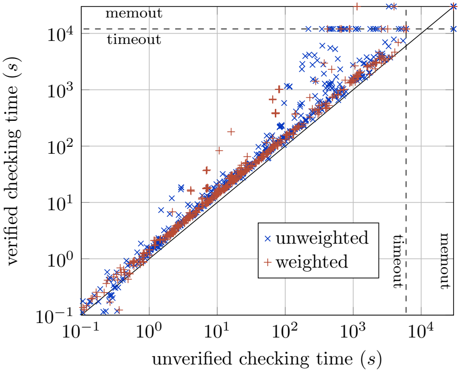
Taking a closer look at the proof checking overhead...





Formal verification is not the bottleneck

- Elaborating proof to formally verified format causes no major slowdown for VeriPB
- Nor is CakePB verified checking massively slower than VeriPB unverified checking

So why is proof checking for MaxSAT preprocessing slow?

Unverified proof checking with VeriPB seems like the main bottleneck

- Again, lots of engineering could (and should) be done here
- Focus has been on expanding the reach of proof logging to completely new domains

Checked deletion steps take **a lot** of time

- Perhaps not entirely unreasonable
- Equioptimality and equisatisfiability are strong guarantees that don't come for free

But the final variable renaming at the end can take half of the total time!?

- Using redundance & checked deletion for this is a very heavy hammer
- Maybe have dedicated rule stating the obvious that variable names don't matter?
- Related earlier observation for objective update: Stating not the new objective O^{NEW} but the **difference** $O^{\mathrm{NEW}} - O^{\mathrm{OLD}}$ speeds up proof logging dramatically

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Conclusion

Our contribution:

- VeriPB proof logging for standalone MaxSAT preprocessor
  - 15+ preprocessing techniques implemented in MaxPre
- Proofs of equioptimality
  - First practical tool for even verifying equisatisfiability
- Formally verified end-to-end proof checking with CakePB

Future work:

- Implement more features
  - solution reconstruction
  - proof trimming
  - proof composition
- Optimize proof checker
- Provide efficient proof logging to even more combinatorial optimization paradigms!

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Thank you for your attention!

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