Graph Colouring Is Hard on Average for Polynomial Calculus and Nullstellensatz

Jakob Nordström

University of Copenhagen and Lund University

IRN CLoVe Workshop on Complexity Theory
University of Copenhagen
January 8, 2025

Joint work with Jonas Conneryd, Susanna de Rezende, Shuo Pang, and Kilian Risse

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Thanks for the slides!

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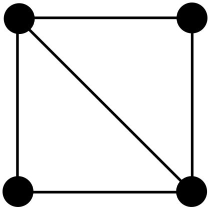
Can vertices of graph G *be coloured with* k *colours so that all neighbours get distinct colours?*

One of Karp's 21 NP-complete problems



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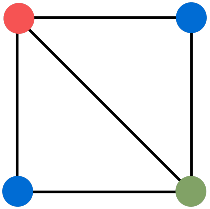

Is Graph Colouring Hard?

Colouring seems hard even to approximate:

- If G k-colourable, best efficient algorithm uses $k \cdot \widetilde{\Omega}(n)$ colours [Halldorsson 93]
- If G 3-colourable, best algorithm uses $n^{0.199\cdots}$ colours [Kawarabayashi–Thorup 17]
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However, applied algorithms appear to do well:

- Backtracking and SAT-based algorithms
  [San Segundo 12, Hebrard–Katsirelos 20, Heule–Karahalios–van Hoeve 22]
- Integer programming
  [Mehortra–Trick 95, Gualandi–Malucelli 12]
- Algebraic algorithms
  [DeLoera–Lee–Malkin–Margulies 08 & 11, DeLoera–Lee–Margulies–Onn 09, DeLoera–Margulies–Pernpeinter–Riedl–Rolnick–Spencer–Stasi–Swenson 15]

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Can we prove that graph colouring is hard for these algorithms?

Hardness for Algebraic Algorithms

- Exponential lower bounds known for explicit graphs
  [Lauria–Nordström 17, Atserias–Ochremiak 19]
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Perhaps graph colouring is *easy on most graphs?*

To rule this out, want average-case hardness results

SAT-based algorithms **[Beame–Culberson–Mitchell–Moore 05]**

Conflict-driven clause learning (CDCL) SAT solvers need exponential time for k-colouring on random graphs for $k \geq 3$

Our Result

Theorem

Algorithms based on Hilbert's Nullstellensatz and/or Gröbner bases require exponential time to solve k-colouring on random graphs for $k \geq 3$

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Established via proof complexity:

- Formalise reasoning method in algorithm as a proof system
- Fast execution for graph G with chromatic number $\chi(G) > k$
  $\Longrightarrow$ short proof of statement "G is not k-colourable"
- Show that such short proofs do not exist

Nullstellensatz Proof System

To show polynomials $p_1, \ldots, p_m$ in $\mathbb{F}[\vec{x}]$ have no common root in $\mathbb{F}$, suffices to find polynomials $q_1, \ldots, q_m$ in $\mathbb{F}[\vec{x}]$ such that

$$\sum_{i=1}^{m} q_i(\vec{x}) \cdot p_i(\vec{x}) = 1$$

This is a Nullstellensatz proof of unsatisfiability

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Soundness: if such polynomials q_i exist, then clearly $\{p_i\}$ have no common root

Completeness (Boolean variables): special case of Hilbert's Nullstellensatz

Polynomial Calculus Proof System [Clegg–Edmonds–Impagliazzo 96]

Dynamic version: given $\{p_1, \ldots, p_m\}$, derive new polynomials using two rules

(linear combination) $\dfrac{p \quad q}{\alpha p + \beta q}$ $\quad \alpha, \beta \in \mathbb{F}$

(multiplication) $\dfrac{p}{x \cdot p}$ $\quad x$ variable

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Polynomial calculus proof system models Gröbner basis computations

- Polynomial = linear combination of monomials
- **Proof size:** # monomials in derivation
  Make proof system stronger by allowing dual variables $\overline{x}_i$ for negative literals
  [Alekhnovich–Ben-Sasson–Razborov–Wigderson 02]
- **Proof degree:** max total degree of polynomial in derivation

Encoding k-Colouring as Polynomials

Variables $x_{v,i}$ = “vertex v gets colour i”, $v \in V(G)$, $i \in [k]$

Axiom polynomials for graph G:

Each vertex gets a colour	$\sum_{i=1}^{k} x_{v,i} - 1$	
Colours are unique	$x_{v,i} \cdot x_{v,i'}$	$i \neq i'$
Distinct colours for neighbours	$x_{u,i} \cdot x_{v,i}$	$(u,v) \in E(G)$
Variables are Boolean	$x_{v,i}^2 - x_{v,i}$	

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Common root of polynomials ⇔ k-colouring of G

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Common root of polynomials $\Leftrightarrow$ k-colouring of G

Other important encoding used in computational algebra [Bayer 82]:

- Colours X_v are kth roots of unity $\{1, \zeta, \zeta^2, \cdots, \zeta^{k-1}\}$ (assuming $\mathrm{char}(\mathbb{F}) \nmid k$)
- Linear substitution from X_v to $x_{v,1}, \ldots, x_{v,k}$ $\Rightarrow$ (roughly) same proof degree

More Formal Statement of Result

Theorem

For G random sparse graph on n vertices, with probability $1 - o(1)$ any polynomial calculus proof of fact “G is not 3-colourable” has size $\exp\left(\Omega(n)\right)$

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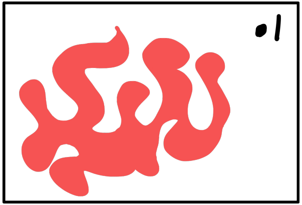
- Lower bound holds over any field
- For both random regular graphs and Erdős–Rényi random graphs (with appropriately chosen parameters)
- Obtained by showing $\Omega(n)$ degree lower bound
- Implies exponential size lower bound for Boolean encoding

[Impagliazzo–Pudlák–Sgall 99]

Degree Lower Bound: Framework

Task: **separate** 1 from {polynomials derivable in degree D}



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1. $R(\text{axiom}) = 0$
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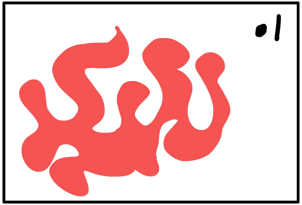

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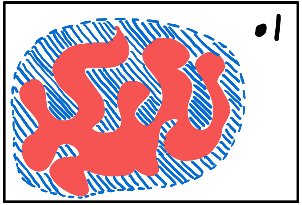
Kernel of R overapproximates what is derivable in degree D



🟥 Derivable in degree D
▨ $\ker(R)$

Quick Recap: Polynomial Ideals

Given set of polynomials $\mathcal{P}$, ideal $\langle \mathcal{P} \rangle$ is smallest set such that

- $\mathcal{P} \subseteq \langle \mathcal{P} \rangle$
- $p, q \in \langle \mathcal{P} \rangle \Rightarrow p + q \in \langle \mathcal{P} \rangle$
- $p \in \langle \mathcal{P} \rangle \Rightarrow r \cdot p \in \langle \mathcal{P} \rangle$ for all polynomials r

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Connection to polynomial calculus:

- $\langle \mathcal{P} \rangle$ contains all polynomial implied by $\mathcal{P}$
- Which is exactly what is derivable by polynomial calculus
- $1 \in \langle \mathcal{P} \rangle \Leftrightarrow \langle \mathcal{P} \rangle =$ all polynomials $\Leftrightarrow \mathcal{P}$ is unsatisfiable

Polynomial Ideal Reductions

- Impose total order on monomials (with 1 smallest)
- Order polynomials by largest monomial (leading monomial)
- **Reduction modulo ideal $\langle \mathcal{P} \rangle$:** Operator $R_{\langle \mathcal{P} \rangle} : \mathbb{F}[\vec{x}] \to \mathbb{F}[\vec{x}]$ defined as

$$R_{\langle \mathcal{P} \rangle}(q) := \text{minimum polynomial in } \{q - r \mid r \in \langle \mathcal{P} \rangle\}$$

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Properties of $R_{\langle \mathcal{P} \rangle}$:

- well-defined
- linear
- $\ker\left(R_{\langle \mathcal{P} \rangle}\right) = \langle \mathcal{P} \rangle$
- $R^2_{\langle \mathcal{P} \rangle} = R_{\langle \mathcal{P} \rangle}$

Example of Polynomial Reduction

Consider $\mathbb{F}[x, y]$ and ideal generated by $\{x + y\}$.

- Order $x \succ y$ extended to all monomials (lexicographically, say)
- $\mathcal{R}_{\langle x+y \rangle}$ maps $x^a y^b$ to $(-1)^a y^{a+b}$

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Alekhnovich-Razborov 03

- For each monomial m, reduce m modulo ideal of subset $S(m)$ of axioms
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Intuition:

- $S(m)$ contains axioms "closely related" to variables in m
- R indistinguishable from polynomial ideal reduction in low degree, but $R(1) \neq 0$
- Think of R as pseudo-reduction modulo fake ideal claiming that $\mathcal{P}$ is satisfiable

From Pseudo-reductions to Degree Lower Bounds

Recall that we want three properties from linear operator R:

1. $R(\text{axiom}) = 0$
2. $R(xp) = R(xR(p))$ for any p of degree $\leq D - 1$
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This would show:

- All input axioms in $\mathcal{P}$ are in $\ker(R)$
- All polynomials derivable from $\mathcal{P}$ in degree $\leq D$ are in $\ker(R)$
- But $1 \notin \ker(R)$
- So degree lower bound $> D$ follows

Getting Pseudo-reductions to Behave Well

- Concretely, for axiom polynomial $p = m_1 + m_2$ want $R(p) = 0$

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$$R(p) = R(m_1) + R(m_2) = R_{\langle S(m_1)\rangle}(m_1) + R_{\langle S(m_2)\rangle}(m_2)$$

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- Dream scenario: Show that there exists ideal $\mathcal{I}$ such that
  - $p \in \mathcal{I}$ for our axiom $p = m_1 + m_2$
  - $S(m_i) \subseteq \mathcal{I}$ for $i = 1, 2$
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- Then

$$R(p) = R_{\langle S(m_1)\rangle}(m_1) + R_{\langle S(m_2)\rangle}(m_2) = R_{\mathcal{I}}(m_1) + R_{\mathcal{I}}(m_2) = R_{\mathcal{I}}(m_1 + m_2) = 0$$

Why Aren’t We Done Already?

- All of this is old news…
  - Proposed in [Alekhnovich–Razborov 03]
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- Crucial new ideas in [Romero-Tunçel 22] — more about that later

Degree Lower Bounds for Colouring

- For colouring, associate to each monomial m a vertex set V_m
- Slightly abuse notation $R_{V_{m_i}}$ to mean reduction modulo ideal generated by axioms “induced graph $G[V_{m_i}]$ is k-colourable”
- Define $R\big(\sum_i c_i m_i\big) := \sum_i c_i R_{V_{m_i}}(m_i)$
- Technical challenge: construct V_m so that R satisfies required properties

Vertex Set V_m

Say that monomial $m = x_{u,2}x_{v,3}x_{w,1}$ mentions vertices u, v, w

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Vertices V_m related to m

- Define closure $\mathrm{Cl}(U) \supseteq U$ of vertex sets U
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Desired properties of closure:

1. **Subset-preserving:** $U' \subseteq \mathrm{Cl}(U) \Rightarrow \mathrm{Cl}(U') \subseteq \mathrm{Cl}(U)$
2. **Size-preserving:** $|U| \leq D \Rightarrow |\mathrm{Cl}(U)| = O(D)$
3. **Reduction-preserving:** For any monomial m mentioning only vertices in $\mathrm{Cl}(U)$ and any vertex set J of size $O(D)$ it holds that

$$R_{\mathrm{Cl}(U)}(m) = R_{\mathrm{Cl}(U) \cup J}(m)$$

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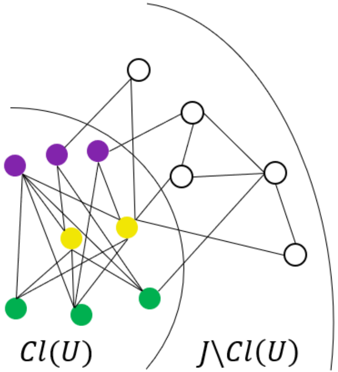

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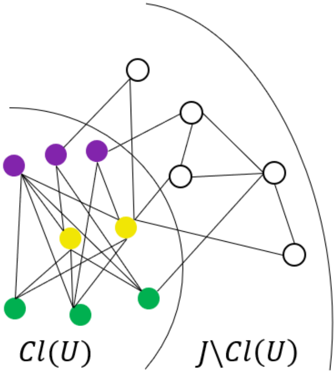
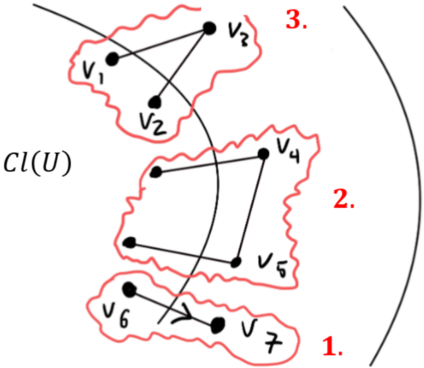
- each colouring of $G[\mathsf{Cl}(U)]$ can be extended to $G[\mathsf{Cl}(U) \cup J]$
- ... in **order-decreasing** way: for each v in $J \setminus \mathsf{Cl}(U)$, colour can be determined based on colouring of $\{w \in \mathsf{Cl}(U) : w < v\}$



Construction of Closure (1/2)

Sufficient to prevent certain structures in neighbourhood outside Cl(U) such as



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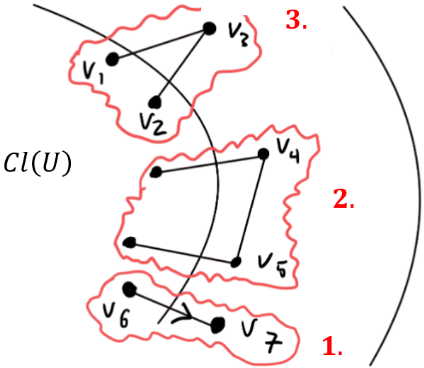

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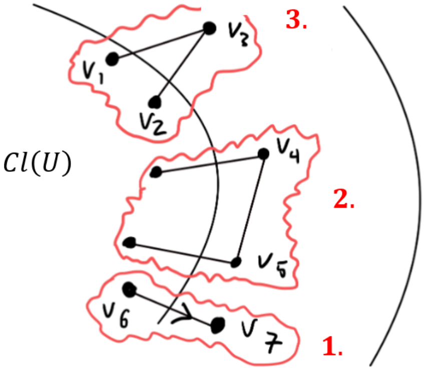
Similar structures identified in [Romero-Tunçel 22] *in colouring lower bound for large-girth graphs!*



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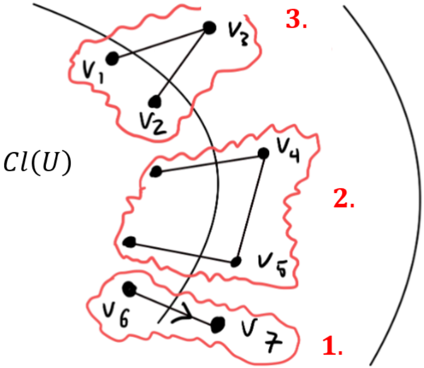

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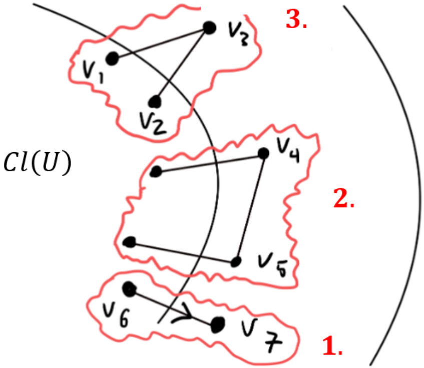

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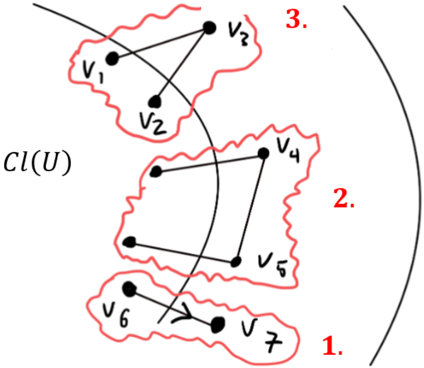

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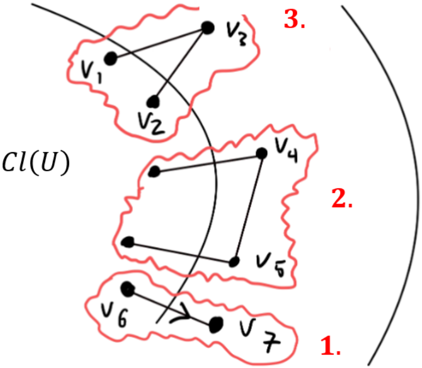

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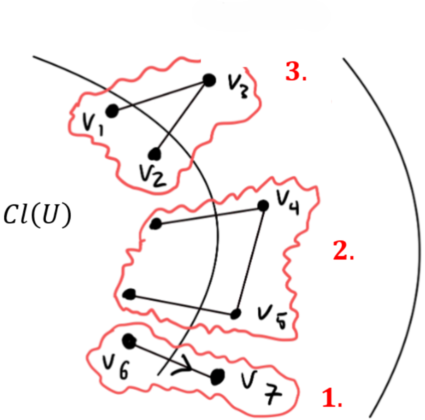

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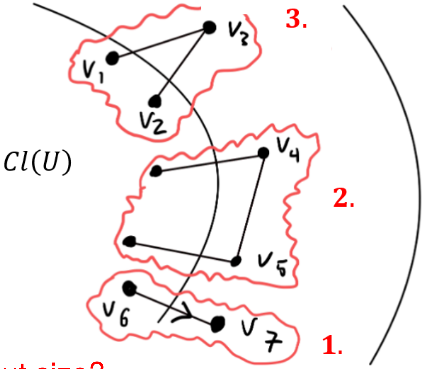

Not hard to show Cl(U) well-defined, but what about size?

Keeping the Closure Small Enough

Size lemma [CdRNPR 23]

For random n-vertex graph with max vertex degree d, it holds for any vertex set U with $|U| \leq 2^{-d^{O(1)}} \cdot n$ that

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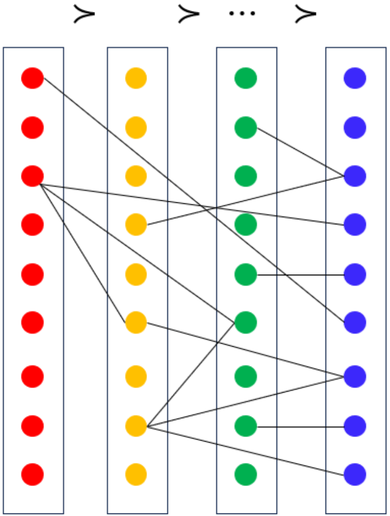
- Proof relies on “good” vertex order introduced by [Romero-Tunçel 22]:

Order vertices according to a valid colouring of G

- Chromatic number of random graph G is $\chi(G) = O(d/\log d) = O(1)$
  $\Rightarrow$ order-decreasing paths have length $O(1)$



Use any vertex order that respects colour classes

Completing the Proof (Sketch) of the Colouring Lower Bound

Size lemma: $|\mathrm{Cl}(U)| = O(|U|)$ for all U of small size

- Intuition: Closure $\mathrm{Cl}(U)$ obtained from sequence of vertex sets $U \subset U_1 \subset U_2 \subset \ldots$ of increasing edge density
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Pseudo-reduction operator properties:

- $R(\text{axiom}) = 0$ since each axiom p mentions vertex set U_p of size ≤ 2 and $R(m) = R_{\mathrm{Cl}(U_p)}(m)$ for each monomial m in p
- $R(xp) = R(xR(p))$ for all p of degree $\leq D - 1$ since closure is size- and reduction-preserving
- $R(1) = 1$ since $\mathrm{Cl}(\emptyset) = \emptyset$ and $R_{\mathrm{Cl}(1)}(\cdot)$ hence does nothing

Some Future Research Directions

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4. Connections between pseudo-reductions and other lower bound operators
    - Designs for Nullstellensatz
    - Pseudo-expectations for sums-of-squares

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Thank you for your attention!