A (Biased) Proof Complexity Survey for SAT Practitioners

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SAT Solving and Proof Complexity

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  - Sometimes extended resolution
- How can we analyze the power of these methods?
  This is the research area of **proof complexity**

Outline of This Presentation

This talk: crash course in proof complexity

Focus on proof systems behind some current approaches to SAT solving:

- Conflict-driven clause learning — resolution
- Algebraic Gröbner basis computations — polynomial calculus
- Geometric pseudo-Boolean solvers — cutting planes
- Will also briefly mention extended resolution

Survey (some of) what is known about these proof systems

Show some of the "benchmark formulas" used

By necessity, selective and somewhat subjective coverage — apologies in advance for omissions

Some Notation and Terminology

- Literal a: variable x or its negation $\overline{x}$
- Clause $C = a_1 \lor \cdots \lor a_k$: disjunction of literals
  (Consider as sets, so no repetitions and order irrelevant)
- CNF formula $F = C_1 \land \cdots \land C_m$: conjunction of clauses
- k-CNF formula: CNF formula with clauses of size $\leq k$
  (where k is some constant)
- Mostly assume formulas k-CNFs (for simplicity of exposition)
  Conversion to 3-CNF (most often) doesn't change much
- N denotes size of formula (# literals, which is $\approx$ # clauses)

The Resolution Proof System

Goal: refute **unsatisfiable** CNF

Start with clauses of formula (axioms)

Derive new clauses by resolution rule

$$\frac{C \vee x \qquad D \vee \overline{x}}{C \vee D}$$

Refutation ends when empty clause $\bot$ derived

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Refutation ends when empty clause $\perp$ derived

1. $x \lor y$
2. $x \lor \overline{y} \lor z$
3. $\overline{x} \lor z$
4. $\overline{y} \lor \overline{z}$
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Can represent refutation as

- annotated list or
- DAG

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2.	$x \vee \overline{y} \vee z$	Axiom
3.	$\overline{x} \vee z$	Axiom
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5.	$\overline{x} \vee \overline{z}$	Axiom
6.	$x \vee \overline{y}$	Res$(2, 4)$
7.	x	Res$(1, 6)$
8.	$\overline{x}$	Res$(3, 5)$
9.	$\bot$	Res$(7, 8)$

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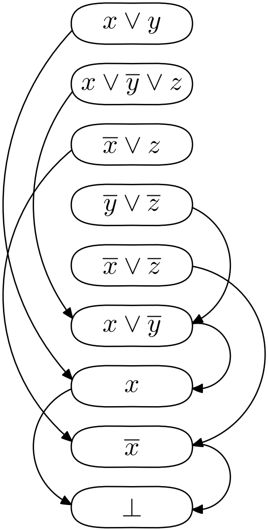

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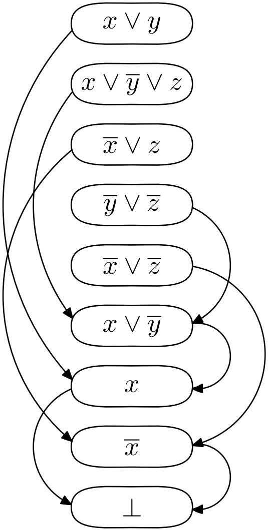
Tree-like resolution if DAG is tree



Resolution Size/Length

Size/length = # clauses in refutation

Most fundamental measure in proof complexity

Lower bound on CDCL running time
(can extract resolution proof from execution trace)

Never worse than $\exp(\mathcal{O}(N))$

Matching $\exp(\Omega(N))$ lower bounds known

Examples of Hard Formulas w.r.t Resolution Length (1/3)

Pigeonhole principle (PHP) [Hak85]*
"$n+1$ pigeons don't fit into n holes"

Variables $p_{i,j}$ = "pigeon i goes into hole j"

$p_{i,1} \lor p_{i,2} \lor \cdots \lor p_{i,n}$	every pigeon i gets a hole
$\overline{p}_{i,j} \lor \overline{p}_{i',j}$	no hole j gets two pigeons $i \neq i'$

Can also add "functionality" and "onto" axioms

$\overline{p}_{i,j} \lor \overline{p}_{i,j'}$	no pigeon i gets two holes $j \neq j'$
$p_{1,j} \lor p_{2,j} \lor \cdots \lor p_{n+1,j}$	every hole j gets a pigeon

Even onto functional PHP formula is hard for resolution
But only length lower bound $\exp\left(\Omega\left(\sqrt[3]{N}\right)\right)$ in terms of formula size

(*) A full list of references is given at the end of the slides

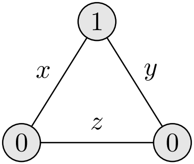
Examples of Hard Formulas w.r.t Resolution Length (2/3)

Tseitin formulas [Urq87]
"Sum of degrees of vertices in graph is even"

Variables = edges (in undirected graph of bounded degree)

- Label every vertex 0/1 so that sum of labels odd
- Write CNF requiring parity of edges around vertex = label

Requires length $\exp(\Omega(N))$ on well-connected so-called expanders



$$
\begin{array}{ll}
(x \lor y) & \land\ (\overline{x} \lor z) \\
\land\ (\overline{x} \lor \overline{y}) & \land\ (y \lor \overline{z}) \\
\land\ (x \lor \overline{z}) & \land\ (\overline{y} \lor z)
\end{array}
$$

Examples of Hard Formulas w.r.t Resolution Length (3/3)

Random k-CNF formulas [CS88]
Δn randomly sampled k-clauses over n variables

($\Delta \gtrsim 4.5$ sufficient to get unsatisfiable 3-CNF almost surely)

Again lower bound $\exp(\Omega(N))$

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And more...

- k-colourability [BCMM05]
- Independent sets and vertex covers [BIS07]
- Zero-one designs [Spe10, VS10, MN14]
- Et cetera...

Resolution Width

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Proof: at most $(2 \cdot \#\text{variables})^{\text{width}}$ distinct clauses
(This simple counting argument is essentially tight [ALN14])

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Width lower bound $\Rightarrow$ length lower bound

Much less obvious. . .

Width Lower Bounds Imply Length Lower Bounds

Theorem ([BW01])

$$\textit{length} \geq \exp\left(\Omega\left(\frac{\textit{width}^2}{\textit{formula size } N}\right)\right)$$

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- Almost all known lower bounds on length derivable via width

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For tree-like resolution have length $\geq 2^{\text{width}}$ [BW01]

General resolution: width up to $\mathcal{O}\left(\sqrt{N \log N}\right)$ implies no length lower bounds — possible to tighten analysis? **No!**

Optimality of the Length-Width Lower Bound

Ordering principles [Stå96, BG01]
"Every (partially) ordered set $\{e_1, \ldots, e_n\}$ has minimal element"

Variables $x_{i,j} =$ "$e_i < e_j$"

$\overline{x}_{i,j} \lor \overline{x}_{j,i}$	anti-symmetry; not both $e_i < e_j$ and $e_j < e_i$
$\overline{x}_{i,j} \lor \overline{x}_{j,k} \lor x_{i,k}$	transitivity; $e_i < e_j$ and $e_j < e_k$ implies $e_i < e_k$
$\bigvee_{1 \leq i \leq n,\, i \neq j} x_{i,j}$	e_j is not a minimal element

Can also add "total order" axioms

$x_{i,j} \lor x_{j,i}$	totality; either $e_i < e_j$ or $e_j < e_i$

Reuftable in resolution in length $\mathcal{O}(N)$
Requires resolution width $\Omega\left(\sqrt[3]{N}\right)$ (3-CNF version)

Resolution Space

Space $=$ max $\#$ clauses in memory when performing refutation

Motivated by SAT solver memory usage (but also intrinsically interesting for proof complexity)

Can be measured in different ways — focus here on most common measure clause space

Space at step t: $\#$ clauses at steps $\leq t$ used at steps $\geq t$

1.	$x \lor y$	Axiom
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Example: Space at step 7 . . .

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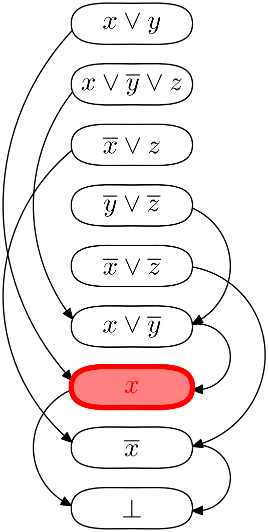

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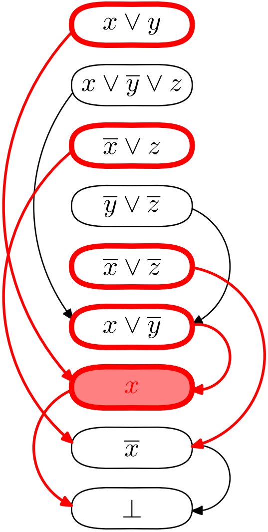
Example: Space at step 7 is 5



Bounds on Resolution Space

Space always at most $N + \mathcal{O}(1)$ [ET01]

Lower bounds for

- Pigeonhole principle [ABRW02, ET01]
- Tseitin formulas [ABRW02, ET01]
- Random k-CNFs [BG03]

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Results always matching width bounds

And proofs of very similar flavour... What is going on?

Space vs. Width

Theorem ([AD08])

$$\textit{space} \geq \textit{width} + \mathcal{O}(1)$$

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Are space and width asymptotically always the same? **No!**

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Pebbling formulas [BN08]

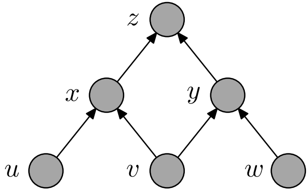
- Can be refuted in width $\mathcal{O}(1)$
- May require space $\Omega(N/\log N)$

A bit more involved to describe than previous benchmarks. . .

Pebbling Formulas: Vanilla Version

CNF formulas encoding so-called pebble games on DAGs

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



- sources are true
- truth propagates upwards
- but sink is false

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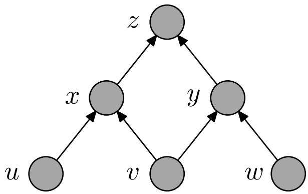

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Pebbling Formulas: Vanilla Version

CNF formulas encoding so-called pebble games on DAGs

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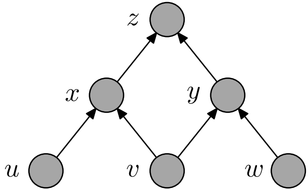

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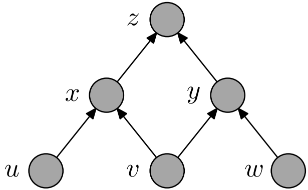

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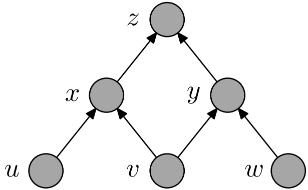

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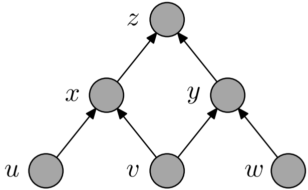

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Have been useful in proof complexity before in various contexts

Hope that pebbling properties of DAG somehow carry over to resolution refutations of pebbling formulas. **Except...**

Substituted Pebbling Formulas

Won't work — solved by unit propagation, so supereasy

Make formula harder by substituting $x_1 \oplus x_2$ for every variable x (also works for other Boolean functions with "right" properties):

$$\overline{x} \vee y$$

$$\Downarrow$$

$$\neg(x_1 \oplus x_2) \vee (y_1 \oplus y_2)$$

$$\Downarrow$$

$$\begin{aligned} & (x_1 \vee \overline{x}_2 \vee y_1 \vee y_2) \\ \wedge\ & (x_1 \vee \overline{x}_2 \vee \overline{y}_1 \vee \overline{y}_2) \\ \wedge\ & (\overline{x}_1 \vee x_2 \vee y_1 \vee y_2) \\ \wedge\ & (\overline{x}_1 \vee x_2 \vee \overline{y}_1 \vee \overline{y}_2) \end{aligned}$$

Now CNF formula inherits pebbling graph properties!

Space-Width Trade-offs

Given a formula easy w.r.t. these complexity measures, can refutations be optimized for two or more measures?

For space vs. width, the answer is a strong no

Theorem ([Ben09])

There are formulas for which

- *exist refutations in width $\mathcal{O}(1)$*
- *exist refutations in space $\mathcal{O}(1)$*
- *optimization of one measure causes (essentially) worst-case behaviour for other measure*

Holds for vanilla version of pebbling formulas

Length-Space Trade-offs

Theorem ([BN11, BBI12, BNT13])

There are formulas for which

- *exist refutations in short length*
- *exist refutations in small space*
- *optimization of one measure* causes *dramatic blow-up for other measure*

Holds for

- Substituted pebbling formulas over the right graphs
- Tseitin formulas over long, narrow rectangular grids

So no meaningful simultaneous optimization possible for length and space in the worst case

Length-Width Trade-offs?

What about length versus width? [BW01] transforms short refutation to narrow one, but blows up length exponentially

- Is this blow-up inherent?
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Very recent news (solved after problem was advertised at *SAT '14*):

Theorem ([Tha14])

There are formulas for which

- *exist refutations in short length*
- *exist refutations in small width*
- *optimization of one measure causes dramatic blow-up for other measure*

Minor issue: formulas have logarithmic width — would like k-CNFs

Recap of Complexity Measures for Resolution

Recall that N = size of formula

Length

clauses in refutation — at most $\exp(N)$

Width

Size of largest clause in refutation — at most N

Space

Max # clauses one needs to remember when "verifying correctness of refutation" — at most N (!)

Proof Complexity Measures and CDCL Hardness

Recall $\log(\text{length}) \lesssim \text{width} \lesssim \text{space}$

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- CDCL polynomially simulates resolution [PD11]
- But short proofs may be worst-case intractable to find [AR08]

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- Searching in small width known heuristic in AI community
- Small width $\Rightarrow$ CDCL solver will run fast [AFT11]

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Space

- In practice, memory consumption important bottleneck
- Space complexity gives lower bound on clause database size
- Plus assumes solver knows **exactly** which clauses to keep $\Rightarrow$ in reality, probably (much) more memory needed

Relations Between Theoretical and Practical Hardness?

1. Are width or even space lower bounds relevant indicators of CDCL hardness?
2. Or is it true in practice that CDCL does essentially as well as resolution w.r.t. length/running time?
3. Can CDCL even do as well as resolution w.r.t. time and space simultaneously?

Relations Between Theoretical and Practical Hardness?

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3. Can CDCL even do as well as resolution w.r.t. time and space simultaneously?

Not mathematically well-defined questions. . .

But perhaps still possible to perform experiments and draw interesting conclusions?

Some preliminary work along these lines — see slides from talk on Monday Feb 2 at `http://www.csc.kth.se/~jakobn/research/`

Practical Conclusions So Far?

- No firm conclusions — messy reality not easily captured by nice theories
- CDCL performance on combinatorial benchmarks sometimes surprising; e.g.:
  - For PHP, worse behaviour with heuristics than without
  - Sometimes "easy" formulas harder than "hard" ones?! [MN14]
  - Sometimes small changes in VSIDS decay factor makes all the difference between supereasy and totally impossible

Open Problems

- *Could explanations of above phenomena help us understand CDCL better?*
- *Could controlled experiments on easily scalable theoretical benchmarks yield other interesting insights?*

Polynomial Calculus (or Actually PCR)

Introduced in [CEI96]; below modified version from [ABRW02]

Clauses interpreted as polynomial equations over finite field
Any field in theory; $\mathrm{GF}(2)$ in practice
Example: $x \vee y \vee \overline{z}$ gets translated to $xy\overline{z} = 0$
(Think of $0 \equiv true$ and $1 \equiv false$)

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(Think of $0 \equiv \mathit{true}$ and $1 \equiv \mathit{false}$)

Derivation rules

Boolean axioms $\dfrac{}{x^2 - x = 0}$

Negation $\dfrac{}{x + \overline{x} = 1}$

Linear combination $\dfrac{p = 0 \qquad q = 0}{\alpha p + \beta q = 0}$

Multiplication $\dfrac{p = 0}{xp = 0}$

Goal: Derive $1 = 0 \Leftrightarrow$ no common root $\Leftrightarrow$ formula unsatisfiable

Size, Degree and Space

Write out all polynomials as sums of monomials
W.l.o.g. all polynomials multilinear (because of Boolean axioms)

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Write out all polynomials as sums of monomials
W.l.o.g. all polynomials multilinear (because of Boolean axioms)

Size — analogue of resolution length
total # monomials in refutation (counted with repetitions)
Can also define length measure — might be much smaller

Degree — analogue of resolution width
largest degree of monomial in refutation

(Monomial) space — analogue of resolution (clause) space
max # monomials in memory during refutation (with repetitions)

Polynomial Calculus Simulates Resolution

Polynomial calculus can simulate resolution proofs efficiently with respect to length/size, width/degree, and space simultaneously

- Can mimic resolution refutation step by step
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Example: Resolution step:

$$\frac{x \lor \overline{y} \lor z \qquad \overline{y} \lor \overline{z}}{x \lor \overline{y}}$$

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$$\frac{x \lor \overline{y} \lor z \qquad \overline{y} \lor \overline{z}}{x \lor \overline{y}}$$

simulated by polynomial calculus derivation:

$$\dfrac{x\overline{y}z = 0 \qquad \dfrac{\dfrac{\overline{y}\,\overline{z} = 0}{x\overline{y}\,\overline{z} = 0} \qquad \dfrac{\dfrac{z + \overline{z} - 1 = 0}{\overline{y}z + \overline{y}\,\overline{z} - \overline{y} = 0}}{x\overline{y}z + x\overline{y}\,\overline{z} - x\overline{y} = 0}}{-x\overline{y}z + x\overline{y} = 0}}{x\overline{y} = 0}$$

Polynomial Calculus Strictly Stronger than Resolution

Polynomial calculus strictly stronger w.r.t. size and degree

- Tseitin formulas on expanders (just do Gaussian elimination)
- Onto functional pigeonhole principle [Rii93]

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Open Problem

Show that polynomial calculus is strictly stronger than resolution w.r.t. space

Size vs. Degree

- Degree upper bound $\Rightarrow$ size upper bound [CEI96]
  Qualitatively similar to resolution bound
  A bit more involved argument
  Again essentially tight by [ALN14]
- Degree lower bound $\Rightarrow$ size lower bound [IPS99]
  Precursor of [BW01] — can do same proof to get same bound
- Size-degree lower bound essentially optimal [GL10]
  Example: again ordering principle formulas
- Most size lower bounds for polynomial calculus derived via degree lower bounds (but machinery less developed)

Examples of Hard Formulas w.r.t. Size (and Degree)

Pigeonhole principle formulas

Follows from [AR03]

Earlier work on other encodings in [Raz98, IPS99]

Hard even with functionality axioms added [MN15]

Tseitin formulas with "wrong modulus"

Can define Tseitin-like formulas counting $\bmod p$ for $p \neq 2$

Hard if $p \neq$ characteristic of field [BGIP01]

Random k-CNF formulas

Hard in all characteristics except 2 [BI99]

Lower bound for all characteristics in [AR03]

Bounds on Polynomial Calculus Space

Lower bound for PHP with wide clauses [ABRW02]

k-CNFs much trickier — sequence of lower bounds for

- Obfuscated 4-CNF versions of PHP [FLN+12]
- Random 4-CNFs [BG13]
- Tseitin formulas in 4-CNF on (some) expanders [FLM+13]
- Random 3-CNFs [BGHW14] (but bound is log factor off)

Open Problems

- *Prove space lower bounds for Tseitin on any expander*
- *Prove tight lower bounds on random 3-CNFs*
- *Prove space lower bound on 3-CNF version of PHP formulas*

Space vs. Degree

Open Problem (analogue of [AD08])

Is it true that space $\geq$ degree $+ \mathcal{O}(1)$?

Partial progress: if formula requires large resolution width, then XOR-substituted version requires large space [FLM+13]

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Partial progress: if formula requires large resolution width, then XOR-substituted version requires large space [FLM^{+}13]

Optimal separation of space and degree in [FLM^{+}13] using flavour of Tseitin formulas which

- can be refuted in degree $\mathcal{O}(1)$
- require space $\Omega(N)$
- but separating formulas depend on characteristic of field

Open Problem

Prove space lower bounds for substituted pebbling formulas (would give space-degree separation independent of characteristic)

Trade-offs for Polynomial Calculus

- Strong, essentially optimal space-degree trade-offs [BNT13]
  Same vanilla pebbling formulas as for resolution
  Same parameters

- Strong size-space trade-offs [BNT13]
  Same formulas as for resolution
  Some loss in parameters

Open Problem

Are there size-degree trade-offs in polynomial calculus?

[Tha14] works only for resolution (so far)

Algebraic SAT Solvers?

- Quite some excitement about Gröbner basis approach to SAT solving after [CEI96]
- Promise of performance improvement failed to deliver
- Meanwhile: the CDCL revolution...

Algebraic SAT Solvers?

- Quite some excitement about Gröbner basis approach to SAT solving after [CEI96]
- Promise of performance improvement failed to deliver
- Meanwhile: the CDCL revolution...
- Some current SAT solvers do Gaussian elimination, but this is only very limited form of polynomial calculus
- Is it harder to build good algebraic SAT solvers, or is it just that too little work has been done (or both)?
- Some shortcut seems to be needed — full Gröbner basis computation does too much work

Cutting Planes

Introduced in [CCT87]

Clauses interpreted as linear inequalities over the reals with integer coefficients

Example: $x \lor y \lor \overline{z}$ gets translated to $x + y + (1 - z) \geq 1$

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Derivation rules

Variable axioms $\dfrac{}{0 \leq x \leq 1}$

Multiplication $\dfrac{\sum a_i x_i \geq A}{\sum c a_i x_i \geq cA}$

Addition $\dfrac{\sum a_i x_i \geq A \quad \sum b_i x_i \geq B}{\sum (a_i + b_i) x_i \geq A + B}$

Division $\dfrac{\sum c a_i x_i \geq A}{\sum a_i x_i \geq \lceil A/c \rceil}$

Goal: Derive $0 \geq 1 \Leftrightarrow$ formula unsatisfiable

Size, Length and Space

Length = total # lines/inequalities in refutation

Size = sum also size of coefficients

Space = max # lines in memory during refutation

No (useful) analogue of width/degree

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- is strictly stronger w.r.t. length/size — can refute PHP efficiently [CCT87]

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Hard Formulas w.r.t Cutting Planes Length

Clique-coclique formulas [Pud97]
"A graph with a k-clique is not $(k-1)$-colourable"

Lower bound via interpolation and circuit complexity

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Lower bound via interpolation and circuit complexity

Open Problems

Prove length lower bounds for cutting planes

- *for Tseitin formulas*
- *for random k-CNFs*
- *for any formula using other technique than interpolation*

Size-Space Trade-offs for Cutting Planes?

- Short cutting planes refutations of Tseitin formulas on expanders require large space [GP14]
  (But such short refutations probably don't exist anyway)
- Short cutting planes refutations of (some) pebbling formulas require large space [HN12, GP14] (such refutations exist)

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Open Problems

- *Are there trade-offs where the space-efficient CP refutations have small coefficients? (Say, of polynomial size)*
- *Are there space lower bounds for CP refutations with polynomial-size coefficients?*
- *Already coefficients of absolute size* ≤ 2 *quite powerful — can refute PHP formulas [GPT14]*

Geometric SAT Solvers?

- Some work on pseudo-Boolean solvers using (subset of) cutting planes
- Seems hard to make competitive with CDCL on CNFs
- One key problem to recover cardinality constraints
- **But...** If cardinality constraints can be detected, then solvers can do really well (at least on combinatorial benchmarks)
- E.g., PHP formulas and also zero-one design formulas become easy [BBLM14]

Building SAT Solvers on Extended Resolution?

- Resolution + introduce new variables to name subformulas
- Without restrictions, corresponds to extended Frege
- Extremely strong — pretty much no lower bounds known
- In order to study extended resolution, would need to:
  - Describe heuristics/rules actually used
  - See if possible to reason about such restricted proof system

Summing up

- Overview of resolution, polynomial calculus and cutting planes (More details in *SAT '14* proceedings [Nor14] or survey [Nor13])
- Resolution fairly well understood
- Polynomial calculus less so
- Cutting planes almost not at all
- Could there be interesting connections between proof complexity measures and hardness of SAT?
- How can we build efficient SAT solvers on stronger proof systems than resolution?

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Thank you for your attention!

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