Conflict-Driven Solving for Boolean Satisfiability and Beyond

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Mathematics and Algorithms for Intelligent Decision-Making Summer Seminar 2026
Linköping University
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This Is Me...

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Academic path:

- PhD at KTH Royal Institute of Technology 2008
- Postdoc at MIT 2008-2010
- Faculty at KTH 2011-2019
- In Copenhagen and Lund since 2019

. . . And This Is What I Do for a Living

$$(x_{1,1} \vee x_{1,2} \vee x_{1,3} \vee x_{1,4} \vee x_{1,5} \vee x_{1,6} \vee x_{1,7}) \wedge (x_{2,1} \vee x_{2,2} \vee x_{2,3} \vee x_{2,4} \vee x_{2,5} \vee x_{2,6} \vee x_{2,7}) \wedge (x_{3,1} \vee x_{3,2} \vee x_{3,3} \vee x_{3,4} \vee x_{3,5} \vee x_{3,6} \vee x_{3,7}) \wedge (x_{4,1} \vee x_{4,2} \vee x_{4,3} \vee x_{4,4} \vee x_{4,5} \vee x_{4,6} \vee x_{4,7}) \wedge (x_{5,1} \vee x_{5,2} \vee x_{5,3} \vee x_{5,4} \vee x_{5,5} \vee x_{5,6} \vee x_{5,7}) \wedge (x_{6,1} \vee x_{6,2} \vee x_{6,3} \vee x_{6,4} \vee x_{6,5} \vee x_{6,6} \vee x_{6,7}) \wedge (x_{7,1} \vee x_{7,2} \vee x_{7,3} \vee x_{7,4} \vee x_{7,5} \vee x_{7,6} \vee x_{7,7}) \wedge (x_{8,1} \vee x_{8,2} \vee x_{8,3} \vee x_{8,4} \vee x_{8,5} \vee x_{8,6} \vee x_{8,7}) \wedge (\neg x_{1,1} \vee \neg x_{2,1}) \wedge (\neg x_{1,1} \vee \neg x_{3,1}) \wedge (\neg x_{1,1} \vee \neg x_{4,1}) \wedge (\neg x_{1,1} \vee \neg x_{5,1}) \wedge (\neg x_{1,1} \vee \neg x_{6,1}) \wedge (\neg x_{1,1} \vee \neg x_{7,1}) \wedge (\neg x_{1,1} \vee \neg x_{8,1}) \wedge (\neg x_{2,1} \vee \neg x_{3,1}) \wedge (\neg x_{2,1} \vee \neg x_{4,1}) \wedge (\neg x_{2,1} \vee \neg x_{5,1}) \wedge (\neg x_{2,1} \vee \neg x_{6,1}) \wedge (\neg x_{2,1} \vee \neg x_{7,1}) \wedge (\neg x_{2,1} \vee \neg x_{8,1}) \wedge (\neg x_{3,1} \vee \neg x_{4,1}) \wedge (\neg x_{3,1} \vee \neg x_{5,1}) \wedge (\neg x_{3,1} \vee \neg x_{6,1}) \wedge (\neg x_{3,1} \vee \neg x_{7,1}) \wedge (\neg x_{3,1} \vee \neg x_{8,1}) \wedge (\neg x_{4,1} \vee \neg x_{5,1}) \wedge (\neg x_{4,1} \vee \neg x_{6,1}) \wedge (\neg x_{4,1} \vee \neg x_{7,1}) \wedge (\neg x_{4,1} \vee \neg x_{8,1}) \wedge (\neg x_{5,1} \vee \neg x_{6,1}) \wedge (\neg x_{5,1} \vee \neg x_{7,1}) \wedge (\neg x_{5,1} \vee \neg x_{8,1}) \wedge (\neg x_{6,1} \vee \neg x_{7,1}) \wedge (\neg x_{6,1} \vee \neg x_{8,1}) \wedge (\neg x_{7,1} \vee \neg x_{8,1}) \wedge (\neg x_{1,2} \vee \neg x_{2,2}) \wedge (\neg x_{1,2} \vee \neg x_{3,2}) \wedge (\neg x_{1,2} \vee \neg x_{4,2}) \wedge (\neg x_{1,2} \vee \neg x_{5,2}) \wedge (\neg x_{1,2} \vee \neg x_{6,2}) \wedge (\neg x_{1,2} \vee \neg x_{7,2}) \wedge (\neg x_{1,2} \vee \neg x_{8,2}) \wedge (\neg x_{2,2} \vee \neg x_{3,2}) \wedge (\neg x_{2,2} \vee \neg x_{4,2}) \wedge (\neg x_{2,2} \vee \neg x_{5,2}) \wedge (\neg x_{2,2} \vee \neg x_{6,2}) \wedge (\neg x_{2,2} \vee \neg x_{7,2}) \wedge (\neg x_{2,2} \vee \neg x_{8,2}) \wedge (\neg x_{3,2} \vee \neg x_{4,2}) \wedge (\neg x_{3,2} \vee \neg x_{5,2}) \wedge (\neg x_{3,2} \vee \neg x_{6,2}) \wedge (\neg x_{3,2} \vee \neg x_{7,2}) \wedge (\neg x_{3,2} \vee \neg x_{8,2}) \wedge (\neg x_{4,2} \vee \neg x_{5,2}) \wedge (\neg x_{4,2} \vee \neg x_{6,2}) \wedge (\neg x_{4,2} \vee \neg x_{7,2}) \wedge (\neg x_{4,2} \vee \neg x_{8,2}) \wedge (\neg x_{5,2} \vee \neg x_{6,2}) \wedge (\neg x_{5,2} \vee \neg x_{7,2}) \wedge (\neg x_{5,2} \vee \neg x_{8,2}) \wedge (\neg x_{6,2} \vee \neg x_{7,2}) \wedge (\neg x_{6,2} \vee \neg x_{8,2}) \wedge (\neg x_{7,2} \vee \neg x_{8,2}) \wedge (\neg x_{1,3} \vee \neg x_{2,3}) \wedge (\neg x_{1,3} \vee \neg x_{3,3}) \wedge (\neg x_{1,3} \vee \neg x_{4,3}) \wedge (\neg x_{1,3} \vee \neg x_{5,3}) \wedge (\neg x_{1,3} \vee \neg x_{6,3}) \wedge (\neg x_{1,3} \vee \neg x_{7,3}) \wedge (\neg x_{1,3} \vee \neg x_{8,3}) \wedge (\neg x_{2,3} \vee \neg x_{3,3}) \wedge (\neg x_{2,3} \vee \neg x_{4,3}) \wedge (\neg x_{2,3} \vee \neg x_{5,3}) \wedge (\neg x_{2,3} \vee \neg x_{6,3}) \wedge (\neg x_{2,3} \vee \neg x_{7,3}) \wedge (\neg x_{2,3} \vee \neg x_{8,3}) \wedge (\neg x_{3,3} \vee \neg x_{4,3}) \wedge (\neg x_{3,3} \vee \neg x_{5,3}) \wedge (\neg x_{3,3} \vee \neg x_{6,3}) \wedge (\neg x_{3,3} \vee \neg x_{7,3}) \wedge (\neg x_{3,3} \vee \neg x_{8,3}) \wedge (\neg x_{4,3} \vee \neg x_{5,3}) \wedge (\neg x_{4,3} \vee \neg x_{6,3}) \wedge (\neg x_{4,3} \vee \neg x_{7,3}) \wedge (\neg x_{4,3} \vee \neg x_{8,3}) \wedge (\neg x_{5,3} \vee \neg x_{6,3}) \wedge (\neg x_{5,3} \vee \neg x_{7,3}) \wedge (\neg x_{5,3} \vee \neg x_{8,3}) \wedge$$

In a Bit More Detail...

$$(x \vee z) \wedge (y \vee \neg z) \wedge (x \vee \neg y \vee u) \wedge (\neg y \vee \neg u)$$
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- $\wedge$ means all constraints should hold simultaneously
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For operations research listeners:

- Think of $x \vee \neg y \vee u$ as $x + (1 - y) + u \geq 1$
- Just a special type of 0–1 linear program (without objective)

Some Highly Concrete Applications of Such Very Abstract Problems

- Software analysis, testing, and synthesis [DMB11]
- Hardware verification [Sha09]
- Air and train traffic control [ABFP12, FFH+16, ZR14]
- Smart crypto contracts [AGRS20, AGH+22]
- Gene regulatory network inference [PBD+22]
- Computational protein design [AAB+14, HD19]
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(Requires fleshing out quite a bit of details, but in the interest of time we will take this on trust. . .)

A Deep, Challenging Problem...

- Real-world problems correspond to **humongous** formulas (with 100,000s or even 1,000,000s of variables)
- Can we use computers to solve these problems efficiently?
- Question mentioned already in Gödel's famous letter in 1956 to von Neumann (the "father of computer science")
- Topic of intense research in computer science ever since 1960s
- Problem known to be computationally very challenging (NP-complete or worse) [Coo71, Lev73]
- Widely believed to be exponentially hard in the worst case [IP01, CIP09]

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  - How can they be so good in practice?
  - When they fail to be efficient, can we understand why?

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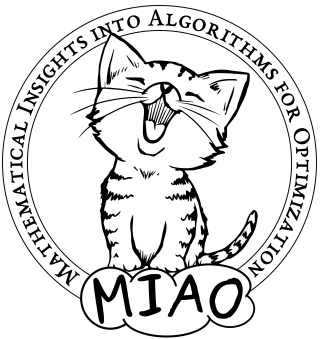
. . . And in the process also touch on some of the research conducted in the Mathematical Insights into Algorithms for Optimization (MIAO) group in Copenhagen and Lund



Outline of Survey on Conflict-Driven Solving for SAT and Beyond

1. Boolean Satisfiability (SAT) Solving
   - Davis-Putnam-Logemann-Loveland (DPLL) Method
   - Conflict-Driven Clause Learning (CDCL)
   - Analyzing SAT Solvers Using Proof Complexity
2. Conflict-Driven Search for Stronger Combinatorial Solving Paradigms
   - Mixed Integer Programming (MIP) and Constraint Programming (CP)
   - Conflict Analysis Beyond SAT
   - Opportunities and Challenges
3. Research Outlook
   - Research in the MIAO Group
   - Pointers for Further Study

Formal Description of SAT Problem

- Variable x: takes value 1 (**true**) or 0 (**false**)
- Literal ℓ: variable x or its negation $\overline{x}$ (write $\overline{x}$ instead of $\neg x$)
- Clause $C = \ell_1 \vee \cdots \vee \ell_k$: disjunction of literals
  (Consider as sets, so no repetitions and order irrelevant)
- Conjunctive normal form (CNF) formula $F = C_1 \wedge \cdots \wedge C_m$: conjunction of clauses

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For instance, what about our example formula?

$$(x \vee z) \wedge (y \vee \overline{z}) \wedge (x \vee \overline{y} \vee u) \wedge (\overline{y} \vee \overline{u})$$
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Solving SAT by Case Analysis: DPLL

So the SAT problem is NP-complete and maybe exponentially hard, but what do you do if you have to solve it anyway?

Chances are you'll use some variant of the DPLL method developed by Davis, Putnam, Logemann & Loveland [DP60, DLL62]

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- If F contains no clauses, report "satisfiable" and terminate
- Otherwise pick some variable x in F
- Set $x = 0$, simplify F and make recursive call
- Set $x = 1$, simplify F and make recursive call
- If result in both cases "unsatisfiable", then report "unsatisfiable" and return

A DPLL Toy Example

$$(x \vee z) \wedge (y \vee \overline{z}) \wedge (x \vee \overline{y} \vee u) \wedge (\overline{y} \vee \overline{u})$$
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Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

"Simplify formula" by (mentally) removing

- satisfied clauses
- falsified literals

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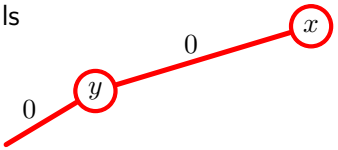

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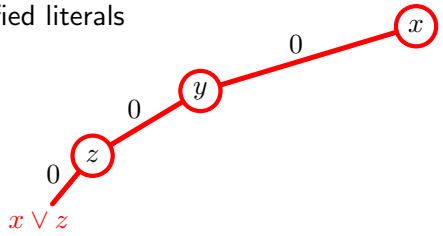

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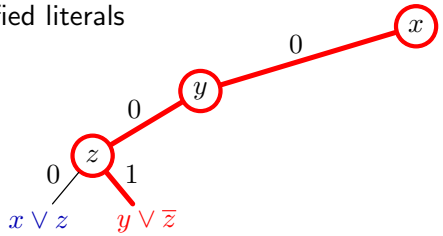

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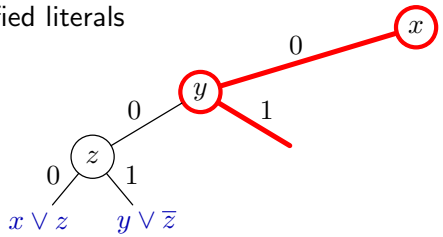

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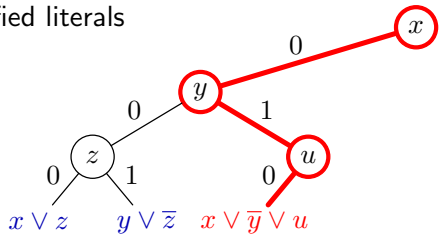

A DPLL Toy Example

$$(\quad z) \wedge (y \vee \overline{z}) \wedge (\qquad u) \wedge (\overline{y} \vee \overline{u})$$
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Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

"Simplify formula" by (mentally) removing

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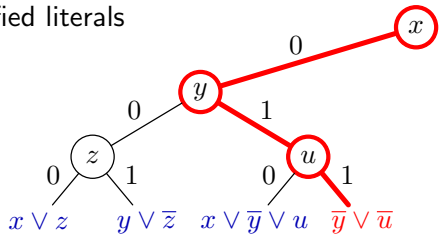

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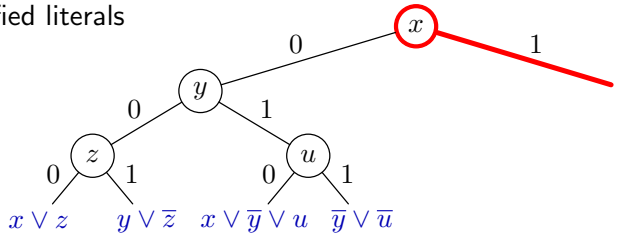

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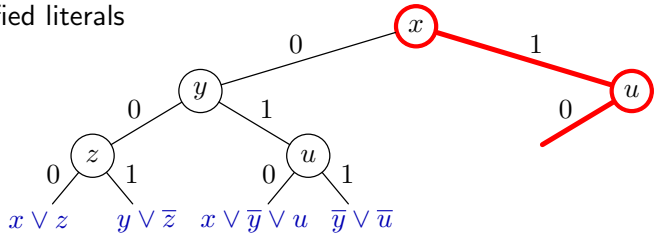

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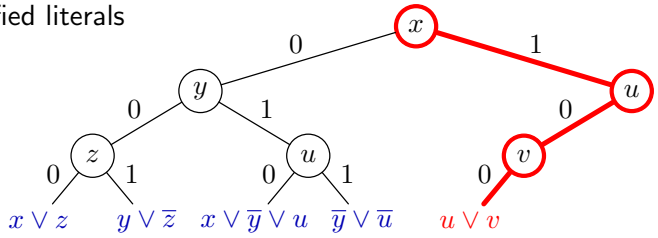

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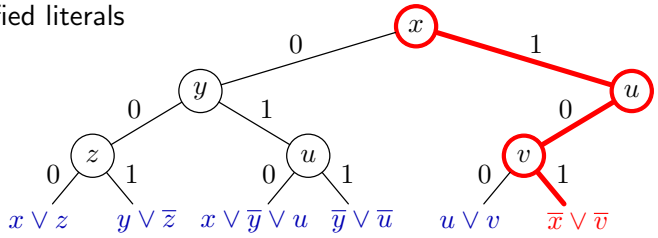

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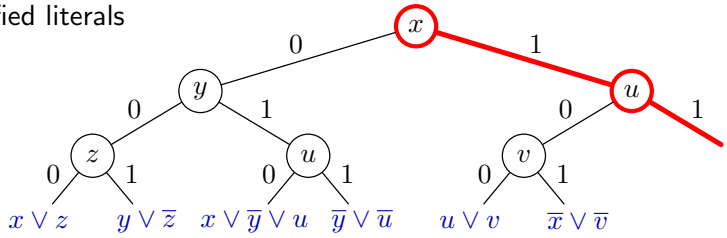

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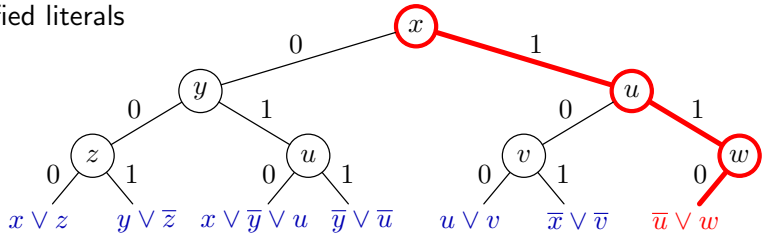

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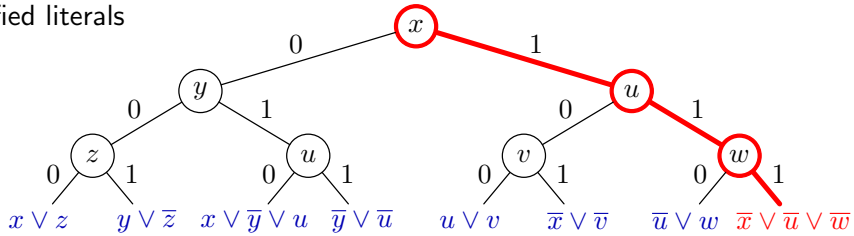

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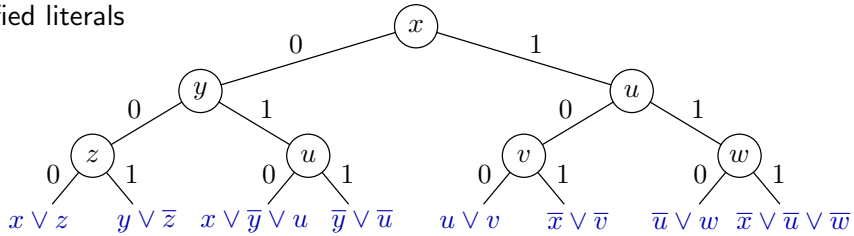

State-of-the-art SAT Solvers: Ingredients

Many more ingredients in modern conflict-driven clause learning (CDCL) SAT solvers (as pioneered in [BS97, MS99, MMZ$^+$01]), e.g.:

- Branching or decision heuristic (choice of pivot variables crucial)
- When reaching leaf, compute explanation for conflict and add to formula as new clause (clause learning)
- Every once in a while, restart from beginning (but save computed info)
- Preprocessing the formula before the search even starts

Let us discuss some of these ingredients

Variable Assignment Heuristics

Unit propagation

- Suppose current assignment ρ falsifies all literals in $C = \ell_1 \vee \ell_2 \vee \cdots \vee \ell_k$ except one (say ℓ_k) — C is unit under ρ
- Then ℓ_k has to be true, so set it to true
- Known as unit progagation or Boolean constraint progagation
- Always propagate if possible — in modern solvers aim for ≈99% of assignments being unit propagations

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VSIDS (Variable state independent decaying sum)

- When backtracking, score $+1$ for variables "causing conflict"
- Also multiply all scores with factor $\kappa < 1$ — exponential filter rewarding variables involved in recent conflicts
- When no propagations, decide on variable with highest score

Clause Learning

- At conflict, want to add clause avoiding same part of search tree being explored again

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- Nowadays, more sophisticated learning schemes starting with [MS99, MMZ$^+$01]
- Often described in terms of cuts in conflict graph
- More helpful to view conflict analysis as syntactic derivation applied on clauses unit propagating to conflict

Decisions, Unit Propagations, and Conflict

Two kinds of assignments — illustrate on example formula:

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

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Free choice to assign value to variable

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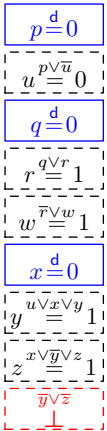

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Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

$p \overset{\mathsf{d}}{=} 0$
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decision level 1

$q \overset{\mathsf{d}}{=} 0$
$r \overset{q \vee r}{=} 1$
$w \overset{\overline{r} \vee w}{=} 1$

decision level 2

$x \overset{\mathsf{d}}{=} 0$
$y \overset{u \vee x \vee y}{=} 1$
$z \overset{x \vee \overline{y} \vee z}{=} 1$
$\overline{y} \vee \overline{z}$
$\bot$

decision level 3

Conflict Analysis

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$p \overset{\mathsf{d}}{=} 0$ — decision level 1

$u \overset{p \vee \overline{u}}{=} 0$ — decision level 1

$q \overset{\mathsf{d}}{=} 0$ — decision level 2

$r \overset{q \vee r}{=} 1$ — decision level 2

$w \overset{\overline{r} \vee w}{=} 1$ — decision level 2

$x \overset{\mathsf{d}}{=} 0$ — decision level 3

$y \overset{u \vee x \vee y}{=} 1$ — decision level 3

$z \overset{x \vee \overline{y} \vee z}{=} 1$ — decision level 3

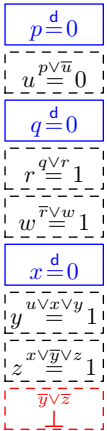
$\overset{\overline{y} \vee \overline{z}}{\bot}$ — decision level 3

Could backtrack by erasing conflict level & flipping last decision (this is what DPLL does)

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$



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But want to learn from conflict and cut away as much of search space as possible

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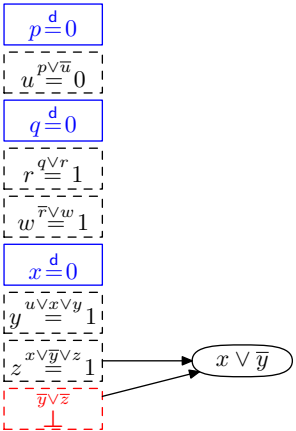


Could backtrack by erasing conflict level & flipping last decision (this is what DPLL does)

But want to learn from conflict and cut away as much of search space as possible

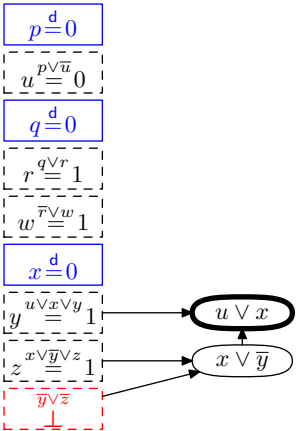
Case analysis for last two clauses over propagated variable:

- $x \vee \overline{y} \vee z$ wants $z = 1$
- $\overline{y} \vee \overline{z}$ wants $z = 0$
- Merge clauses & remove z — must satisfy $x \vee \overline{y}$

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$



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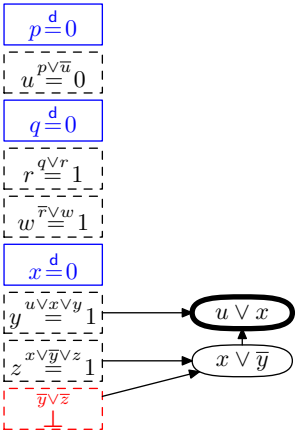
- Merge clauses & remove z — must satisfy $x \vee \overline{y}$

Repeat until UIP clause with only 1 variable at conflict level after last decision — learn and backjump

Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$



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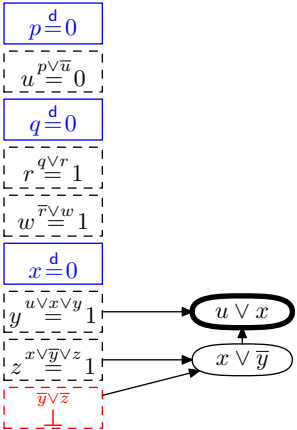
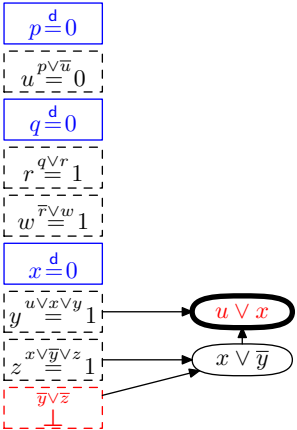


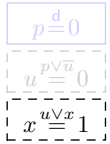
Assertion level 1 (2nd largest level in learned clause) — trim trail to that level

Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$





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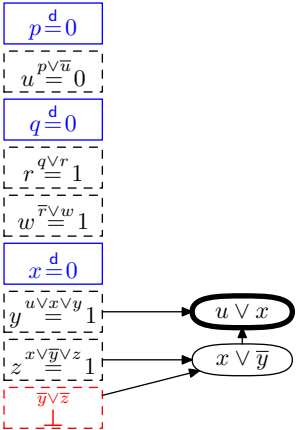
Now UIP literal guaranteed to flip (assert) — but this is a propagation, not a decision

Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$



Assertion level 1 (2nd largest level in learned clause) — trim trail to that level

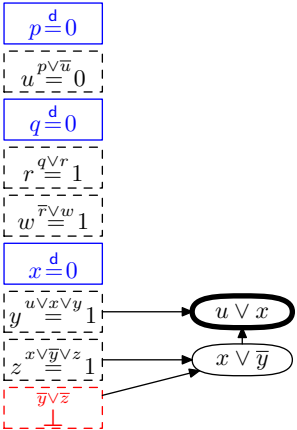
Now UIP literal guaranteed to flip (assert) — but this is a propagation, not a decision

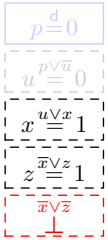
Then continue as before...

Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$





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Backjump: undo max #decisions while learned clause propagates

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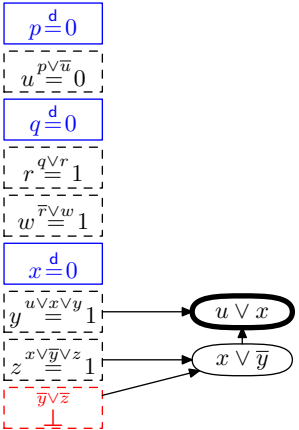

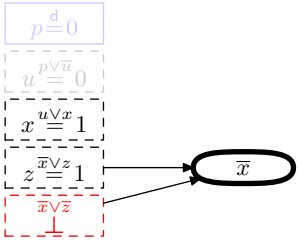

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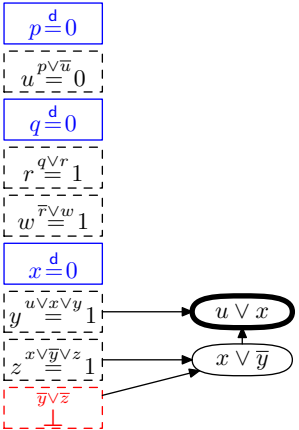

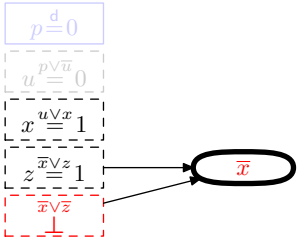

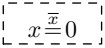

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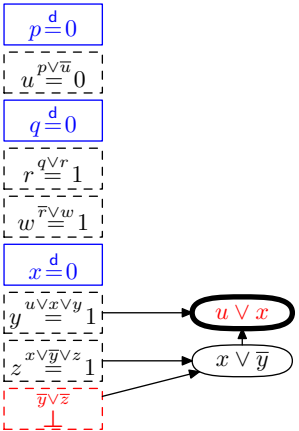

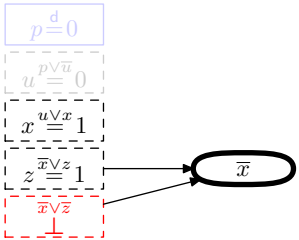

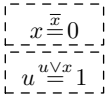

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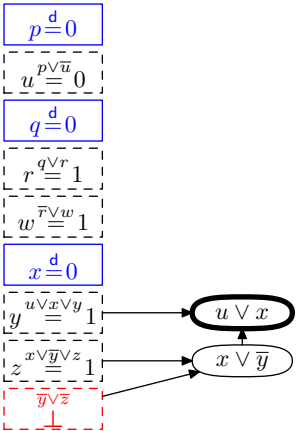

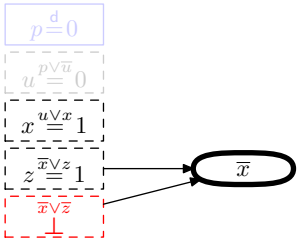

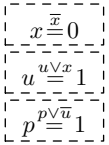

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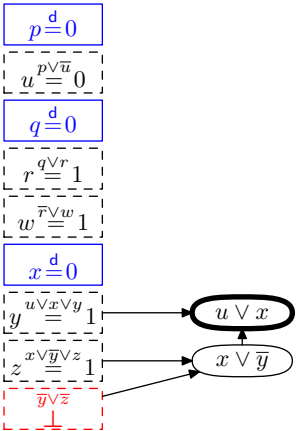

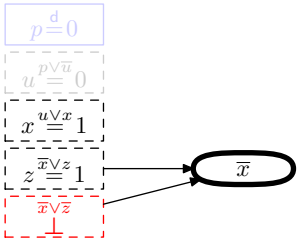

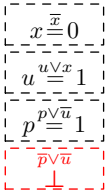

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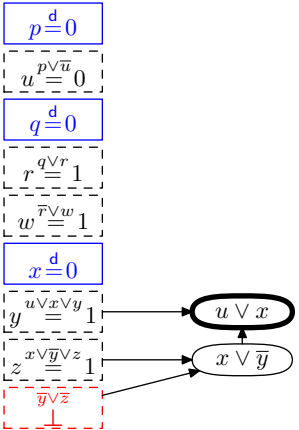

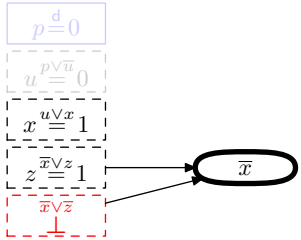

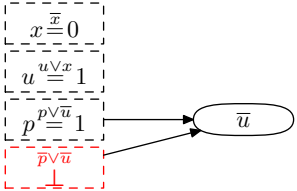

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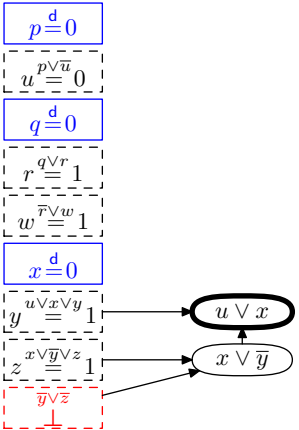

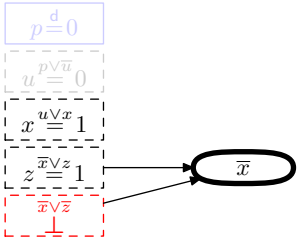

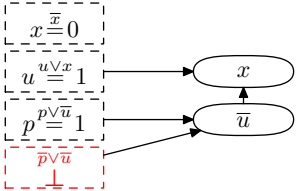

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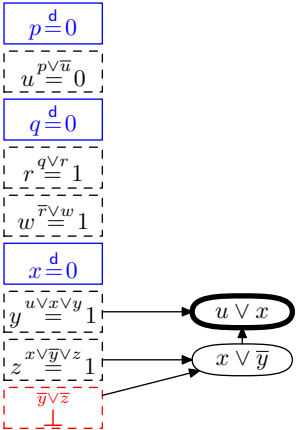

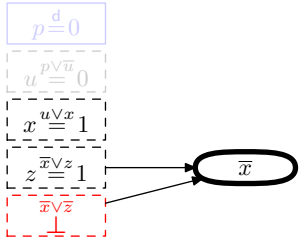

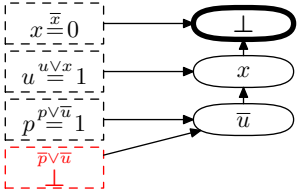

Clause Database Reduction

- In addition to learning clauses, also erase learned clauses that don't seem useful
- Modern solvers do this very aggressively
- Speeds up CDCL search (in particular, unit propagation, which dominates running time)
- But erasing too aggressively can throw away clauses that would have made solver terminate faster [EGG+18]
- So potential trade-off between search speed and search quality
- Except sometimes getting rid of clauses improves search quality too! [KN20]

Restarts

- Fairly frequently, start search all over (but keep learned clauses)
- Original intuition: stuck in bad part of search tree — go somewhere else
- Not the reason this is done now
- Popular variables with high VSIDS scores get set again [MMZ$^+$01]
- Are even set to same values (phase saving) [PD07]
- Current intuition: improves the search by focusing on important variables
- Restart at fixed intervals or (better) make adaptive restarts depending on "quality" of learned clauses [AS09, AS12]

Conflict-Driven Clause Learning in Pseudocode (Slightly Simplified)

CDCL(F)

$\mathcal{D} \leftarrow F$; // initialize clause database to contain formula
$\rho \leftarrow \emptyset$; // initialize assignment trail to empty
forever do
  if ρ *falsifies some clause* $C \in \mathcal{D}$ **then**
    $A \leftarrow \text{analyzeConflict}(\mathcal{D}, \rho, C)$;
    if $A = \bot$ **then** output UNSATISFIABLE and exit ;
    else add learned clause A to $\mathcal{D}$ and backjump by shrinking ρ ;
  else if *exists clause* $C \in \mathcal{D}$ *unit propagating* x *to* $b \in \{0, 1\}$ *under* ρ **then**
    add propagated assignment $x \overset{C}{=} b$ to ρ ;
  else if *time to restart* **then** $\rho \leftarrow \emptyset$;
  else if *time for clause database reduction* **then**
    erase (roughly) half of learned clauses in $\mathcal{D} \setminus F$ from $\mathcal{D}$
  else if *all variables assigned* **then** output SATISFIABLE and exit ;
  else
    use decision scheme to choose x and b and add assignment $x \overset{\text{d}}{=} b$ to ρ ;

Conflict-Driven Clause Learning in Pseudocode (Slightly Simplified)

CDCL(F)

1 $\mathcal{D} \leftarrow F$; // initialize clause database to contain formula
2 $\rho \leftarrow \emptyset$; // initialize assignment trail to empty
3 **forever do**
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6 **if** $A = \bot$ **then** output UNSATISFIABLE and exit ;
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10 **else if** *time to restart* **then** $\rho \leftarrow \emptyset$;
11 **else if** *time for clause database reduction* **then**
12 erase (roughly) half of learned clauses in $\mathcal{D} \setminus F$ from $\mathcal{D}$
13 **else if** *all variables assigned* **then** output SATISFIABLE and exit ;
14 **else**
15 use decision scheme to choose x and b and add assignment $x \overset{\text{d}}{=} b$ to ρ ;

Conflict Analysis Pseudocode

analyzeConflict$(\mathcal{D}, \rho, C_{\text{confl}})$

```
1 C_learn ← C_confl ;
2 while C_learn not UIP clause and C_learn ≠ ⊥ do
3 |   ℓ ← literal assigned last on trail ρ ;
4 |   if ℓ propagated and ¬ℓ occurs in C_learn then
5 |   |   C_reason ← reason(ℓ, ρ, D) ;
6 |   |   C_learn  ← resolve(C_learn, C_reason) ;
7 |   ρ ← ρ \ {ℓ} ;
8 return C_learn ;
```

State-of-the-art SAT solvers: What About the Recipe?

List of ingredients again (not exhaustive):

- Variable decisions & propagations
- Conflict analysis
- Restarts
- Clause database reduction
- Preprocessing

State-of-the-art SAT solvers: What About the Recipe?

List of ingredients again (not exhaustive):

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Some natural questions:

- How best to combine these ingredients into a recipe?
- When and why does this recipe work?

State-of-the-art SAT solvers: What About the Recipe?

List of ingredients again (not exhaustive):

- Variable decisions & propagations
- Conflict analysis
- Restarts
- Clause database reduction
- Preprocessing

Some natural questions:

- How best to combine these ingredients into a recipe?
- When and why does this recipe work?

Why SAT solvers actually work so well is poorly understood

Plenty of research has been done to comprehend this better
(Among other places in the MIAO group)

SAT Solver Analysis and the Resolution Proof System

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So focus instead on underlying method of reasoning

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Resolution proof system [Bla37, Rob65]

- Start with clauses of CNF formula (axioms)
- Derive new clauses by resolution rule

$$\frac{C_1 \vee x \qquad C_2 \vee \overline{x}}{C_1 \vee C_2}$$

Resolution Proofs by Contradiction

Resolution rule:

$$\frac{C_1 \vee x \qquad C_2 \vee \overline{x}}{C_1 \vee C_2}$$

Observation

If F is a satisfiable CNF formula and D is derived from clauses $D_1, D_2 \in F$ by the resolution rule, then $F \wedge D$ is satisfiable.

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Such proof by contradiction also called resolution refutation

DPLL and Resolution Proofs

A DPLL execution is essentially a resolution proof

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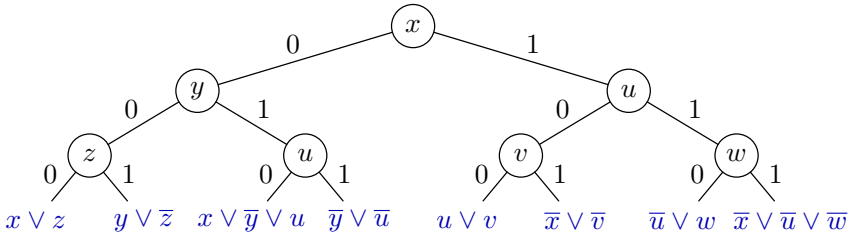
Look at our example again



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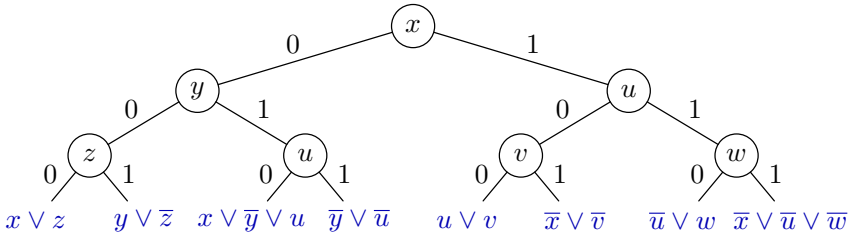

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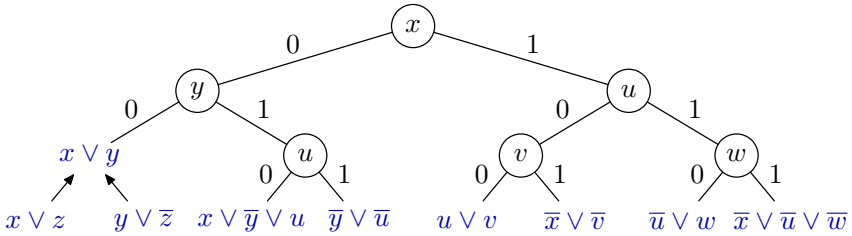

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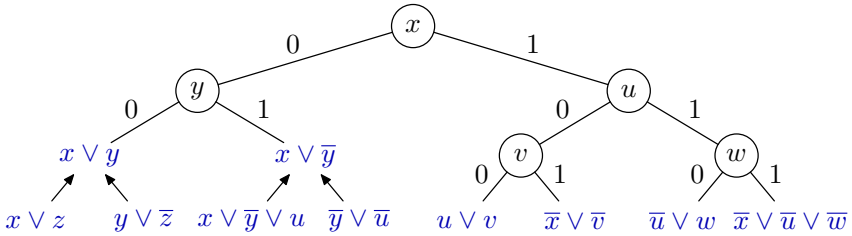

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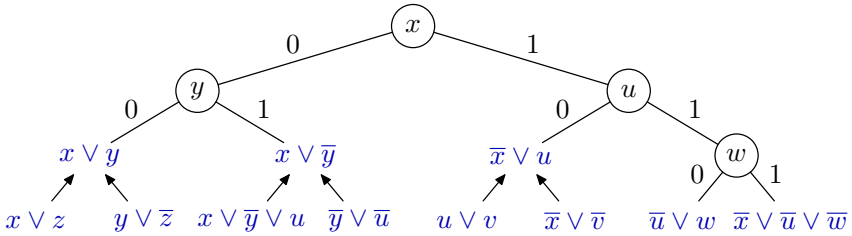

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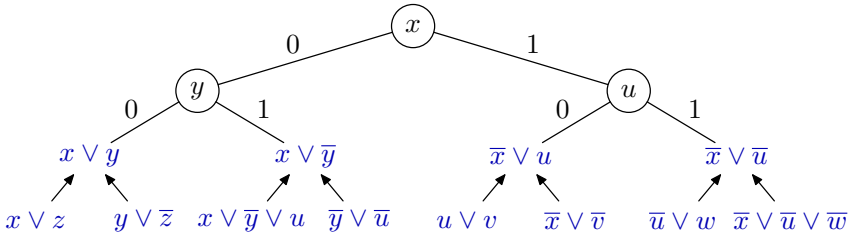

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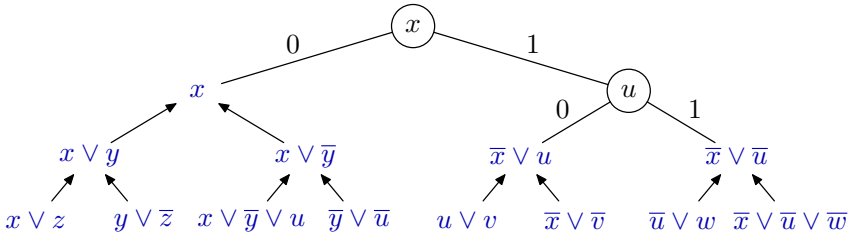

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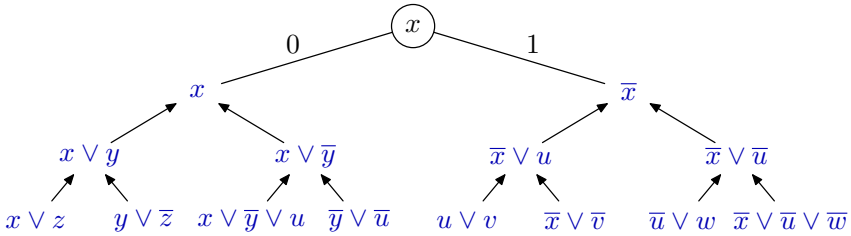

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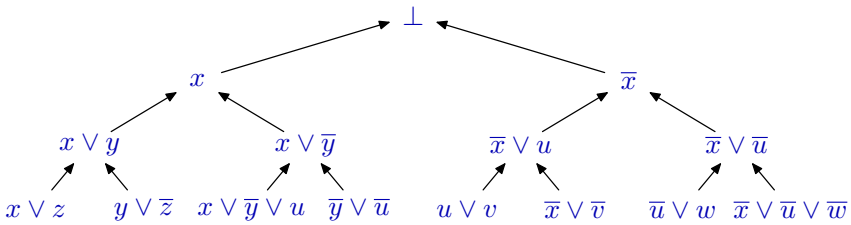

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- Conflict-driven clause learning adds "shortcut edges" in tree, but still yields resolution proof

CDCL and Resolution Proofs

Obtain resolution proof...

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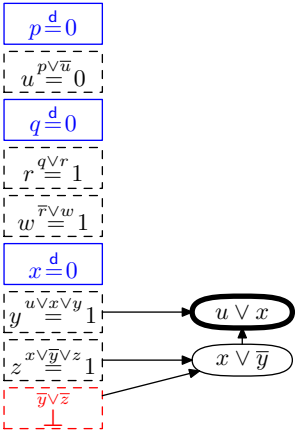
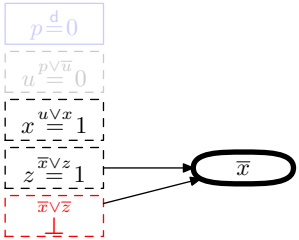
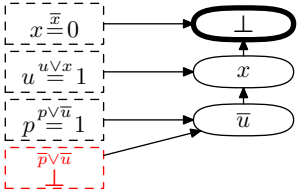
Obtain resolution proof from our example CDCL execution...







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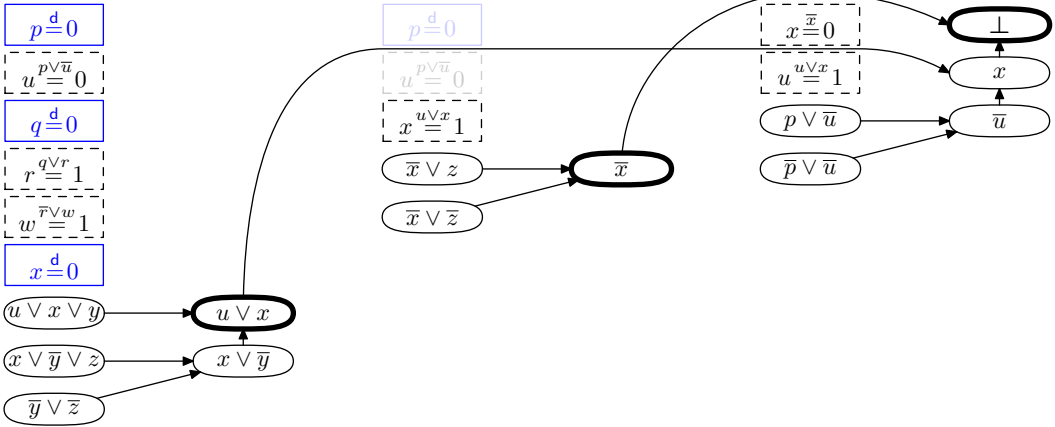

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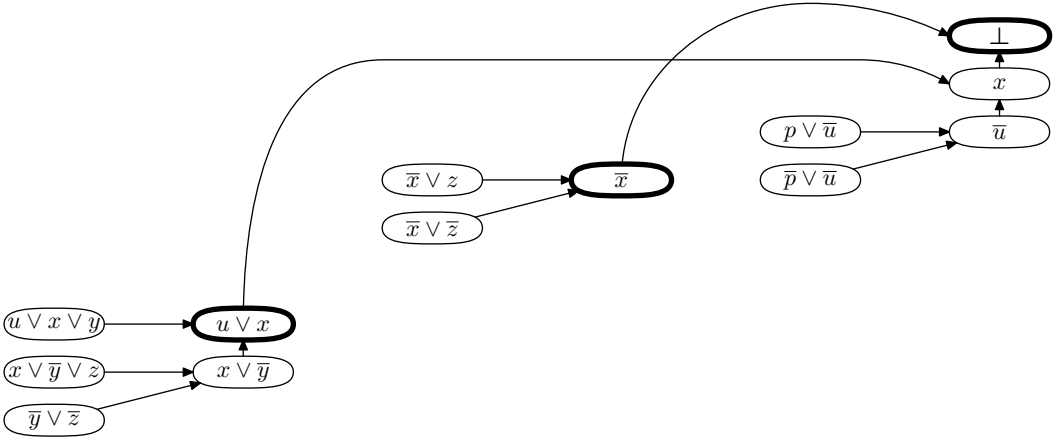

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(*) Except for some preprocessing techniques, which is an important omission, but this gets complicated and we don't have time to go into details...

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- See Chapter 7 on *Proof Complexity and SAT Solving* in the *Handbook of Satisfiability* [BN21] or the lecture `youtu.be/9NR_RGIs1no` for more details

Mixed Integer Linear Programming

Mixed integer linear program

- Minimize $\sum_j a_j x_j$
- Subject to $\sum_j a_{i,j} x_j \leq A_i$, $i = 1, \ldots, m$
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- No real-valued variables: integer linear program (ILP)
- $0 \leq x_j \leq 1$ for all j: 0-1 ILP
- Vacuous objective $\sum_j 0 \cdot x_j$: decision problem
- But MIP makes most sense for optimization

Constraint Programming

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Example problem:

$$X, Y, Z \in \{1, 2, 3, 4\}$$
$$2Y \neq X + Z$$
$$2X + 3Y + 4Z \leq 16$$
$$\mathit{alldiff}\,(X, Y, Z)$$

Comparing SAT, CP, and MIP

Academia vs. industry

- Best SAT and CP solvers found in academia and/or are open-source — e.g., SAT solvers KISSAT [Kis] and CADICAL [CaD] and CP solver OR-TOOLS [ORT]
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So maybe MIP and CP solvers could be improved by adding conflict analysis?
Has been tried, but with much less success than in SAT solving — why?

Approaches for Generalizing Conflict Analysis

Standard approach: Implication graph perspective

- Clauses encode implications between literals
- Learned clauses are cuts in the implication graph
- Extract implications/clauses from more expressive reason constraints
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From now on focus on MIP and restricted case of 0–1 ILP

The Cutting Planes Proof System

Cutting planes [CCT87] operates on pseudo-Boolean constraints (a.k.a. 0–1 integer linear inequalities) $\sum_i a_i \ell_i \geq A$ with

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and has derivation rules

Literal axioms $\dfrac{}{\ell_i \geq 0}$

Linear combination $\dfrac{\sum_i a_i \ell_i \geq A \qquad \sum_i b_i \ell_i \geq B}{\sum_i (c_A a_i + c_B b_i)\ell_i \geq c_A A + c_B B} \quad [c_A, c_B > 0]$

Division $\dfrac{\sum_i a_i \ell_i \geq A}{\sum_i \lceil a_i/c \rceil \ell_i \geq \lceil A/c \rceil} \quad [c > 0]$

MIP View of SAT Conflict Analysis

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- Apply coefficient tightening along the way just to keep constraints nicer

Cancelling Linear Combination of 0–1 Inequalities Doesn't Work!

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- But $x_4 \geq 1$ is satisfiable — does **not** explain why solver reached a conflict!

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- Add C_1 and C_2 to cancel x_3 and get

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- But $x_4 \geq 1$ is satisfiable — does **not** explain why solver reached a conflict!
- Problem: Constraints don't propagate tightly, so LP is not infeasible

Cancelling Linear Combination of 0–1 Inequalities Doesn't Work!

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- Need to tighten propagation to remove solution $\{x_1 = 0, x_2 = x_4 = 1, x_3 = \frac{1}{2}\}$

Tightening Propagation with Division / Chvátal-Gomory Cuts

$$C_1 \doteq 2x_1 + 2x_2 + 2x_3 + x_4 \geq 4$$
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(*) Finding such a cut is always possible, but this requires an argument

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Conflict analysis in MIP solvers [Ach07, ABKW08, AW13]

- Perform derivation not on reason constraints R as described above
- Instead use disjunctive clauses extracted from reason constraints
- Incurs exponential loss in theoretical reasoning power compared to operating on actual linear constraints (follows from [BKS04, CCT87, Hak85])
- Proof-of-concept implementation shows that new conflict analysis can be better also in practice [MBGN23, MSB+25]

Open Problems for Conflict Analysis in Stronger Paradigms than SAT

Pseudo-Boolean (PB) conflict analysis on 0–1 linear inequalities

- Many more degrees of freedom than SAT conflict analysis
- Many (most?) aspects still unexplored
- Which other MIP cut rules could be useful for tightening propagations?

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General MIP conflict analysis

- PB conflict analysis can be extended to 0–1 mixed linear programs [MSB$^+$25]
- But room for lots of additional tuning
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Conflict analysis for constraint programming

- Lazy clause generation CP solvers [OSC09] suffer from same problem as MIP solvers
- Reasons for propagations can also be generated as 0–1 linear constraints and used to certify correctness of solver reasoning [EGMN20, GMN22, MM23, MMN24, MM25]
- Could such constraints also be used efficiently for pseudo-Boolean conflict analysis?

Research Goals in the MIAO Group (1/2)

Strengthen the mathematical analysis of algorithmic methods

- Study methods of reasoning powerful enough to capture state-of-the-art algorithms used in practice
- Prove theorems about their power and limitations
- E.g., look for analogues of result that resolution proof system captures CDCL reasoning

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Construct stronger algorithms for combinatorial problems

- Use insights into stronger mathematical methods of reasoning to build algorithms for SAT and other combinatorial problems
- Aiming for exponential speed-ups over state of the art
- E.g., use cutting-planes-based reasoning to build pseudo-Boolean solvers

Research Goals in the MIAO Group (2/2)

Improve understanding of efficient computation in practice

- Use computational complexity theory to study "real-world" (not worst-case) problems
- Combine theoretical study and empirical experiments
- E.g., take "crafted formulas" with provable theoretical properties and investigate correlation with practical solver performance

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Certify correctness for modern combinatorial solvers

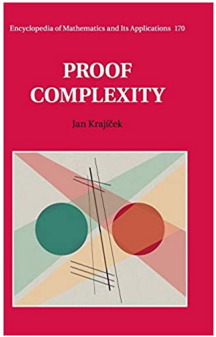
- In many combinatorial optimization paradigms, state-of-the-art solvers are known to be buggy (definitely holds for MIP and CP)
- Develop methods to make solvers output not just answer but machine-verifiable proof of correctness of this answer
- Work on, e.g., SAT-based optimization (MaxSAT), pseudo-Boolean solving, subgraph solving, constraint programming, and automated planning

Some References for In-Depth Reading

Handbook of Satisfiability
(Especially chapter 7 ☺)

[BHvMW21]

Proof Complexity
by Jan Krajíček



[Kra19]

And survey papers, slides, and videos at `www.jakobnordstrom.se`

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  (we're hiring! ☺)

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Thanks for listening!

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