Unconditional Lower Bounds for Algebraic Approaches to Graph Colouring

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Celebration of Complexity
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Based on joint work with Jonas Conneryd, Susanna de Rezende, Shuo Pang, and Kilian Risse

Facets of Complexity

Facets of Johan

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- Postulate P $\neq$ NP or similar assumptions
- Use this to prove (extremely) strong computational infeasibility results
- Obtain web of (highly plausible) interconnected claims

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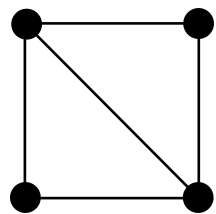
"Real-world" computation results

- Proof complexity can model algorithmic methods of reasoning
- Tools for proving lower (and upper) bounds for concrete, applied algorithms

Graph Colouring

Can vertices of graph G be coloured with k colours so that all neighbours get distinct colours?

One of Karp's 21 NP-complete problems



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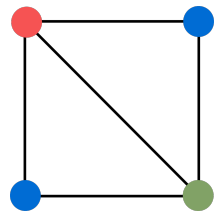

Is Graph Colouring Hard?

Colouring seems hard even to approximate:

- If G k-colourable, best efficient algorithm uses $k \cdot \widetilde{\Omega}(n)$ colours [Halldorsson 93]
- If G 3-colourable, best algorithm uses $n^{0.199\cdots}$ colours [Kawarabayashi–Thorup 17]
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However, applied algorithms appear to do well:

- Backtracking and SAT-based algorithms
  [San Segundo 12, Hebrard–Katsirelos 20, Heule–Karahalios–van Hoeve 22]
- Integer programming
  [Mehortra–Trick 95, Gualandi–Malucelli 12]
- Algebraic algorithms
  [DeLoera–Lee–Malkin–Margulies 08 & 11, DeLoera–Lee–Margulies–Onn 09, DeLoera–Margulies–Pernpeinter–Riedl–Rolnick–Spencer–Stasi–Swenson 15]

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Can we prove unconditionally that graph colouring is hard for these algorithms?

Hardness for Algebraic Algorithms

- Exponential lower bounds known for explicit graphs

  [Lauria–Nordström 17, Atserias–Ochremiak 19]

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Perhaps graph colouring is *easy on most graphs?*

To rule this out, want average-case hardness results

SAT-based algorithms [Beame–Culberson–Mitchell–Moore 05]

Conflict-driven clause learning (CDCL) SAT solvers need exponential time for k-colouring on random graphs for $k \geq 3$

Main Focus of This Presentation

Theorem [Conneryd–deRezende–Nordström–Pang–Risse 23]

Algorithms based on Hilbert's Nullstellensatz and/or Gröbner bases require exponential time to solve k-colouring on random graphs for $k \geq 3$

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Established via proof complexity:

- Formalise reasoning method in algorithm as a proof system
- Fast execution for graph G with chromatic number $\chi(G) > k$
  $\Longrightarrow$ short proof of statement "G is not k-colourable"
- Show that such short proofs do not exist

Nullstellensatz Proof System

To show polynomials $p_1, \ldots, p_m$ in $\mathbb{F}[\vec{x}]$ have no common root in $\mathbb{F}$, suffices to find polynomials $q_1, \ldots, q_m$ in $\mathbb{F}[\vec{x}]$ such that

$$\sum_{i=1}^{m} q_i(\vec{x}) \cdot p_i(\vec{x}) = 1$$

This is a Nullstellensatz proof of unsatisfiability

[Beame–Impagliazzo–Krajíček–Pitassi–Pudlák 96]

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Soundness: if such polynomials q_i exist, then clearly $\{p_i\}$ have no common root

Completeness (Boolean variables): special case of Hilbert's Nullstellensatz

Polynomial Calculus Proof System [Clegg–Edmonds–Impagliazzo 96]

Dynamic version: given $\{p_1, \ldots, p_m\}$, derive new polynomials using two rules

$$\text{(linear combination)} \quad \frac{p \quad q}{\alpha p + \beta q} \qquad \alpha, \beta \in \mathbb{F}$$

$$\text{(multiplication)} \quad \frac{p}{x \cdot p} \qquad x \text{ variable}$$

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Polynomial calculus proof system models Gröbner basis computations

- Polynomial = linear combination of monomials
- **Proof size:** # monomials in derivation
- **Proof degree:** max total degree of polynomial in derivation

Encoding k-Colouring as Polynomials

Variables $x_{v,i}$ = "vertex v gets colour i", $v \in V(G)$, $i \in [k]$

Axiom polynomials for graph G:

Each vertex gets a colour	$\sum_{i=1}^{k} x_{v,i} - 1$	
Colours are unique	$x_{v,i} \cdot x_{v,i'}$	$i < i'$
Distinct colours for neighbours	$x_{u,i} \cdot x_{v,i}$	$(u,v) \in E(G)$
Variables are Boolean	$x_{v,i}^2 - x_{v,i}$	

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Other important encoding used in computational algebra [Bayer 82]:

- Colours X_v are kth roots of unity $\{1, \zeta, \zeta^2, \cdots, \zeta^{k-1}\}$ (assuming $\mathrm{char}(\mathbb{F}) \nmid k$)
- Affine substitution from X_v to $x_{v,1}, \ldots, x_{v,k} \Rightarrow$ (roughly) same proof degree

More Formal Statement of Result

Theorem

For G random sparse graph on n vertices, with probability $1 - o(1)$ any polynomial calculus proof of fact "G is not 3-colourable" has size $\exp(\Omega(n))$

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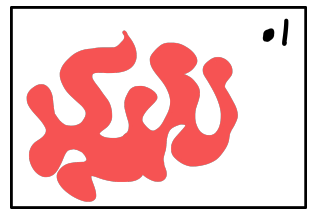
- Lower bound holds over any field
- For both random regular graphs and Erdős–Rényi random graphs (with appropriately chosen parameters)
- Obtained by showing $\Omega(n)$ degree lower bound
- Implies exponential size lower bound for Boolean encoding

[Impagliazzo–Pudlák–Sgall 99]

Degree Lower Bound: Framework

Task: **separate** 1 from $\{$polynomials derivable in degree $D\}$



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[Razborov 98]: suffices to find linear $R : \mathbb{F}[\vec{x}] \to \mathbb{F}[\vec{x}]$ such that

1. $R(\text{axiom}) = 0$
2. $R(xp) = R(xR(p))$ for any p of degree $\leq D - 1$
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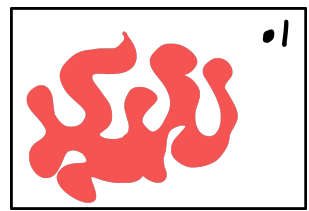

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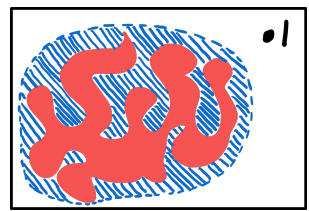
Kernel of R overapproximates what is derivable in degree D



Derivable in degree D

$\ker(R)$

Quick Recap: Polynomial Ideals

Ideal $\langle \mathcal{P} \rangle$ generated by set of polynomials $\mathcal{P}$ is smallest set such that

- $\mathcal{P} \subseteq \langle \mathcal{P} \rangle$
- $p, q \in \langle \mathcal{P} \rangle \Rightarrow p + q \in \langle \mathcal{P} \rangle$
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Connection to polynomial calculus:

- $\langle \mathcal{P} \rangle$ contains all polynomial implied by $\mathcal{P}$
- Which is exactly what is derivable by polynomial calculus
- $1 \in \langle \mathcal{P} \rangle \Leftrightarrow \langle \mathcal{P} \rangle$ = all polynomials $\Leftrightarrow \mathcal{P}$ is unsatisfiable

Polynomial Ideal Reductions

- Impose total order on monomials (with 1 smallest)
- Order polynomials by largest monomial (leading monomial)
- **Reduction modulo ideal $\langle \mathcal{P} \rangle$:** Operator $R_{\langle \mathcal{P} \rangle} : \mathbb{F}[\vec{x}] \to \mathbb{F}[\vec{x}]$ defined as

$$R_{\langle \mathcal{P} \rangle}(q) := \text{minimum polynomial in } \{q - p \mid p \in \langle \mathcal{P} \rangle\}$$

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Properties of $R_{\langle \mathcal{P} \rangle}$:

- well-defined
- linear
- $\ker\left(R_{\langle \mathcal{P} \rangle}\right) = \langle \mathcal{P} \rangle$

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Intuition:

- $S(m)$ contains axioms "closely related" to variables in m
- R indistinguishable from polynomial ideal reduction in low degree, but $R(1) \neq 0$
- Think of R as pseudo-reduction modulo fake ideal claiming that $\mathcal{P}$ is satisfiable

From Pseudo-Reductions to Degree Lower Bounds

Recall that we want three properties from linear operator R:

1. $R(\text{axiom}) = 0$
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This would show:

- All input axioms in $\mathcal{P}$ are in $\ker(R)$
- All polynomials derivable from $\mathcal{P}$ in degree $\leq D$ are in $\ker(R)$
- But $1 \notin \ker(R)$
- So degree lower bound $> D$ follows

Getting Pseudo-Reductions to Behave Well

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- Dream scenario: Show that there exists ideal $\mathcal{I}$ such that
  - $p \in \mathcal{I}$ for our axiom $p = m_1 + m_2$
  - $S(m_i) \subseteq \mathcal{I}$ for $i = 1, 2$
  - $R_{\langle S(m_i)\rangle}(m_i) = R_{\mathcal{I}}(m_i)$ for $i = 1, 2$

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- Then

$$R(p) = R_{\langle S(m_1)\rangle}(m_1) + R_{\langle S(m_2)\rangle}(m_2) = R_{\mathcal{I}}(m_1) + R_{\mathcal{I}}(m_2) = R_{\mathcal{I}}(m_1 + m_2) = 0$$

Why Aren't We Done Already?

- All of this is old news...
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  - Further developed in, e.g., [Galesi–Lauria 10a, 10b; Mikša–Nordström 15]

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  - But partial colouring propagates constraints throughout whole graph!?
- Average-case polynomial calculus lower bounds for colouring open since [Beame–Culberson–Mitchell–Moore 05]

Degree Lower Bounds for Colouring

- For colouring, associate to each monomial m a vertex set V_m
- Let $S(m)$ = axioms "induced graph $G[V_m]$ is k-colourable"
- Slightly abuse notation R_{V_m} to mean reduction modulo ideal generated by $S(m)$
- Define $R\big(\sum_i c_i m_i\big) \coloneqq \sum_i c_i R_{V_{m_i}}(m_i)$
- Technical challenge: construct V_m so that R satisfies required properties

Vertex Set V_m

Say that monomial $m = x_{u,2}x_{v,3}x_{w,1}$ mentions vertices u, v, w

Vertices V_m related to m

- Define closure $\mathrm{Cl}(U) \supseteq U$ of vertex sets U
- Set $V_m := \mathrm{Cl}(\{\text{vertices mentioned in } m\})$

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Desired properties of closure:

1. **Subset-preserving:** $U' \subseteq \mathsf{Cl}(U) \Rightarrow \mathsf{Cl}(U') \subseteq \mathsf{Cl}(U)$
2. **Size-preserving:** $|U| \leq D \Rightarrow |\mathsf{Cl}(U)| = O(D)$
3. **Reduction-preserving:** For any monomial m mentioning only vertices in $\mathsf{Cl}(U)$ and any vertex set J of size $O(D)$ it holds that

$$R_{\mathsf{Cl}(U)}(m) = R_{\mathsf{Cl}(U)\cup J}(m)$$

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Reduction lemma [CdRNPR 23]

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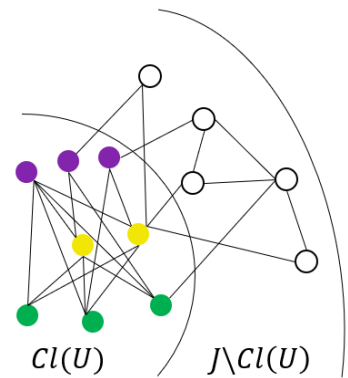


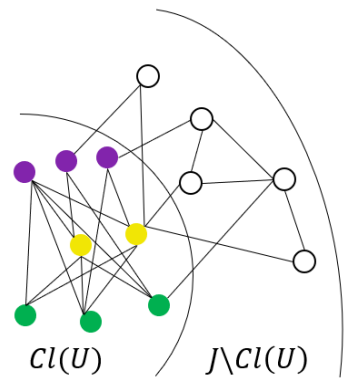
Reduction-Preserving Property of Closure

Reduction-preserving: For any monomial m mentioning only vertices in $\mathsf{Cl}(U)$ and any vertex set J of size $O(D)$ it holds that $R_{\mathsf{Cl}(U)}(m) = R_{\mathsf{Cl}(U)\cup J}(m)$

Reduction lemma [CdRNPR 23]

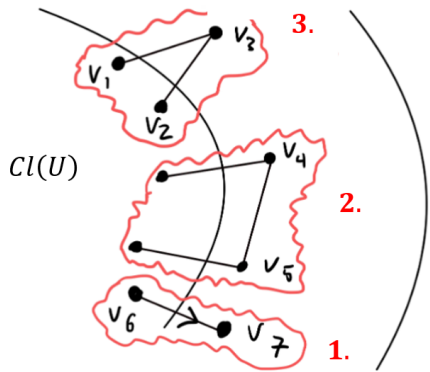
For fixed order on vertices (and variables), can achieve this property if:

- each colouring of $G[\mathsf{Cl}(U)]$ can be extended to $G[\mathsf{Cl}(U) \cup J]$
- ... in **order-decreasing** way: for each v in $J \setminus \mathsf{Cl}(U)$, colour can be determined based on colouring of $\{w \in \mathsf{Cl}(U) : w < v\}$



Construction of Closure (1/2)

Sufficient to prevent certain structures in neighbourhood outside Cl(U) such as



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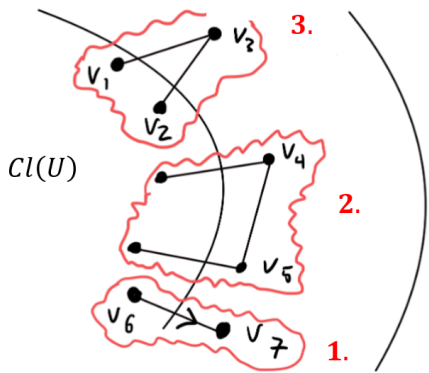

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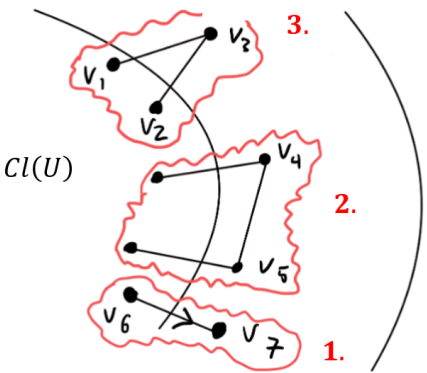
Similar structures identified in [Romero-Tunçel 22] *in colouring lower bound for large-girth graphs!*



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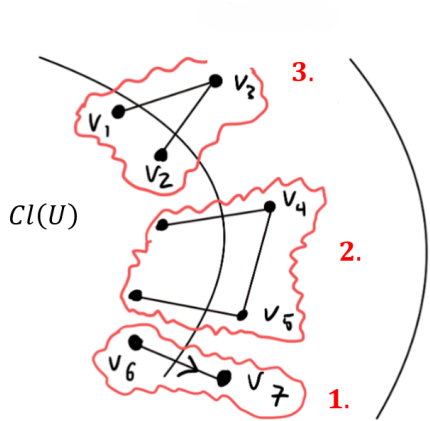

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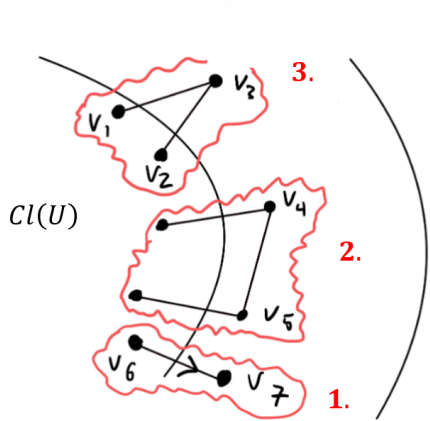

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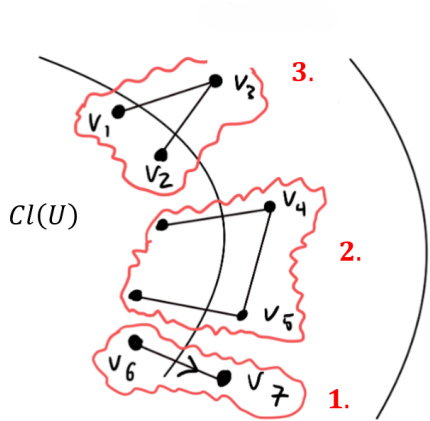

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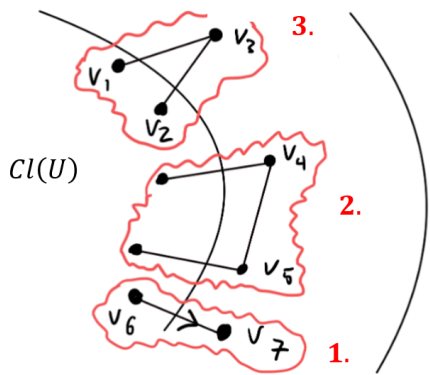

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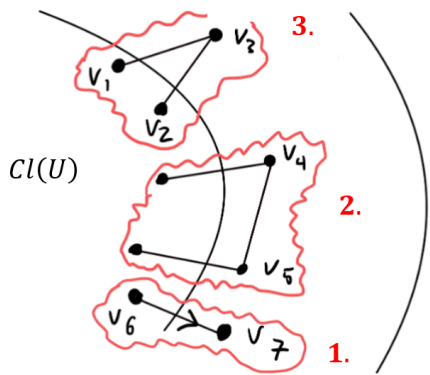

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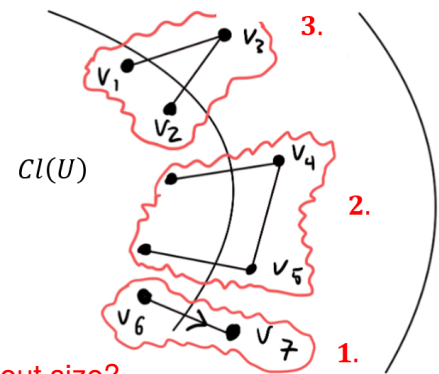

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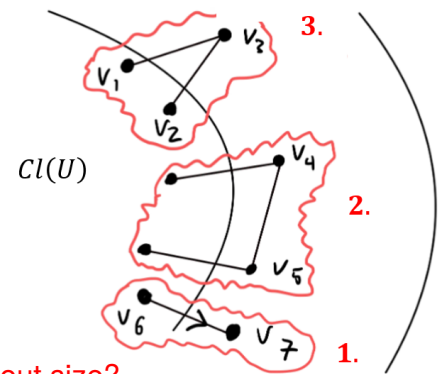

Not hard to show Cl(U) well-defined, but what about size?
OK for random graph if *vertices ordered according to colouring* [Romero-Tunçel 22]

Some Future Research Directions

1. Colouring lower bounds for other proof systems
   - Sherali–Adams (linear programming hierarchies)
   - Sum-of-squares (semidefinite programming hierarchies)
   - Cutting planes (integer linear programming)

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   - Designs for Nullstellensatz
   - Pseudo-expectations for sums-of-squares

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Thank you for your attention! And thank you, Johan!