End-to-End Verification for Subgraph Solving

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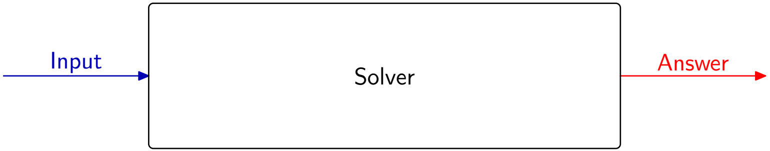
Joint work with Stephan Gocht, Ciaran McCreesh, Magnus O. Myreen, Andy Oertel, and Yong Kiam Tan

The Success of Combinatorial Solving (and the Dirty Little Secret)

- Astounding progress last couple of decades on combinatorial solvers for, e.g.:
  - Boolean satisfiability (SAT) solving and optimization [BHvMW21]
  - Constraint programming [RvBW06]
  - Mixed integer linear programming [AW13, BR07]
  - Satisfiability modulo theories (SMT) solving [BHvMW21]
- Solvers very fast, but sometimes wrong (even most mature ones in industry and academia) [BLB10, CKSW13, AGJ+18, GSD19, GS19, BMN22, BBN+23]
- Only currently realistic solution: Proof logging
  Make solver certifying [ABM+11, MMNS11] by adding code so that it outputs
  1. not only answer but also
  2. simple, machine-verifiable proof that answer is correct

Proof Logging with Certifying Solvers: Workflow



1. Run combinatorial solving algorithm on problem input

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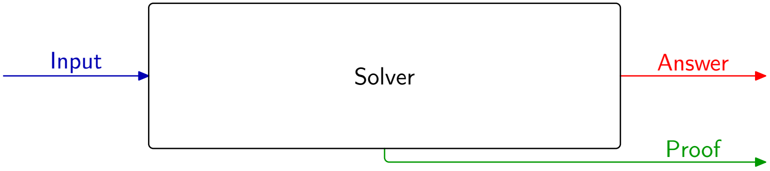

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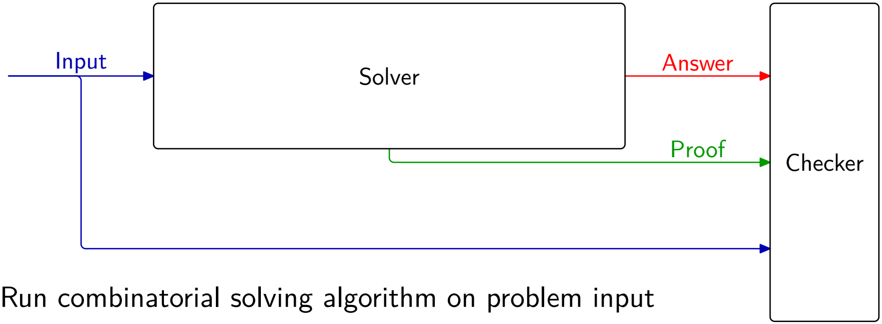

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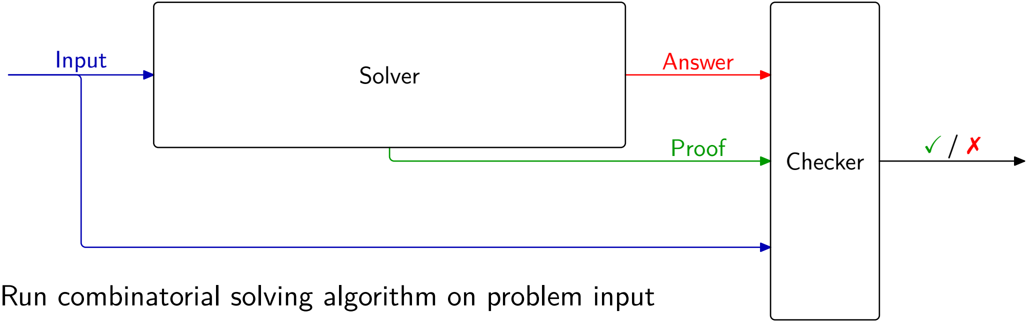

1. Run combinatorial solving algorithm on problem input
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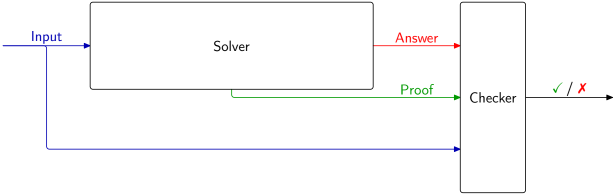
4. Verify that proof checker says answer is correct

Proof Logging Desiderata

Proof format for certifying solver should be



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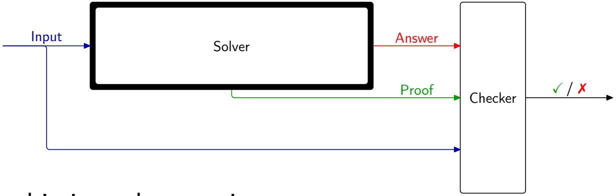

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- **very powerful:** minimal overhead for sophisticated reasoning

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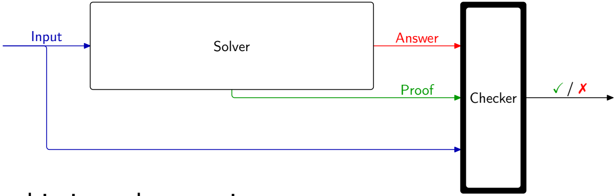

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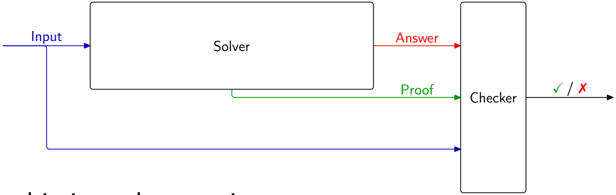

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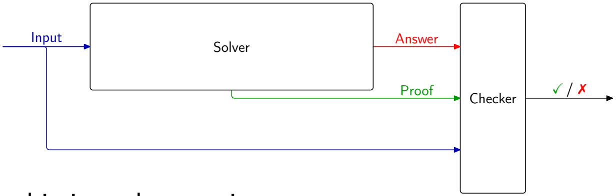

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Earlier proof logging approaches for SAT, MaxSAT, constraint programming, et cetera have struggled with this trade-off (and failed to master it)

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- Build on successes in proof logging for SAT solving with proof formats such as DRAT [HHW13a, HHW13b, WHH14], GRIT [CMS17], LRAT [CHH+17], ...
- But represent constraints as 0–1 integer linear inequalities
- Formalize reasoning using cutting planes [CCT87] proof system
- Add well-chosen strengthening rules [Goc22, GN21, BGMN23]
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Purpose of this talk:

1. Review basic set-up for proof logging beyond SAT
2. Discuss a highly non-obvious application: Subgraph solving
3. Describe a fully formally verified pipeline for such graph problems

Design Principles for Proof Logging

Proof logging implementation

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- Proof logging overhead small constant fraction of running time ($\lessapprox 10\%$)
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Proof system

- Keep language simple — no XOR constraints, CP propagators, symmetries, ...
- But reason efficiently about such notions using power of proof system
- Combine proof logging with formally verified proof checker

Proof Language: Pseudo-Boolean Constraints

Proof consists of 0-1 integer linear inequalities or pseudo-Boolean constraints:

$$\sum_i a_i \ell_i \geq A$$

- $a_i, A \in \mathbb{Z}$
- literals ℓ_i: x_i or $\overline{x}_i$ (where $x_i + \overline{x}_i = 1$)
- variables x_i take values $0 = \mathit{false}$ or $1 = \mathit{true}$

Sometimes convenient to use normalized form [Bar95] with all a_i, A positive (without loss of generality)

Some Types of Pseudo-Boolean Constraints

1. Disjunctive clauses

$$x \lor \overline{y} \lor z \quad \Leftrightarrow \quad x + \overline{y} + z \geq 1$$

2. Cardinality constraints

$$x_1 + x_2 + x_3 + x_4 \geq 2$$

3. General pseudo-Boolean constraints

$$x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$$

Successful Applications of VeriPB Proof Logging

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Surprisingly, pseudo-Boolean reasoning is sufficient to efficiently certify wide range of combinatorial solving techniques:

1. Boolean satisfiability (SAT) solving including advanced techniques such as
   - Gaussian elimination [GN21]
   - symmetry breaking [BGMN23]
2. SAT-based optimization (MaxSAT) [VDB22, BBN+23, BBN+24, IOT+24]
3. (Linear) Pseudo-Boolean solving [GMNO22]
4. **Subgraph solving** [GMN20, GMM+20, GMM+24]
5. Dynamic programming and decision diagrams [DMM+24]
6. Presolving in 0–1 integer linear programming [HOGN24]
7. Constraint programming [EGMN20, GMN22, MM23, MMN24]

Pseudo-Boolean Proof Logging — How and Why?

If problem is (special case of) 0-1 integer linear program

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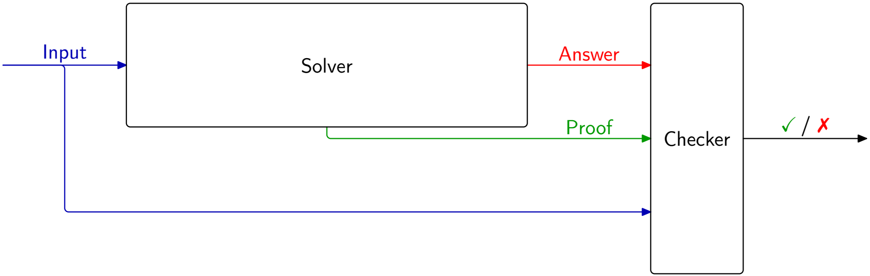
Goldilocks compromise between expressivity and simplicity:

1. 0-1 ILP expressive formalism for combinatorial problems (including objective)
2. Powerful reasoning capturing many combinatorial arguments

Proof Logging with Formally Verified Checking: Full Workflow



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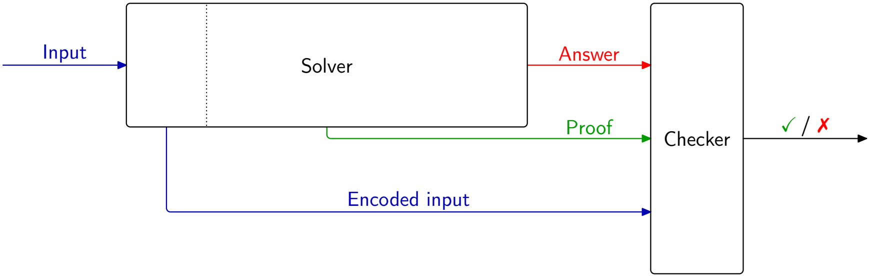

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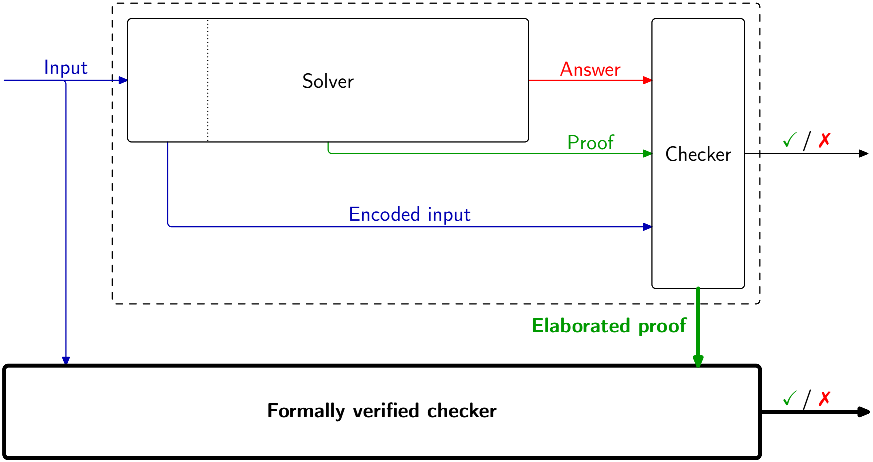

Subgraph Problems and the Glasgow Subgraph Solver

Some important subgraph problems

- Maximum clique in a given graph
- Subgraph isomorphism of pattern graph in target graph
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The Glasgow Subgraph Solver [ADH+19, GSS]

- State-of-the-art solver for such problems
- Sometimes the only solver returning an answer
- Can we trust that such an answer is correct?

Pseudo-Boolean Proof Logging for Subgraph Solving

All reasoning steps in Glasgow Subgraph Solver can be formalized efficiently in the cutting planes proof system [GMN20, GMM+20]

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Let's see how this works for subgraph isomorphism

The Subgraph Isomorphism Problem

Input

- Pattern graph $\mathcal{P}$ with vertices $V(\mathcal{P}) = \{a, b, c, \ldots\}$
- Target graph $\mathcal{T}$ with vertices $V(\mathcal{T}) = \{u, v, w, \ldots\}$

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Task

- Find all subgraph isomorphisms $\varphi : V(\mathcal{P}) \to V(\mathcal{T})$
- I.e., if
  1. $\varphi(a) = u$
  2. $\varphi(b) = v$
  3. $(a, b) \in E(\mathcal{P})$

  then must have $(u, v) \in E(\mathcal{T})$

Subgraph Isomorphism as a 0–1 Integer Linear Program

- Pattern graph $\mathcal{P}$ with $V(\mathcal{P}) = \{a, b, c, \ldots\}$
- Target graph $\mathcal{T}$ with $V(\mathcal{T}) = \{u, v, w, \ldots\}$
- No loops (for simplicity)

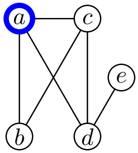
0–1 integer linear (pseudo-Boolean) encoding

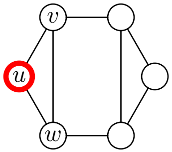
$$\sum_{v \in V(\mathcal{T})} x_{a,v} = 1 \qquad \text{[every pattern vertex } a \text{ maps somewhere]}$$

$$\sum_{b \in V(\mathcal{P})} \overline{x}_{b,u} \geq |V(\mathcal{P})| - 1 \qquad \text{[mapping is one-to-one on target vertices]}$$

$$\overline{x}_{a,u} + \sum_{v \in N(u)} x_{b,v} \geq 1 \qquad \text{[pattern edge } (a,b) \text{ maps to target edge } (u,v)\text{]}$$

Degree Preprocessing Example (Vertex a Cannot Map to Vertex u)





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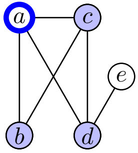

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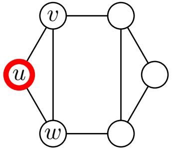

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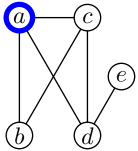

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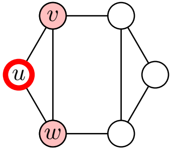

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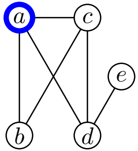

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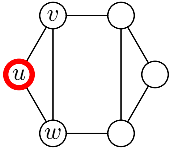

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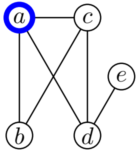

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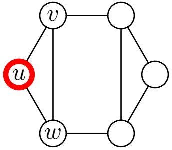

Sum up all constraints & divide by 3 to obtain

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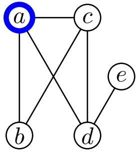

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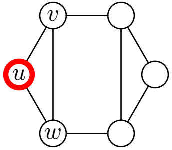

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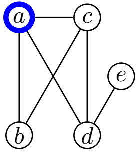

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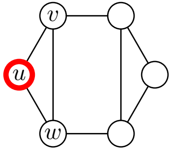

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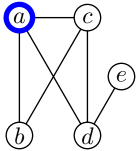

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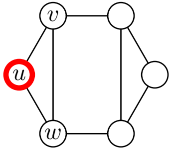

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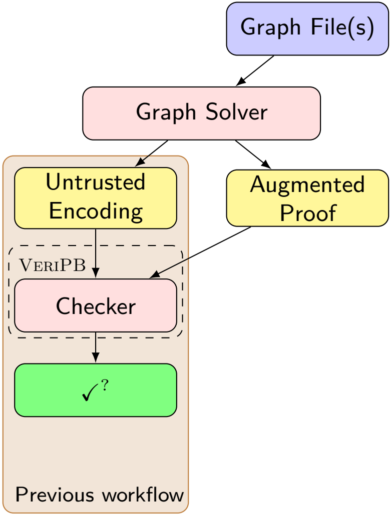
\end{aligned}$$

Workflow for Subgraph Solvers with Pseudo-Boolean Proof Logging



Problem solved!

- Modern graph solvers are complex
- Cannot trust (or maybe even understand) the code
- But now we have pseudo-Boolean proof logging [GMN20, GMM+20]

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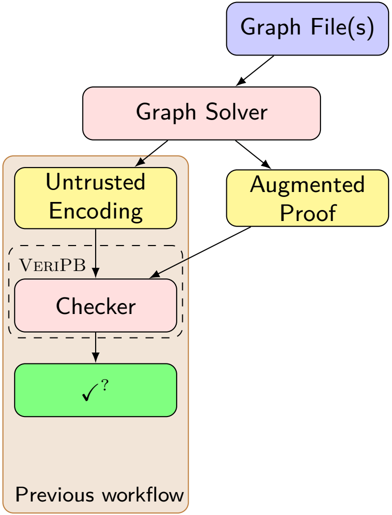

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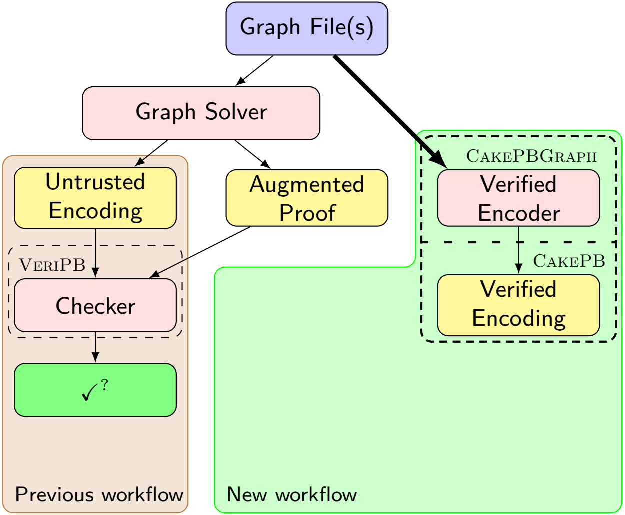
Or is it. . .

- Can we trust the proof checker?
- And how do we know the 0–1 ILP encoding is correct?

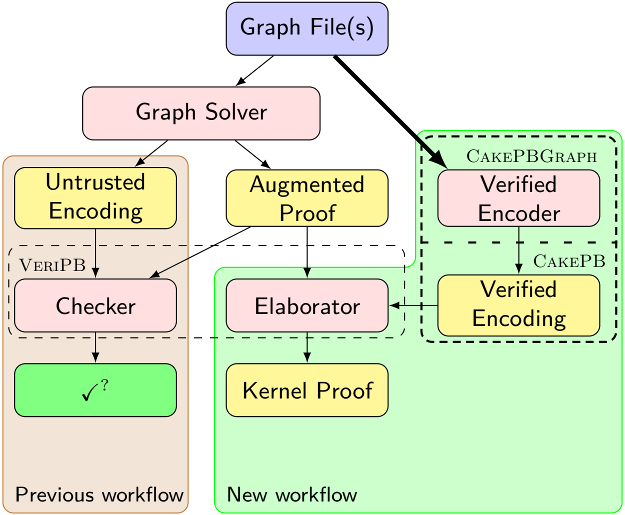
End-to-End Verification Workflow



Verified workflow:

1. Encode problem using formally verified encoder

End-to-End Verification Workflow



Verified workflow:

1. Encode problem using formally verified encoder
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Proof Elaboration

Two versions of the VeriPB proof format:

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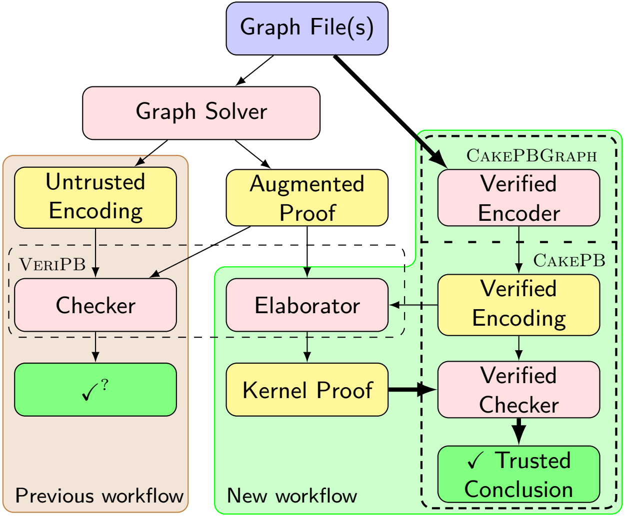
How to get kernel proof?

- VeriPB can elaborate an augmented proof to a kernel proof

End-to-End Verification Workflow (cont.)



Verified workflow:

1. Encode problem using formally verified encoder
2. Elaborate augmented proof to kernel proof
3. Check kernel proof using formally verified checker

Can We Trust This Workflow?

The following needs to be trusted or closely inspected:

- Higher-order logic (HOL) definitions of input parser and problems
  - easy to check
- HOL model of CakeML environment and correspondence to real system
  - validated extensively
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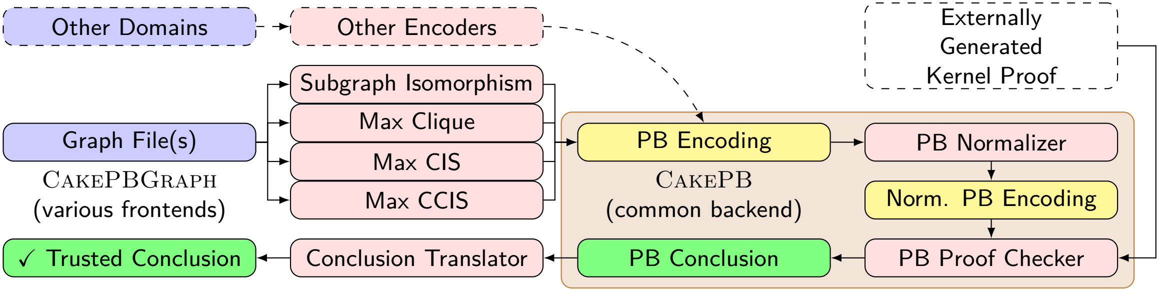
Gives the highest assurance standard for formally verified checker CakePB (`https://gitlab.com/MIAOresearch/software/cakepb`)

Extensible Checking Framework



- Common backend: Performs reasoning on 0–1 ILP (a.k.a. pseudo-Boolean encoding)
- Frontend: Translates specific problem class into 0–1 ILP and back

Future Research Directions

Proof processing

- Trimming proof while verifying (as in DRAT-TRIM [HHW13a])
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- Mixed integer linear programming *(suggested extension of* VeriPB *in [DEGH23])*
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And more. . .

- Use proof logs for algorithm analysis or explainability purposes
- Lots of other challenging problems and interesting ideas
- We're hiring! Talk to me to join the pseudo-Boolean proof logging revolution! ☺

Summing up

- Combinatorial solving and optimization is a true success story
- But ensuring correctness is a crucial, and not yet satisfactorily addressed, concern
- Certifying solvers producing machine-verifiable proofs of correctness seems like the most promising (or only) approach
- Cutting planes reasoning with pseudo-Boolean constraints hits a sweet spot between simplicity and expressivity (for much more general problems)
- Can be combined with formal methods to yield end-to-end verification

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Thank you for your attention!

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