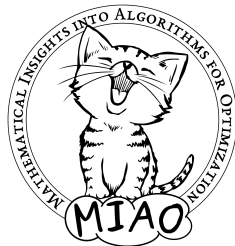


Certifying Combinatorial Solving Using Cutting Planes with Strengthening Rules

Jakob Nordström

University of Copenhagen and Lund University

Oberwolfach Workshop 2413
“Proof Complexity and Beyond”
March 29, 2024



Based on joint work with Jeremias Berg, Bart Bogaerts, Jan Elffers, Ambros Gleixner, Stephan Gocht, Alexander Hoen, Hannes Ihalainen, Matti Järvisalo, Ciaran McCreesh, Matthew McIlree, Magnus O. Myreen, Andy Oertel, Yong Kiam Tan, and Dieter Vandesande

In a Galaxy Far, Far Away from Oberwolfach. . .

- Astounding progress last couple of decades on **combinatorial solvers** for, e.g.:
 - Boolean satisfiability (SAT) solving and optimization [BHvMW21]
 - Constraint programming [RvBW06]
 - Mixed integer linear programming [AW13, BR07]
 - Satisfiability modulo theories (SMT) solving [BHvMW21]
- Solvers very fast, but **sometimes wrong** (even best commercial ones) [BLB10, CKSW13, AGJ⁺18, GSD19, GS19, BMN22, BBN⁺23]
- Even get feasibility of solutions wrong (though this should be straightforward!)
- And how to check the absence of solutions?
- Or that a solution is optimal? (Even off-by-one mistakes can snowball into large errors if solver used as subroutine)

What Can Be Done About Solver Bugs?

- **Software testing**

Hard to get good test coverage for sophisticated solvers

Progress using fuzzing and delta debugging [BB09, BLB10, KB22, NPB22, PB23]

But inherently can only detect presence of bugs, not absence

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Prove that solver implementation adheres to formal specification

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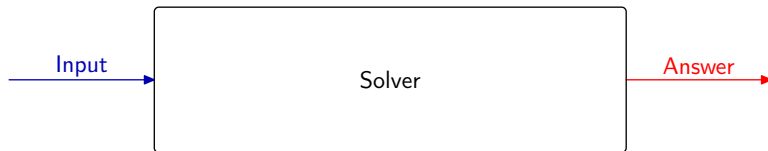
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- **Proof logging**

Make solver **certifying** [ABM⁺11, MMNS11] by adding code so that it outputs

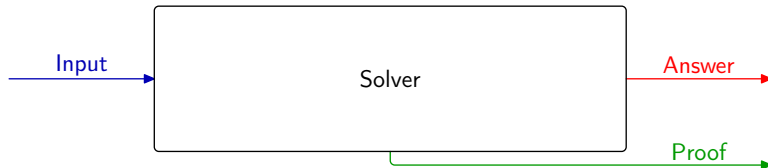
- ① not only **answer** but also
- ② simple, machine-verifiable **proof** that answer is correct

Proof Logging with Certifying Solvers: Workflow



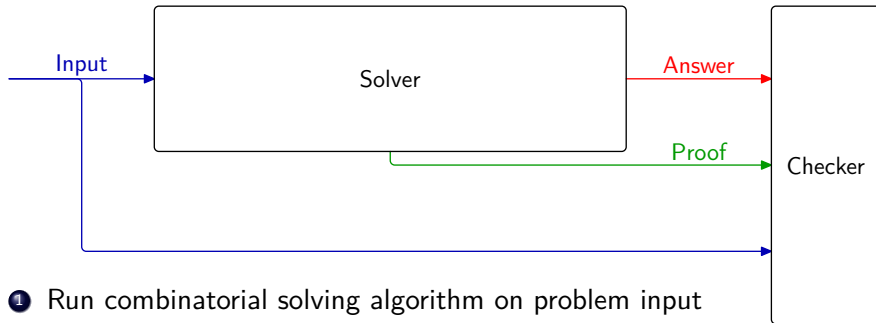
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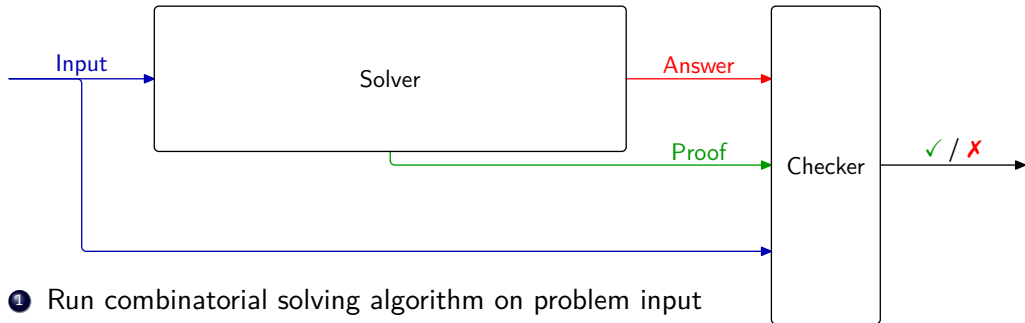
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- ③ Feed input + answer + proof to proof checker

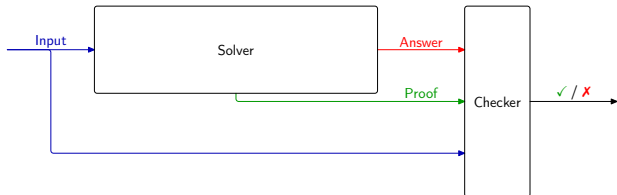
Proof Logging with Certifying Solvers: Workflow



- ① Run combinatorial solving algorithm on problem input
- ② Get as output not only answer but also proof
- ③ Feed input + answer + proof to proof checker
- ④ Verify that proof checker says answer is correct

Proof Logging Desiderata

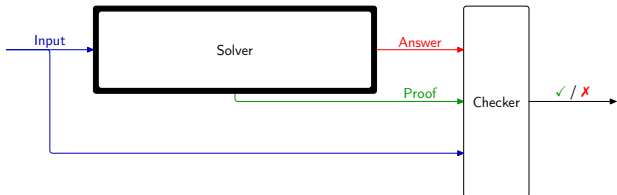
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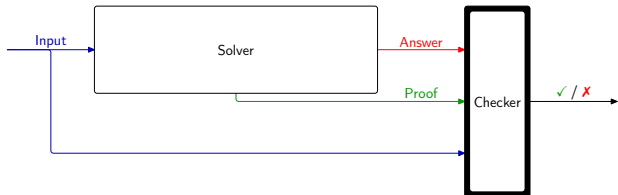
- **very powerful:** minimal overhead for sophisticated reasoning



Proof Logging Desiderata

Proof format for certifying solver should be

- **very powerful:** minimal overhead for sophisticated reasoning
- **dead simple:** checking correctness of proofs should be (almost) trivial

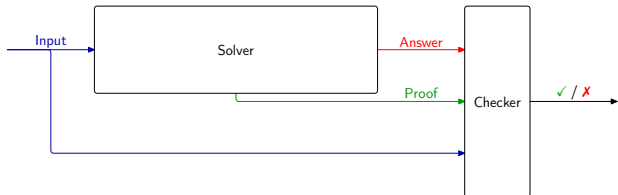


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Clear conflict expressivity vs. simplicity!



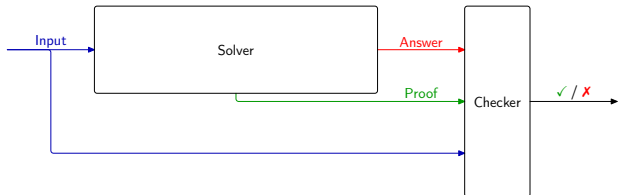
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Asking for both perhaps a little bit too good to be true?



This Talk

Proof logging for combinatorial optimization is possible with **single, unified method!**

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- Build on successes in SAT solving with proof formats such as DRAT [HHW13a, HHW13b, WHH14], GRIT [CMS17], LRAT [CHH⁺17], ...
- But represent constraints as **0–1 integer linear inequalities**
- Formalize reasoning using **cutting planes** [CCT87] proof system
- Add well-chosen **strengthening rules** [Goc22, GN21, BGMN23]
- Implemented in **VERiPB** (<https://gitlab.com/MIA0research/software/VeriPB>)

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- ① Marketing pitch 😊
- ② Highlight some interesting related questions in proof complexity

The Sales Pitch For Proof Logging

- ① Certifies correctness of computed results
- ② Detects errors even if due to compiler bugs, hardware failures, or cosmic rays
- ③ Provides debugging support during software development
[EG21, GMM⁺20, KM21, BBN⁺23]
- ④ Facilitates performance analysis
- ⑤ Helps identify potential for further improvements
- ⑥ Enables auditability
- ⑦ Serves as stepping stone towards explainability

Proof Language: Pseudo-Boolean Constraints

Proof consists of **0-1 integer linear inequalities** or **pseudo-Boolean constraints**:

$$\sum_i a_i \ell_i \geq A$$

- $a_i, A \in \mathbb{Z}$
- **literals** ℓ_i : x_i or \bar{x}_i (where $x_i + \bar{x}_i = 1$)
- variables x_i take values **0 = false** or **1 = true**

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Sometimes convenient to use **normalized form** [Bar95] with **all a_i, A positive** (without loss of generality)

Some Types of Pseudo-Boolean Constraints

① Clauses

$$x \vee \bar{y} \vee z \quad \Leftrightarrow \quad x + \bar{y} + z \geq 1$$

② Cardinality constraints

$$x_1 + x_2 + x_3 + x_4 \geq 2$$

③ General pseudo-Boolean constraints

$$x_1 + 2\bar{x}_2 + 3x_3 + 4\bar{x}_4 + 5x_5 \geq 7$$

Pseudo-Boolean Proof Logging Wishlist

Paradigms

- SAT solving
- (linear) pseudo-Boolean solving
- graph solving
- constraint programming
- automated planning
- mixed integer linear programming
- SMT solving

Problem types

- decision / feasibility
- optimization
- multi-objective optimization
- projected model enumeration
- projected model counting
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Supported in VERIPB **presently**, **Real Soon Now™**, or **hopefully in future extensions**

Design Principles for Proof Logging

Proof logging implementation

- Don't change solver
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- Proof logging overhead small constant fraction ($\lesssim 10\%$)
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Proof system

- Keep proof language maximally simple
- Reason about XOR constraints, CP propagators, symmetries, etc within language
- Combine proof logging with formally verified proof checker

Pseudo-Boolean Proof Logging — How and Why?

If problem is (special case of) 0-1 integer linear program

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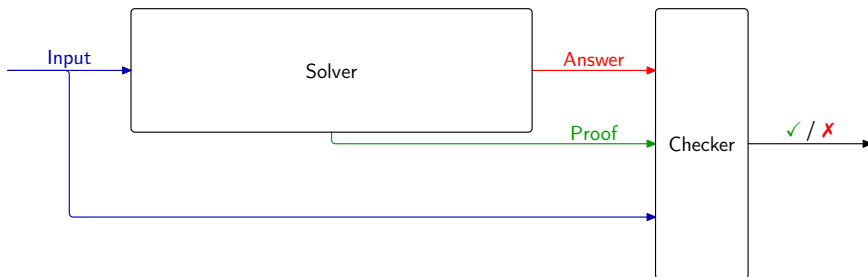
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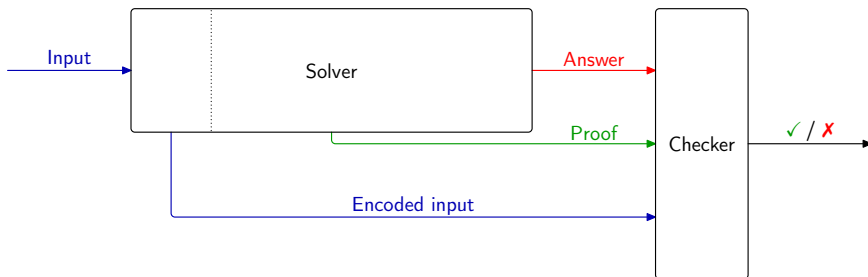
Goldilocks compromise between expressivity and simplicity:

- ① 0-1 ILP **expressive formalism** for combinatorial problems (including objective)
- ② **Powerful reasoning** capturing many combinatorial arguments

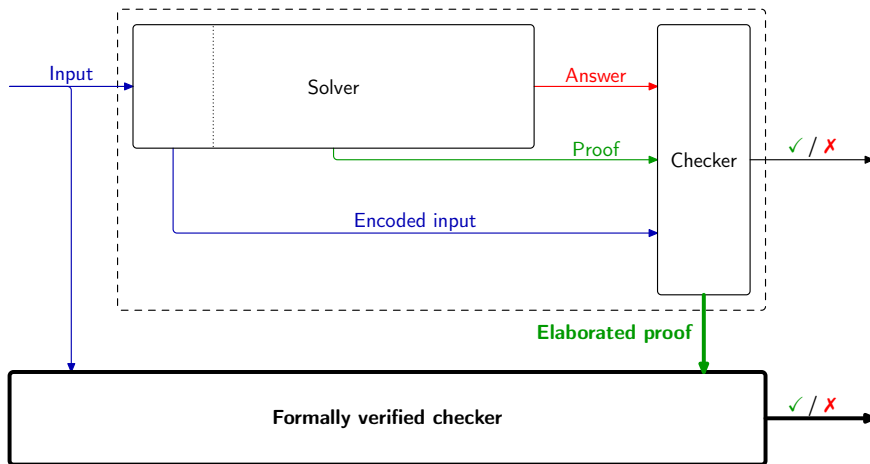
Proof Logging with Formally Verified Checking: Full Workflow



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VERIPB Proof Configuration (Slightly Simplified)

Core set \mathcal{C}

- Contains input formula at the start
- Maintains “equivalence” with input formula

Derived set \mathcal{D}

- All constraints derived during search
- Also intermediate constraints used in proof logging [but not used by solver]

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Order \mathcal{O}

- Pseudo-Boolean formula encoding pre-order (reflexive and transitive)
- Syntactic proof of properties required
- Applied to specified variable set \mathcal{Z}

Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

Input axioms

From the input

Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

Input axioms

Literal axioms

From the input

$$\overline{l_i \geq 0}$$

Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

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Addition

From the input

$$\frac{\overline{\ell_i \geq 0} \quad \sum_i a_i \ell_i \geq A \quad \sum_i b_i \ell_i \geq B}{\sum_i (a_i + b_i) \ell_i \geq A + B}$$

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Multiplication for any $c \in \mathbb{N}^+$

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$$\frac{\sum_i a_i l_i \geq A}{\sum_i c a_i l_i \geq cA}$$

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Multiplication for any $c \in \mathbb{N}^+$

Division for any $c \in \mathbb{N}^+$
 (constraint in normalized form)

From the input

$$\frac{\overline{\ell_i \geq 0} \quad \sum_i a_i \ell_i \geq A \quad \sum_i b_i \ell_i \geq B}{\sum_i (a_i + b_i) \ell_i \geq A + B}$$

$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i c a_i \ell_i \geq c A}$$

$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i \lceil \frac{a_i}{c} \rceil \ell_i \geq \lceil \frac{A}{c} \rceil}$$

Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

Input axioms

Literal axioms

Addition

Multiplication for any $c \in \mathbb{N}^+$

Division for any $c \in \mathbb{N}^+$
(constraint in normalized form)

Saturation
(constraint in normalized form)

From the input

$$\frac{\overline{\ell_i \geq 0} \quad \sum_i a_i \ell_i \geq A \quad \sum_i b_i \ell_i \geq B}{\sum_i (a_i + b_i) \ell_i \geq A + B}$$

$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i c a_i \ell_i \geq c A}$$

$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i \lceil \frac{a_i}{c} \rceil \ell_i \geq \lceil \frac{A}{c} \rceil}$$

$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i \min(a_i, A) \cdot \ell_i \geq A}$$

Cutting Planes Toy Example

$$w + 2x + y \geq 2$$

Cutting Planes Toy Example

Multiply by 2 $\frac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}$

Cutting Planes Toy Example

$$\text{Multiply by 2} \quad \frac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} \quad w + 2x + 4y + 2z \geq 5$$

Cutting Planes Toy Example

$$\begin{array}{rcl}
 \text{Multiply by 2} & \frac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} & \\
 \text{Add} & \frac{2w + 4x + 2y \geq 4 \quad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9} &
 \end{array}$$

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 & & \frac{\bar{z} \geq 0}{2\bar{z} \geq 0} \text{ Multiply by 2}
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 \text{Divide by 3} & \frac{3w + 6x + 6y}{w + 2x + 2y} & \geq 2\frac{1}{3}
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By naming constraints by integers and literal axioms by the literal involved as

$$\text{Constraint 1} \doteq 2x + y + w \geq 2$$

$$\text{Constraint 2} \doteq 2x + 4y + 2z + w \geq 5$$

$$\sim z \doteq \bar{z} \geq 0$$

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such a calculation is written in the proof log in reverse Polish notation as

pol 1 2 * 2 + ~z 2 * + 3 d

Open Problem: Division Versus Saturation

$$\text{Division} \frac{\sum_i a_i \ell_i \geq A}{\sum_i \lceil \frac{a_i}{c} \rceil \ell_i \geq \lceil \frac{A}{c} \rceil}$$

$$\text{Saturation} \frac{\sum_i a_i \ell_i \geq A}{\sum_i \min(a_i, A) \cdot \ell_i \geq A}$$

How do division and saturation rules compare?

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How do division and saturation rules compare?

- Strengths of rules as such incomparable [GNY19]
- Cutting planes with division can be exponentially stronger than cutting planes with saturation
- Unknown whether cutting planes with saturation can be stronger than cutting planes with division

Redundance-Based Strengthening

C is **redundant** with respect to F if F and $F \cup \{C\}$ are **equisatisfiable**

Want to allow adding such “redundant” constraints

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Redundance-based strengthening ([BT19, GN21], inspired by [JHB12])

C is redundant with respect to F if and only if there is a **substitution** ω (mapping variables to truth values or literals), called a **witness**, for which

$$F \cup \{\neg C\} \models (F \cup \{C\}) \upharpoonright_{\omega}$$

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C is redundant with respect to F if and only if there is a **substitution** ω (mapping variables to truth values or literals), called a **witness**, for which

$$F \cup \{\neg C\} \models (F \cup \{C\}) \upharpoonright_{\omega}$$

- Proof sketch for interesting direction: If α satisfies F but falsifies C , then $\alpha \circ \omega$ satisfies $F \cup \{C\}$

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C is **redundant** with respect to F if F and $F \cup \{C\}$ are **equisatisfiable**

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- Proof sketch for interesting direction: If α satisfies F but falsifies C , then $\alpha \circ \omega$ satisfies $F \cup \{C\}$
- In a proof, the implication needs to be **efficiently verifiable** — every $D \in (F \cup \{C\})|_{\omega}$ should follow from $F \cup \{\neg C\}$ either
 - ① “obviously” or
 - ② by explicitly presented derivation

Example: Deriving $a \leftrightarrow (x \wedge y)$ Using the Redundance Rule

Want to derive

$$2\bar{a} + x + y \geq 2 \quad a + \bar{x} + \bar{y} \geq 1$$

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Choose $\omega = \{a \mapsto 1\}$ — F untouched; new constraint satisfied

$\neg(a + \bar{x} + \bar{y} \geq 1)$ forces $x \mapsto 1$ and $y \mapsto 1$, hence $2\bar{a} + x + y \geq 2$ remains satisfied after forcing a to be true

Open Problems: Strength of Restricted Redundance Rules?

Adding redundance rule \Rightarrow proof system polynomially equivalent to extended Frege

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- ① What is the power of the redundance rule if we forbid new variables?
 For resolution + redundance known that:
 - Pigeonhole principle formulas easy
 - Tseitin formulas easy
- ② What is the power of resolution with redundance if we only allow new variables $z \leftrightarrow C$ for previously derived clauses C ?
 - Corresponds (kind of) to reasoning in core-guided MaxSAT solvers

Redundance and Dominance Rules in VERIPB (Slightly Simplified)

Redundance-based strengthening, optimization version [BGMN23]

Add constraint C to derived set \mathcal{D} if exists witness substitution ω such that

$$C \cup \mathcal{D} \cup \{\neg C\} \models (C \cup \mathcal{D} \cup \{C\}) \upharpoonright_{\omega} \cup \{f \upharpoonright_{\omega} \leq f\}$$

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- Applying ω should **strictly decrease** f
- If so, don't need to show that $(\mathcal{D} \cup \{C\})|_{\omega}$ implied!

Soundness of Dominance Rule

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- ⑦ ...
- ⑧ Can't go on forever, so finally reach α' satisfying $\mathcal{C} \cup \{C\}$

Soundness of Dominance Rule (Continued)

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- Same inductive proof as before, but also nested forward induction over derivation
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Further extensions:

- Define dominance rule with respect to order \mathcal{O} independent of objective function
- Switch between different orders in same proof
- See [BGMN23] for details

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Cutting planes with redundance and dominance is at least as strong as extended Frege
Could it be even stronger?!

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Plausibly yes [KT23] — talk by Neil after the break

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Solution: distinguish between deletion from core set \mathcal{C} and derived set \mathcal{D}

Deletion, Core Transfer, and Order Change

Deletion

- ① Deletion of constraint C **always OK** from derived set \mathcal{D}
- ② **OK from core set \mathcal{C} only if C can be rederived from $\mathcal{C} \setminus \{C\}$** with redundance rule
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Change of order

Possible to change order only if $\mathcal{D} = \emptyset$

Parity (XOR) Reasoning in SAT Solving

Given clauses

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Could add XORs to language, but prefer to keep things super-simple

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VERIPB can certify **XOR reasoning** [GN21]

Symmetry Breaking in SAT Solving

- 1 Pretend to **solve optimisation problem** minimizing $f \doteq \sum_{i=1}^n 2^{n-i} \cdot x_i$
(search for lexicographically smallest assignment satisfying formula)

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- ❷ Derive (for proof log only) pseudo-Boolean version of **lex-leader constraint**

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VERIPB can certify fully general **SAT symmetry breaking** [BGMN23]

Open Problem: Symmetry Breaking with Redundance Rule?

Is the dominance rule really needed for fully general symmetry breaking?

Or could the redundance rule be enough?

Weaker DRAT strengthening rule sufficient for “pigeonhole-style” symmetries [HHW15]

Open Problem: Efficient Substitution Proofs?

Can cutting planes with redundance and dominance support **proofs with lemmas/substitution** efficiently?

Special case: **symmetric learning** in SAT solving [DBB17]

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Can cutting planes with redundance and dominance support **proofs with lemmas/substitution** efficiently?

Special case: **symmetric learning** in SAT solving [DBB17]

Seems very finicky. . .

Extension and substitution proof systems don't mix well

Future Research Directions

Performance and reliability of pseudo-Boolean proof logging

- Trim proof while verifying (as in DRAT-TRIM [HHW13a])
- Compress proof file using binary format
- Design formally verified proof checker (*work in progress* [BMM⁺23])

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Proof logging for other combinatorial problems and techniques

- Model counting
- Mixed integer linear programming (*work on SCIP in* [CGS17, EG21, DEGH23])
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And more...

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- Lots of other challenging problems and interesting ideas
- **We're hiring!** Talk to me to join the pseudo-Boolean proof logging revolution! 😊

VERIPB Documentation

VERIPB tutorial at *CP* '22 [BMN22]

- video at youtu.be/s_5BIi4I22w
- updated slides for *IJCAI* '23 tutorial [BMN23]



Description of VERIPB and CAKEPB [BMM⁺23] for SAT 2023 competition

- Available at satcompetition.github.io/2023/checkers.html

Specific details on different proof logging techniques covered in research papers [EGMN20, GMN20, GMM⁺20, GN21, GMN22, GMNO22, VDB22, BBN⁺23, BGMN23, MM23, GMM⁺24, HOGN24, IOT⁺24, MMN24]

Lots of concrete example files at gitlab.com/MIA0research/software/VeriPB

Summing up

- Combinatorial solving and optimization is a true success story
- But ensuring correctness is a crucial, and not yet satisfactorily addressed, concern
- Certifying solvers producing machine-verifiable proofs of correctness seems like most promising approach
- Cutting planes reasoning with pseudo-Boolean constraints seems to hit a sweet spot between simplicity and expressivity
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Thank you for your attention!



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