On Minimal Unsatisfiability and Time-Space Trade-offs for k-DNF Resolution

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Joint work with Alexander Razborov

The Focus of This Talk

Starting point: problem that arises in proof complexity

More precisely, in analysis of proof of time-space trade-offs

Leads to nice and clean combinatorial question

Question

How many variables can a minimally unsatisfiable formula contain measured in the formula size m?

This talk:

Focus on combinatorial question — applications described in paper

CNF Preliminaries

- Literal a: variable x or its negation $\overline{x}$
- Clause $C = a_1 \vee \cdots \vee a_k$: disjunction of literals
- CNF formula $F = C_1 \wedge \cdots \wedge C_m$: conjunction of clauses
  Write as set of clauses for simplicity
- Minimally unsatisfiable CNF formula:
  - unsatisfiable, but
  - removing any clause from set makes the rest satisfiable

Non-example:

x
$\overline{x} \vee y$
$\overline{y} \vee z$
$\overline{z}$
$z \vee w$

Example:

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Upper and Lower Bounds on Minimally Unsatisfiable CNFs

Observation (Lower bound)

Minimally unsatisfiable CNF with m clauses can contain $\geq m - 1$ variables

Proof: Generalize example on previous slide

Lemma (Upper bound)

Minimally unsatisfiable CNF with m clauses must contain $\leq m^2$ variables

Proof:

- Suppose not — then $\exists$ clause C with $\geq m$ variables
- Rest of clauses satisfiable by some truth assignment by minimality
- Pick minimal partial assignment — need only set $m - 1$ variables
- But then $\exists$ unset literal in C that we can satisfy $\Rightarrow$ contradiction

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Tight Bound on Minimally Unsatisfiable CNFs

Which bound of $\geq m - 1$ and $\leq m^2$ is the right one?

Upper bound proof throws away lots of information

Unsatisfiability $\Rightarrow$ overconstrained system
Intuitively, need $n + 1$ clauses to constrain n variables...

Theorem (Tarsi's lemma)

Minimally unsatisfiable CNF with m clauses can contain $m - 1$ variables but not more

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Proof of Tarsi's lemma

- Bipartite graph with clauses F left, variables V right, edge if variable occurs in clause (ignore signs)
- Unsatisfiable $\Rightarrow$ no matching
- So by Hall's thm $\exists\ S \subseteq F$ with $|S| > N(S)$ — fix maximal such S
- If $S = F$ we're done, so suppose not
- Then S satisfiable by minimality
- If $S' \subseteq F \setminus S$ then by maximality $|S'| \leq |N(S') \setminus N(S)|$
- By Hall's thm $\exists$ matching — use to satisfy $F \setminus S$ with variables $V \setminus N(S)$
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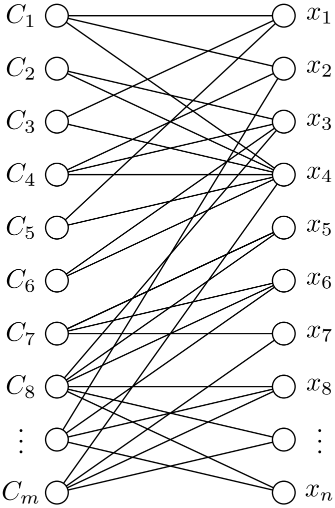

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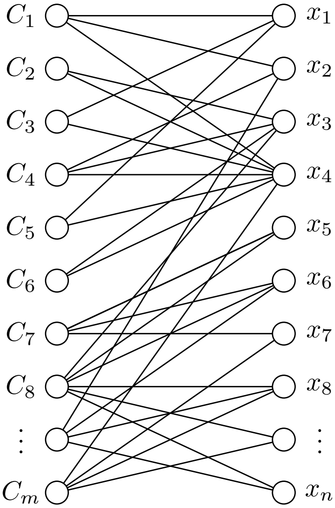

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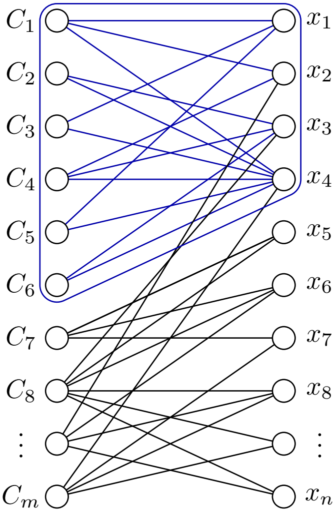

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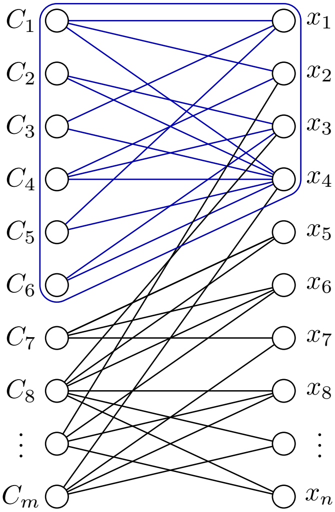

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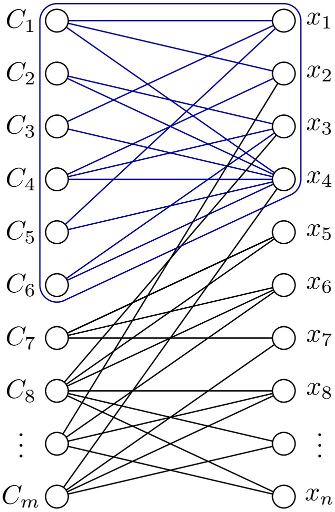

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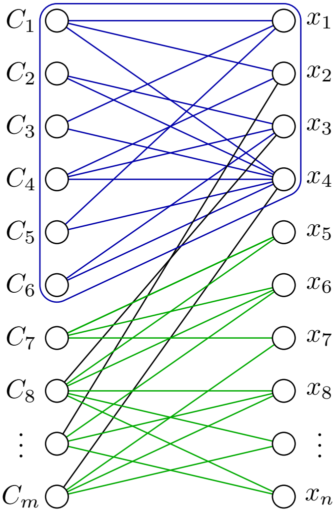

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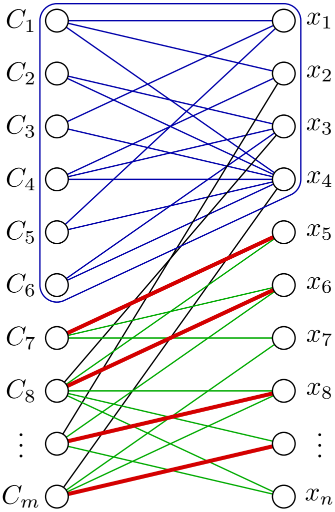

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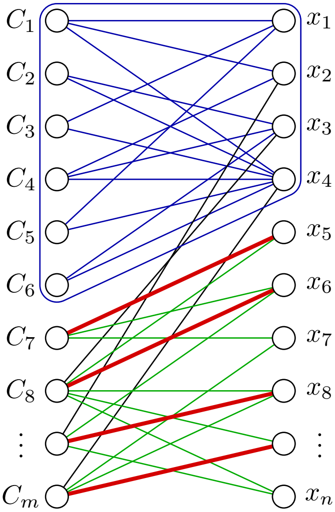

Significance of Tarsi's lemma

- Proof of Tarsi's lemma elementary (Hall's theorem twice)
- But importance in proof complexity hard to overemphasize
- Instrumental for proving results on resolution proof system in e.g.
  - [Chvátal & Szemerédi '88]
  - [Ben-Sasson & Wigderson '99]
  - [Alekhnovich, Ben-Sasson, Razborov & Wigderson '00]
  - [Ben-Sasson & Galesi '03]
  - [Nordström '06]
  - [Nordström & Håstad '08]
  - [Ben-Sasson & Nordström '08]
  - [Ben-Sasson & Nordström '11]
- Study stronger proof systems ⇒ Generalize concept of minimal unsatisfiability
- This work: k-DNF resolution proof systems ⇒ sets of k-DNF formulas

k-DNF Preliminaries

- Term $T = a_1 \wedge \cdots \wedge a_k$: conjunction of literals
- DNF formula $D = T_1 \vee \cdots \vee T_m$: disjunction of terms
- k-DNF formula: DNF formula with terms of size $\leq k$
- Minimally unsatisfiable set of k-DNF formulas:
  - unsatisfiable, but
  - removing any formula makes rest of set satisfiable, and
  - weakening any term anywhere makes set satisfiable

Non-example:

$$(x \wedge y) \vee (x \wedge z)$$
$$(\overline{x} \wedge \overline{y}) \vee (\overline{x} \wedge \overline{z})$$

Example:

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Lower Bound on Minimally Unsatisfiable k-DNF sets

Observation (Ben-Sasson & Nordström '11)

Minimally unsatisfiable set of m k-DNFs can contain $k^2(m-1)$ variables

Proof:

- Take minimally unsatisfiable CNF with m clauses and $m-1$ variables
- Substitute every variable x with

$$(x_1 \wedge \cdots \wedge x_k) \vee (x_{k+1} \wedge \cdots \wedge x_{2k}) \vee \cdots \vee (x_{k^2-k+1} \wedge \cdots \wedge x_{k^2})$$

- Negation of this also expressible as k-DNF
- So expands every clause to a k-DNF formula for total of m formulas
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Minimally unsatisfiable set of m k-DNFs must contain $\leq (km)^{k+1}$ variables

Proof ($k = 2$):

- Suppose $8m^3$ variables; then $\exists$ 2-DNF D with $8m^2$ variables
- Other formulas satisfiable by setting $< 2m$ variables (one 2-term each)
- If D contains $2m$ 2-terms over disjoint variable sets then $\exists$ unset 2-term $\Rightarrow$ contradiction
- By counting $\exists$ literal a occuring in $(a \wedge b_1) \vee (a \wedge b_2) \vee \cdots (a \wedge b_{2m})$
- By minimality can satisfy all other formulas **and set a to true**
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An Obvious Open Question

Summing up, and fixing k for simplicity:

Minimally unsatisfiable k-DNF set with m formulas

- can contain $\Omega(m)$ variables
- must contain $\mathcal{O}(m^{k+1})$ variables

So what's the correct bound?

An Almost Tight Bound

Still don't know correct bound

But we almost close the gap

Improve lower bound from $\Omega(m)$ to $\Omega(m^k)$ variables (fixing k)

Theorem

Minimally unsatisfiable k-DNF sets with m formulas can contain $\geq \left(\frac{m}{4}\left(1-\frac{1}{k}\right)\right)^k \geq \left(\frac{m}{8}\right)^k$ variables

Only off by one in exponent compared to upper bound $\mathcal{O}(m^{k+1})$ (again fixing k)

An Almost Tight Bound

Still don't know correct bound

But we almost close the gap

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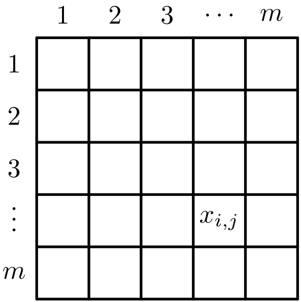
A First Naive Attempt (Again Focusing on $k = 2$)



Formula idea:

- variable $x_{i,j}$ for each cell
- $\exists$ row i with all $x_{i,j} = 1$
- $\exists$ column j with all $x_{i,j} = 0$

m^2 variables — want $\mathcal{O}(m)$ formulas

Naively:

$$\bigvee_{i=1}^{m} \bigwedge_{j=1}^{m} x_{i,j}$$

$$\bigvee_{j=1}^{m} \bigwedge_{i=1}^{m} \overline{x}_{i,j}$$

Great! Minimally unsatisfiable! And even constant # formulas!

Only one problem — these are m-DNFs. . .

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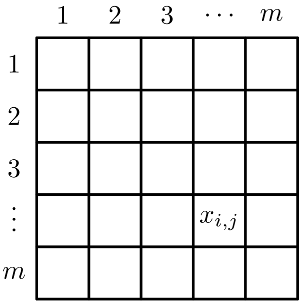

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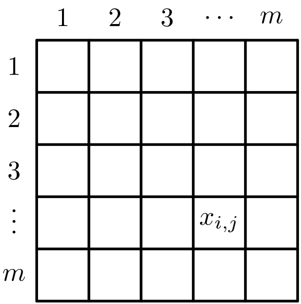

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The Trick

Introduce variables y_i and z_j for $i, j = 1, \ldots, m$

- y_i true if i "chosen row"
- z_j true if j "chosen column"

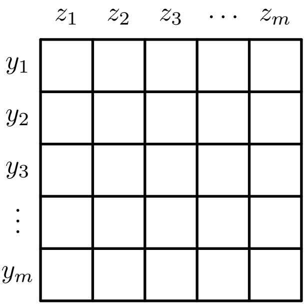
Can write $\mathcal{O}(m)$ clauses (with auxiliary variables) encoding that

- at most one y_i true
- at most one z_j true

Just take linear-sized circuits checking that

- $Weight(y_1, \ldots, y_m) \leq 1$
- $Weight(z_1, \ldots, z_m) \leq 1$

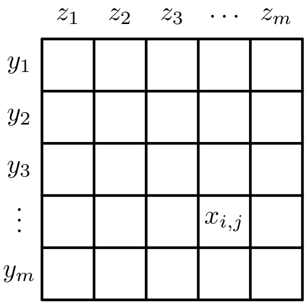
and convert to CNF



The Second (Successful) Attempt

Now take following sets of 2-DNFs

1. $Weight(y_1, \ldots, y_m) \leq 1$
2. $Weight(z_1, \ldots, z_m) \leq 1$
3. $\bigvee_{i=1}^{m}(y_i \wedge x_{i,j})$ for all $j = 1, \ldots, m$
4. $\bigvee_{j=1}^{m}(z_j \wedge \overline{x}_{i,j})$ for all $i = 1, \ldots, m$



3rd line says $\forall$ columns $\exists$ chosen row with 1 in that column

4th line says $\forall$ rows $\exists$ chosen column with 0 in that row

But chosen row and column unique by 1st and 2nd line

$\Rightarrow$ contradiction

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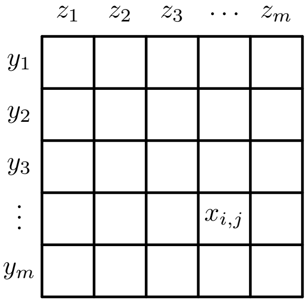

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Wrapping Up the Details

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This set of 2-DNFs:

- has $\mathcal{O}(m)$ formulas
- contains $\Omega(m^2)$ variables
- is minimally unsatisfiable

Proof: Too tedious. . .

General case

Don't know how to extend this construction to $k > 2$

But similar (more involved) ideas yield bound $\geq (m/8)^k$ for any k

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Believe right answer should be $\Theta(m^k)$

Nice combinatorial problem in its own right

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Thank you for your attention!