Trade-offs Between Time and Memory in a Tighter Model of CDCL SAT Solvers

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Joint work with Jan Elffers, Jan Johannsen, Massimo Lauria, Thomas Magnard, and Marc Vinyals

What This Work Is About

The unreasonable effectiveness of SAT solvers

- The Boolean satisfiability problem (SAT) is NP-complete and so should be exponentially hard
- Yet current state-of-the-art conflict-driven clause learning (CDCL) SAT solvers can deal with formulas containing millions of variables
- How can they work so well? What are the limits to what they can do?

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This work

- Driving motivation: **Understand the power of CDCL**
- Tool: **Proof complexity** (don't have much else for rigorous analysis)

What This Talk Is About

- Report on results so far
- Definitely more of "work in progress" than The Final Answer$^{\text{TM}}$
- Also take the opportunity to give my take on some work at intersection of SAT solving and proof complexity
- Believe there is room for improved mutual understanding — hope to stimulate discussions that can remove some misconceptions

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- But CDCL searches for proofs with very special structure — can it match resolution upper bounds?

(*) Ignores preprocessing — our focus on CDCL proof search
Will be happy to elaborate offline on why this is reasonable simplification

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(*) [AFT11] and [PD11] independent but essentially equivalent works
Can use techniques in either paper to establish results in the other

Why Not Completely Happy with [AFT11, PD11]? (1/2)

Learning scheme

- Learned clause assertive but otherwise adversarially chosen
- Very strong aspect of result
- But does not come for free — costs a lot for efficiency of simulation

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Restart policy

- Restarts are *not too frequent* (unless you think Luby is too frequent)
- But no progress at all in between restarts
- Restarts seem important, but not quite *that* important?!

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Decision strategy

- In [PD11], crucially relies on (unknown) resolution proof
- In [AFT11], crucially needs to be (essentially totally) random
- Probably inherent — fully constructive proof search likely to be computationally intractable [AR08]

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Simulation efficiency

- CDCL solvers typically have to run in (close to) linear time $\mathcal{O}(n)$
- But simulation will yield something like $\mathcal{O}(n^5)$ running time

What We Want

More fine-grained and realistic CDCL model. . .

- Capture restarts, clause learning, memory management, et cetera
- Modular design to allow study of different features
- Theoretical analogue of projects in [KSM11, SM11, ENSS16]

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. . . Leading to improved theoretical insights

- Can CDCL proof search be time *and space* efficient?
- And can it be *really* efficient? (No polynomial blow-ups)
- How does memory management affect proof search quality?
- Do restarts increase reasoning power? (Or just a helpful heuristic?)
- How do other heuristics help or hinder proof search?

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(*) So if you see any issues with the model, we definitely want to know
Obviously, must abstract away some features, but we feel we capture the essentials

Some Notation and Terminology

- Literal a: variable x or its negation $\overline{x}$ (or $\neg x$)
- Clause $C = a_1 \vee \cdots \vee a_k$: disjunction of literals
  (Consider as sets, so no repetitions and order irrelevant)
- CNF formula $F = C_1 \wedge \cdots \wedge C_m$: conjunction of clauses
- N denotes size of formula (# literals counted with repetitions)
- $\mathcal{O}(f(N))$ grows at most as quickly as $f(N)$ asymptotically
  $\Omega(g(N))$ grows at least as quickly as $g(N)$ asymptotically

The Resolution Proof System

Goal: refute **unsatisfiable** CNF

Start with clauses of formula (axioms)

Derive new clauses by resolution rule

$$\frac{C \lor x \qquad D \lor \overline{x}}{C \lor D}$$

Proof ends when empty clause $\bot$ derived

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4.	$\overline{y} \vee \overline{z}$	Axiom
5.	$\overline{x} \vee \overline{z}$	Axiom
6.	$x \vee \overline{y}$	Res$(2,4)$
7.	x	Res$(1,6)$
8.	$\overline{x}$	Res$(3,5)$
9.	$\bot$	Res$(7,8)$

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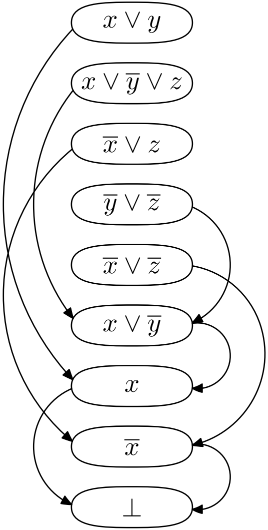

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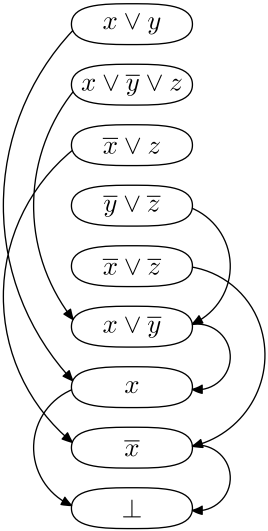

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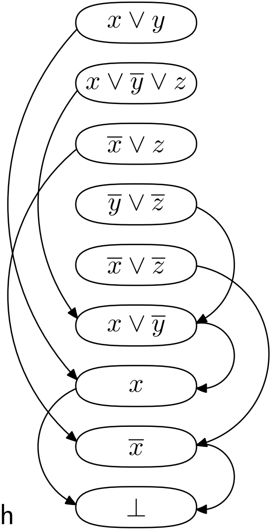
Tree-like if DAG is tree
Regular if resolved variables don't repeat on path



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Size/length of proof = # clauses (9 in example on previous slide)
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Most fundamental measure in proof complexity

Lower bound on CDCL running time
(can extract resolution proof from execution trace)

Never worse than $\exp(\mathcal{O}(N))$

Matching $\exp(\Omega(N))$ lower bounds known [Urq87, CS88, BW01]

Resolution Space

Space = max # clauses in memory when performing refutation

Motivated by SAT solver memory usage (but also intrinsically interesting for proof complexity)

Can be measured in different ways — makes most sense here to focus on clause space

Space at step t = # clauses at steps $\leq t$ used at steps $\geq t$

1.	$x \lor y$	Axiom
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Example: Space at step 7 ...

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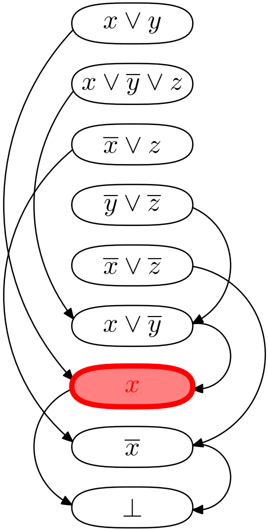

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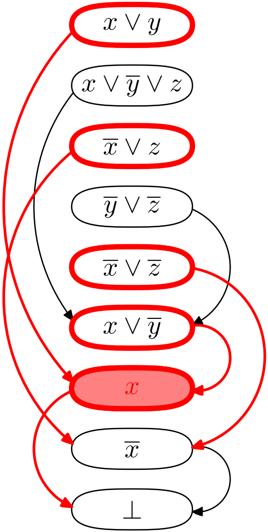

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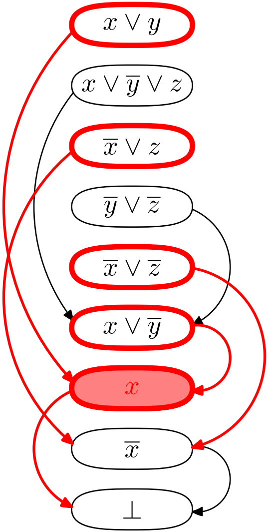

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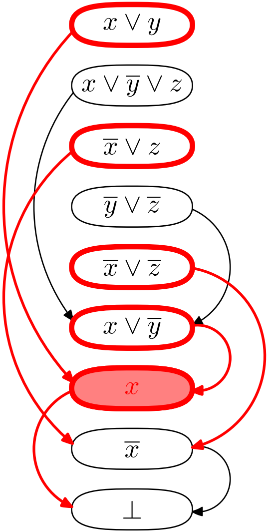
Space of proof $=$ max over all steps
Space of refuting F $=$ min over all proofs



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Space always at most $N + \mathcal{O}(1)$ (!) [ET01]

Matching $\Omega(N)$ lower bounds known [ABRW02, BG03, ET01]

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- So significantly more space might be needed in practice
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Which leads to a natural question...

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Length $\approx$ running time
Space $\approx$ memory consumption
SAT solvers aggressively try to minimize both — is this possible?

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Theorem ([BN11, BBI12, BNT13])

There are formulas for which

- *exist refutations in short length*
- *exist refutations in small space*
- *optimization of one measure* causes *dramatic blow-up for other measure*

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At least for resolution proofs — but what about CDCL proof search?

CDCL Model (1/2)

Trail: a stack of decisions $x_i \stackrel{\mathsf{d}}{=} b$ and unit propagations $x_i \stackrel{C}{=} b$

$$\Big(\underbrace{x_7 \stackrel{\mathsf{d}}{=} 0}_{\text{dec. level 1}}, \underbrace{x_2 \stackrel{\mathsf{d}}{=} 1, x_{12} \stackrel{C_1}{=} 0}_{\text{dec. level 2}}, \underbrace{x_6 \stackrel{\mathsf{d}}{=} 1, x_4 \stackrel{C_2}{=} 1, x_1 \stackrel{C_3}{=} 0}_{\text{dec. level 3}}, \underbrace{x_{11} \stackrel{\mathsf{d}}{=} 0, x_{59} \stackrel{C_4}{=} 1}_{\text{dec. level 4}} \Big)$$

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else solver is in stable state;

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else solver is in stable state; do in sequence:

1. decide whether to restart, i.e., set trail to $()$;
2. decide whether to apply database reduction to $\mathcal{D}$;
3. move to **Decision**

CDCL Model (2/2)

Unit Arbitrarily pick clause $C \in \mathcal{D}$ unit w.r.t. trail
Add propagated assignment $x \stackrel{C}{=} b$ to trail
Move to **Default**

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Conflict **If** trail contains no decisions, output `UNSAT`;
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- apply learning scheme to derive asserting clause C;
- backjump, i.e., remove decision levels $>$ assertion level of C from trail;
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Decision Use decision scheme to add decision $x \stackrel{\mathsf{d}}{=} b$ to trail
Move to **Default**

CDCL Model (2/2)

Unit Arbitrarily pick clause $C \in \mathcal{D}$ unit w.r.t. trail
Add propagated assignment $x \stackrel{C}{=} b$ to trail
Move to **Default**

Conflict **If** trail contains no decisions, output `UNSAT`;
else

- apply learning scheme to derive asserting clause C;
- backjump, i.e., remove decision levels $>$ assertion level of C from trail;
- move to **Unit**

Decision Use decision scheme to add decision $x \stackrel{\mathsf{d}}{=} b$ to trail
Move to **Default**

Model draws heavily on [AFT11, PD11]
Combined with ideas from [BHJ08] to capture memory and restarts

CDCL Cannot Do Better than Resolution

Theorem

CDCL with "standard" learning scheme (e.g., UIP) decides F in time τ and space $s \Rightarrow F$ has resolution proof in length $\leq \tau$ and space $\leq s + \mathcal{O}(1)$

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Means that lower bounds in resolution trade-offs automatically carry over
But can CDCL find time-efficient and space-efficient proofs?

Time-Space Trade-Offs for CDCL (in Math Notation)

We obtain CDCL analogues of (almost all) trade-off results in [BN11, BBI12, BNT13] — here is one sample:

Theorem (slightly informal)

For your favourite $k \in \mathbb{N}^+$ $\exists$ explicit formulas F_N of size $\approx N$ such that

- *CDCL with 1UIP learning and no restarts can decide F_N in time $\mathcal{O}(N^k)$ and space $\mathcal{O}(N^k)$*
- *CDCL with 1UIP learning and no restarts can decide F_N in space $\mathcal{O}(\log^2 N)$ and time $N^{\mathcal{O}(\log N)}$*
- *For any CDCL run in time τ and space s using any learning scheme and restart policy it holds that $\tau \gtrsim (N^k/s)^{\Omega(\log\log N/\log\log\log N)}$*

Time-Space Trade-Offs for CDCL (in English)

Rephrasing theorem on previous slide to convey high-level message:

- The formulas F_N are somewhat tricky (require more than linear time)
- CDCL can solve them efficiently for generous memory management (even without restarts)
- But more aggressive clause erasure policy (such as current MiniSat or Glucose defaults) cause superpolynomial blow-up in running time

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Interpretation:

- This is only a mathematical theorem about asymptotic behaviour for theoretical benchmarks
- But have some indications of similar behaviour for scaled-down versions in practical experiments [ENSS16]

Proof Plan for CDCL Simulation of Resolution

General idea is obvious:

- Given resolution proof $(C_1, C_2, \ldots, C_\tau)$
- Force solver to efficiently learn C_t for $t = 1, 2, 3, \ldots$
- Make sure clause database size $\approx$ space of proof at all times

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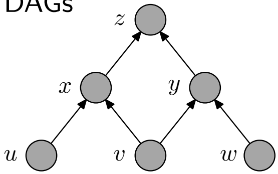
Not as easy as it seems...

- Unit propagation + clause database cause problems
- Suppose have $C \vee x$ and $D \vee \overline{x}$ and now want to learn $C \vee D$
- Easy: decide to make $C \vee D$ false $\Rightarrow$ conflict on x
- But clauses in database can propagate "wrong values"
  $\Rightarrow$ proof search veers off in different direction

Illustrate on One of Benchmarks: Pebbling Formulas

CNF formulas encoding so-called pebble games on DAGs

1. $u_1 \oplus u_2$
2. $v_1 \oplus v_2$
3. $w_1 \oplus w_2$
4. $(u_1 \oplus u_2) \wedge (v_1 \oplus v_2) \rightarrow (x_1 \oplus x_2)$
5. $(v_1 \oplus v_2) \wedge (w_1 \oplus w_2) \rightarrow (y_1 \oplus y_2)$
6. $(x_1 \oplus x_2) \wedge (y_1 \oplus y_2) \rightarrow (z_1 \oplus z_2)$
7. $\neg(z_1 \oplus z_2)$



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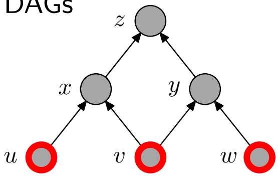

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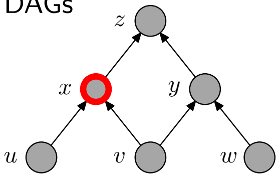

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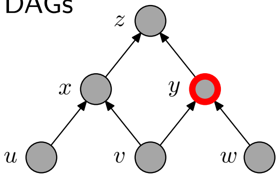

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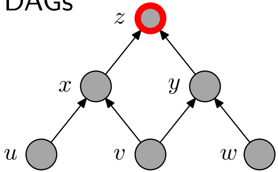

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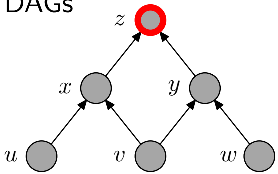

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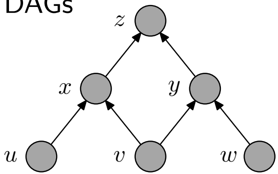

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Rewrite, e.g., $(x_1 \oplus x_2) \rightarrow (y_1 \oplus y_2)$ in CNF as

$$(x_1 \vee \overline{x}_2 \vee y_1 \vee y_2) \wedge (x_1 \vee \overline{x}_2 \vee \overline{y}_1 \vee \overline{y}_2)$$
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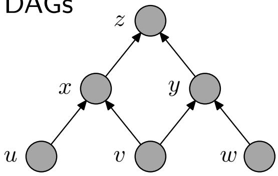

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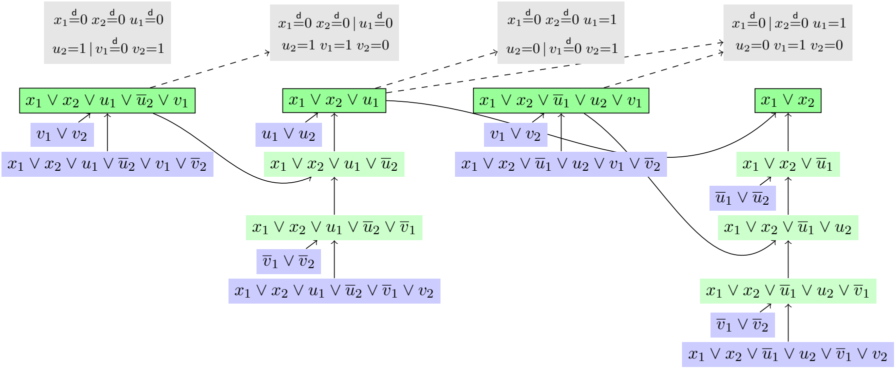
Pebble game trade-offs ⇒ resolution size-space trade-offs [BN08, BN11]

Why Life Without Restarts Might Be Tricky



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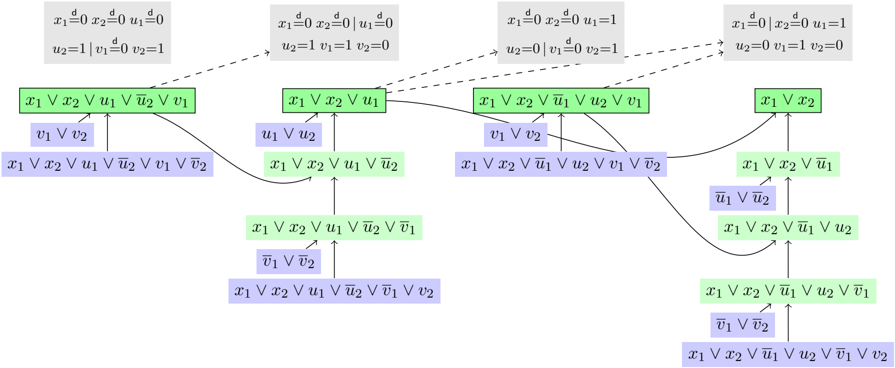

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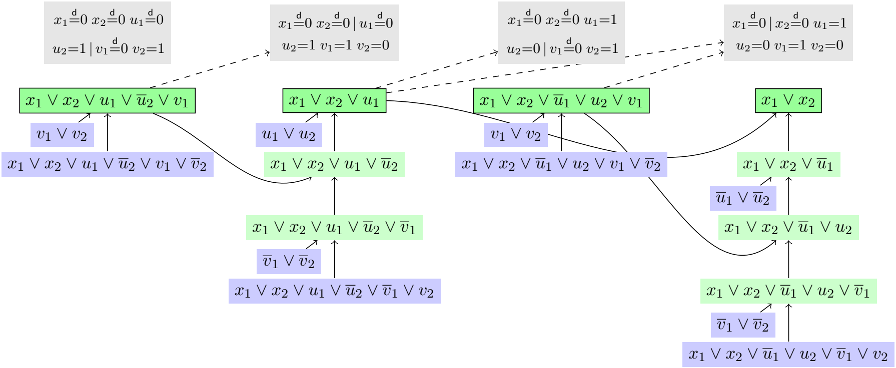

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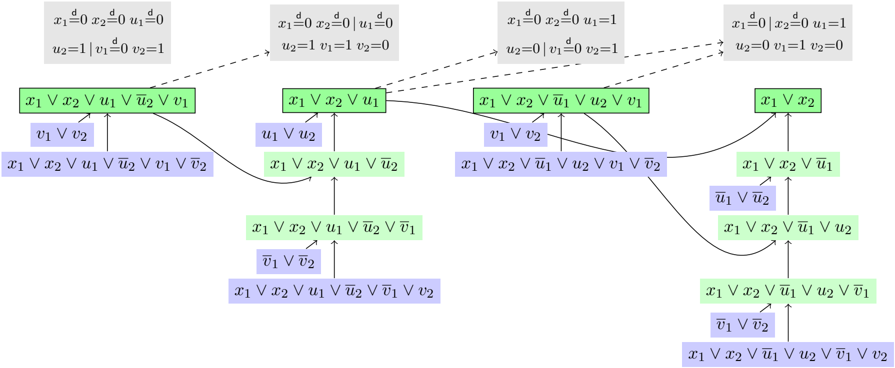

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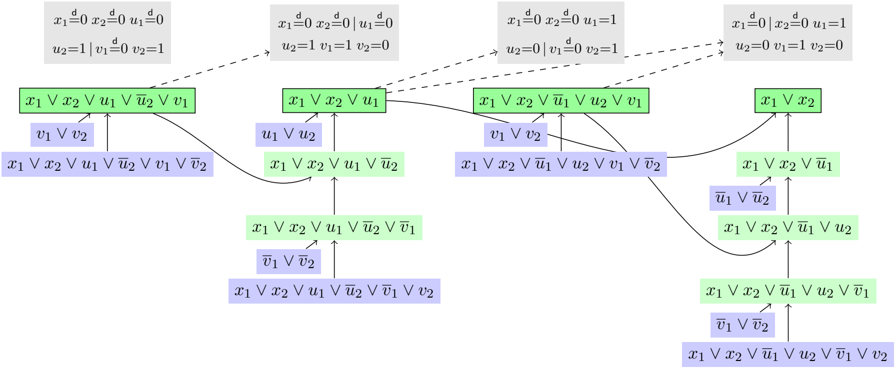

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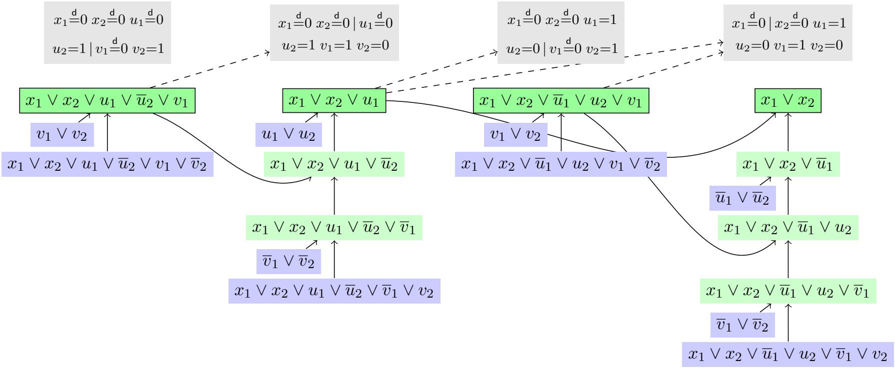

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- Easy with restarts — major pain without…

Open Problems

CDCL vs. resolution

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Theoretical study of power (or weakness) of other heuristics

- How do other heuristics help or hinder proof search?
- Does LBD (literal block distance) measure identify important clauses?
- Prove that VSIDS (variable state independent decaying sum) sometimes goes terribly wrong? (See this on some theory benchmarks)

Summing up This Presentation

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Thank you for your attention!

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