Proof Logging for Combinatorial Optimization Using Pseudo-Boolean Reasoning

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and Lund University

Reunion workshop for
“Satisfiability: Theory, Practice, and Beyond”
Simons Institute of the Theory of Computing
June 15, 2022

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- "$25957 = 101 \cdot 257$; check yourself that these are primes."
  Concise! Primality easy to check [Mil76, Rab80, AKS04]

Key demand: Proofs should be short but efficiently verifiable

Proof System

Proof system for formal language L (adapted from [CR79]):

Deterministic algorithm $P(x, \pi)$ that runs in time polynomial in $|x|$ and $|\pi|$ such that

- for all $x \in L$ there is a string π (a proof) such that $P(x, \pi) = 1$,
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Note that proof π can be very large compared to x
Only have to achieve polynomial time in $|x| + |\pi|$

The Success of Combinatorial Solving and Optimization

- Rich field of math and computer science
- Impact in other areas of science and also industry:
  - airline scheduling
  - logistics
  - hardware verification
  - donor-recipients matching for kidney transplants [MO12, BvdKM$^+$21]
- Typically very challenging problems (NP-complete or worse)
- Lots of effort last decades into developing sophisticated so-called combinatorial solvers that often work surprisingly well in practice
  - Boolean satisfiability (SAT) solving [BHvMW21]
  - Constraint programming [RvBW06]
  - Mixed integer linear programming [AW13, BR07]
  - Satisfiability modulo theories (SMT) solving [BHvMW21]

The Dirty Little Secret...

- Solvers very fast, but sometimes wrong (even best commercial ones) [BLB10, CKSW13, AGJ+18, GSD19, GS19]
- Even worse: No way of knowing for sure when errors happen
- How to check the absence of solutions?
- Or that a solution is optimal?
- And solvers even get feasibility of solutions wrong (though this should be straightforward!)

What Can Be Done About Solver Bugs?

- **Software testing**
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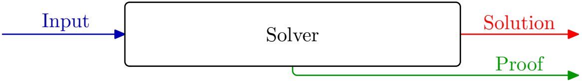
  Current techniques cannot scale to this level of complexity
  (Though there is some exciting recent work on SAT solvers [Fle20])
- **Proof logging**
  Make solver certifying [ABM$^+$11, MMNS11] by outputting
  1. not only solution but also
  2. simple, machine-verifiable proof that solution is correct

Proof Logging with Certifying Solvers: Workflow

1. Run solver on problem input

Proof Logging with Certifying Solvers: Workflow



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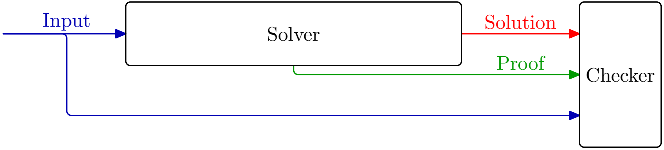

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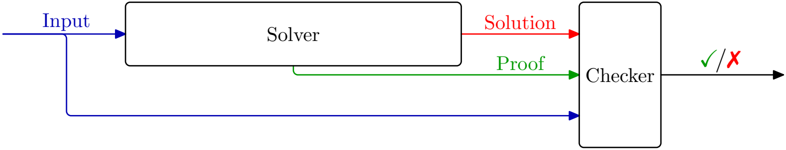

1. Run solver on problem input
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3. Feed input + solution + proof to proof checker
4. Verify that proof checker says solution is correct and/or optimal

Proof Logging with Certifying Solvers: Requirements

Proofs produced by certifying solver [ABM$^+$11, MMNS11] should

- be based on very simple rules
- be powerful enough to allow proof logging with minimal overhead
- not require knowledge of inner workings of solver
- allow verification by stand-alone (formally verified) proof checker

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Does not prove solver correct, but proves solution correct

The Sales Pitch for Proof Logging

1. ***Certifies correctness*** of solutions
2. ***Detects errors*** even if due to compiler bugs, hardware failures, or cosmic rays
3. Provides ***debugging support*** during development [EG21, GMM+20, KM21]
4. Facilitates ***performance analysis***
5. Helps identify potential for ***further improvements***
6. Enables ***auditability***
7. Serves as stepping stone towards ***explainability***

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But has remained out of reach for stronger paradigms
And, in fact, even for some advanced SAT solving techniques

Outline of This Talk

1. Introduction
    - Proofs
    - Combinatorial Solving and Optimization
    - Proof Logging
2. Basic SAT Solving
    - CDCL by Example
    - Resolution
    - Extension Rules
3. Advanced SAT Techniques
    - Cardinality Constraints and Pseudo-Boolean Reasoning
    - Translating Pseudo-Boolean Constraints to CNF
    - Parity Reasoning
4. Beyond SAT
    - Symmetry, Dominance, and Optimization
    - Constraint Programming
    - Further Challenges

A Quick Recap of Modern SAT Solving

DPLL method [DP60, DLL62]

- Assign values to variables (in some smart way)
- Backtrack when conflict with falsified clause

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Variable Assignments

Two kinds of assignments — illustrate on example formula:

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

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Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

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Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

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decision level 1

$\boxed{q \overset{\mathsf{d}}{=} 0}$

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decision level 2

$\boxed{x \overset{\mathsf{d}}{=} 0}$

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decision level 3

Decision

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Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

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decision level 2

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$$\left.\begin{array}{c} \boxed{p \stackrel{\mathsf{d}}{=} 0} \\ u \stackrel{p \vee \overline{u}}{=} 0 \end{array}\right\} \text{decision level } 1$$

$$\left.\begin{array}{c} \boxed{q \stackrel{\mathsf{d}}{=} 0} \\ r \stackrel{q \vee r}{=} 1 \\ w \stackrel{\overline{r} \vee w}{=} 1 \end{array}\right\} \text{decision level } 2$$

$$\left.\begin{array}{c} \boxed{x \stackrel{\mathsf{d}}{=} 0} \\ y \stackrel{u \vee x \vee y}{=} 1 \\ z \stackrel{x \vee \overline{y} \vee z}{=} 1 \\ \stackrel{\overline{y} \vee \overline{z}}{\bot} \end{array}\right\} \text{decision level } 3$$

Could backtrack by removing last decision level & flipping last decision

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$\boxed{p \overset{\mathsf{d}}{=} 0}$
$u \overset{p \vee \overline{u}}{=} 0$
decision level 1

$\boxed{q \overset{\mathsf{d}}{=} 0}$
$r \overset{q \vee r}{=} 1$
$w \overset{\overline{r} \vee w}{=} 1$
decision level 2

$\boxed{x \overset{\mathsf{d}}{=} 0}$
$y \overset{u \vee x \vee y}{=} 1$
$z \overset{x \vee \overline{y} \vee z}{=} 1$
$\overset{\overline{y} \vee \overline{z}}{\bot}$
decision level 3

Could backtrack by removing last decision level & flipping last decision

But want to learn from conflict and cut away as much of search space as possible

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

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Case analysis over z for last two clauses:

- $x \lor \overline{y} \lor z$ wants $z = 1$
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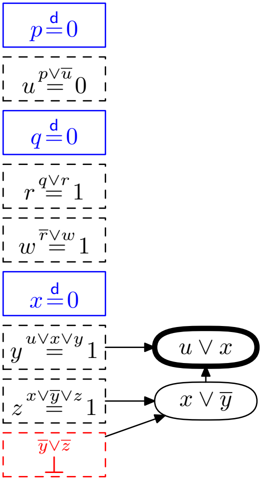

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Repeat until UIP clause with only 1 variable after last decision — learn and backjump

Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

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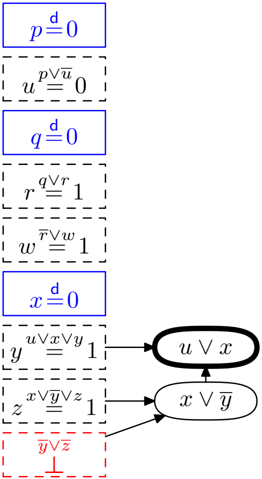
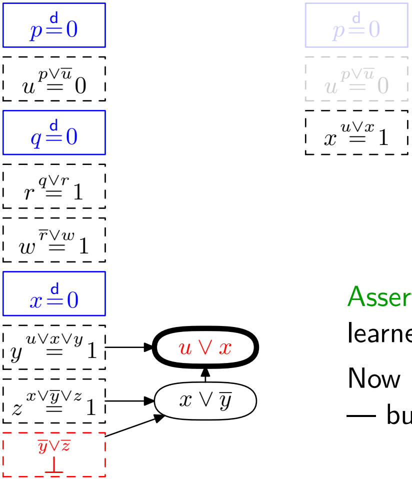


Assertion level 1 (max for non-UIP literal in learned clause) — trim trail to that level

Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$



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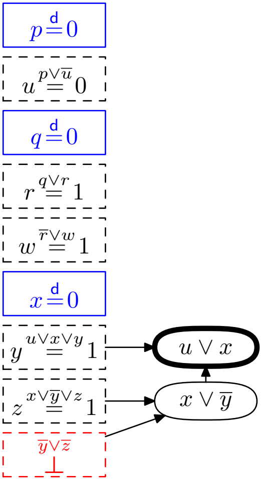

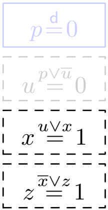

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Then continue as before...

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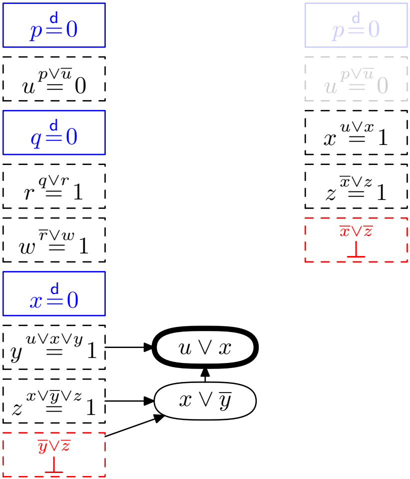

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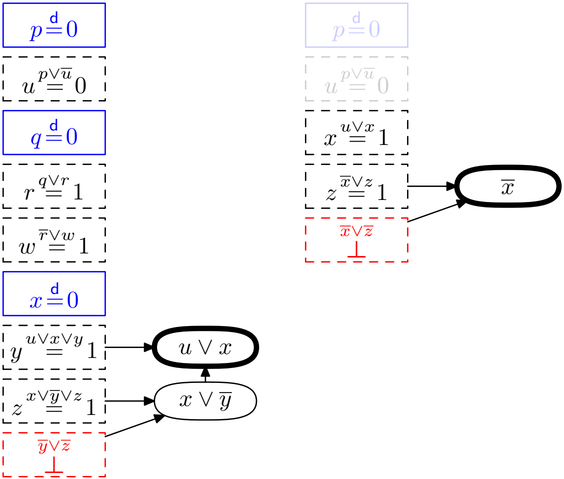

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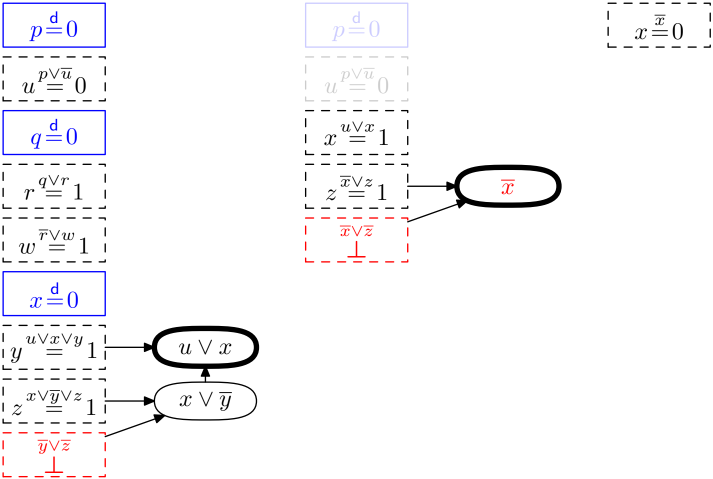

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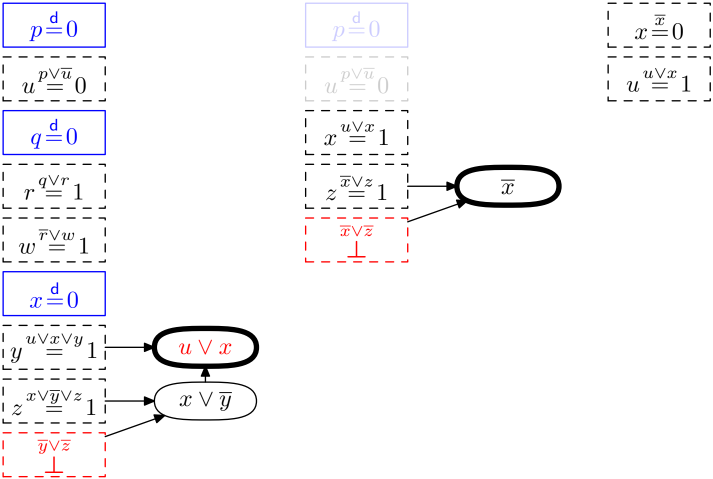

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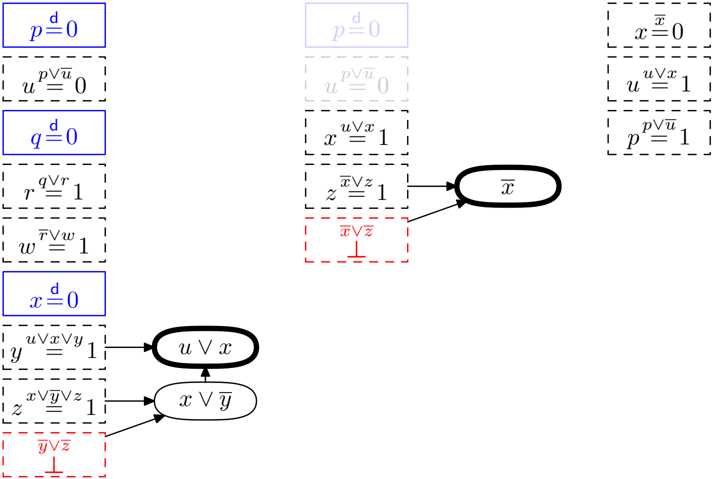

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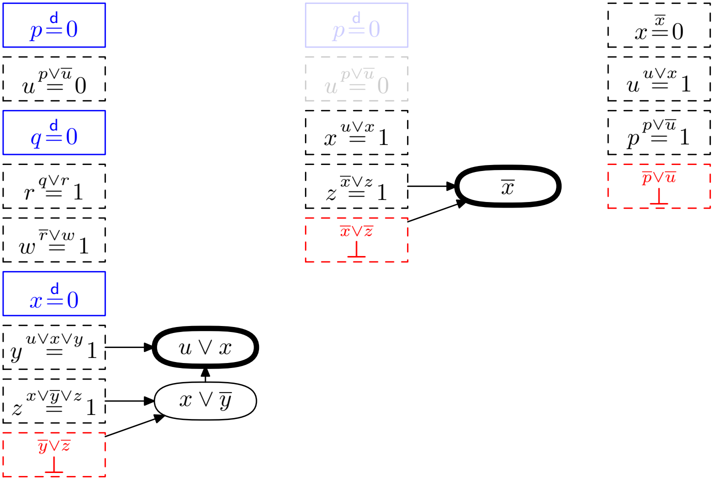

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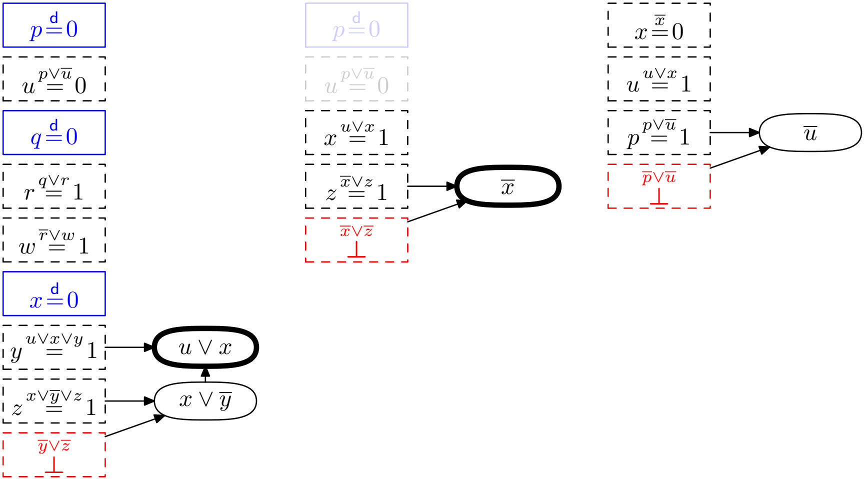

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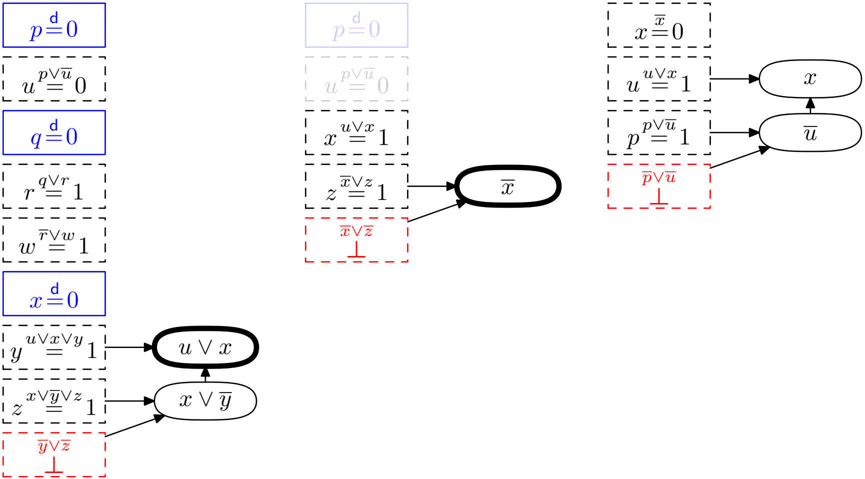

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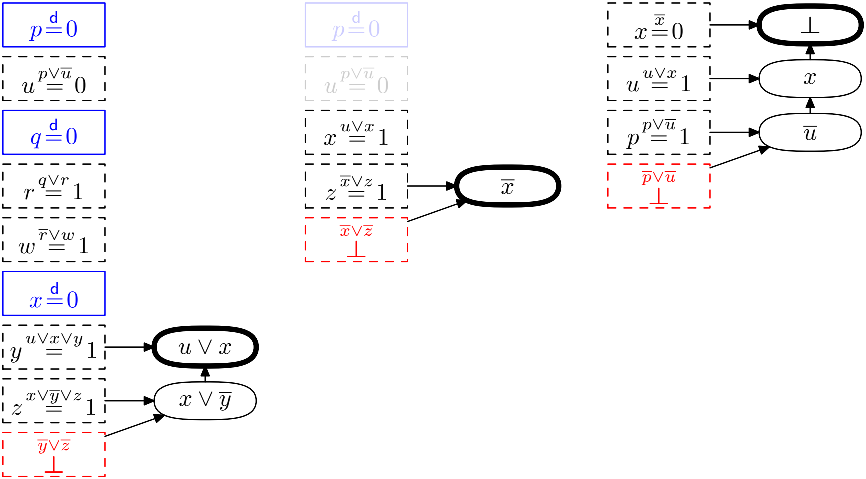

CDCL Reasoning and the Resolution Proof System

For CDCL proof logging, need proof system for unsatisfiable formulas

Focus on underlying method of reasoning

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- Derive new clauses by **resolution rule**

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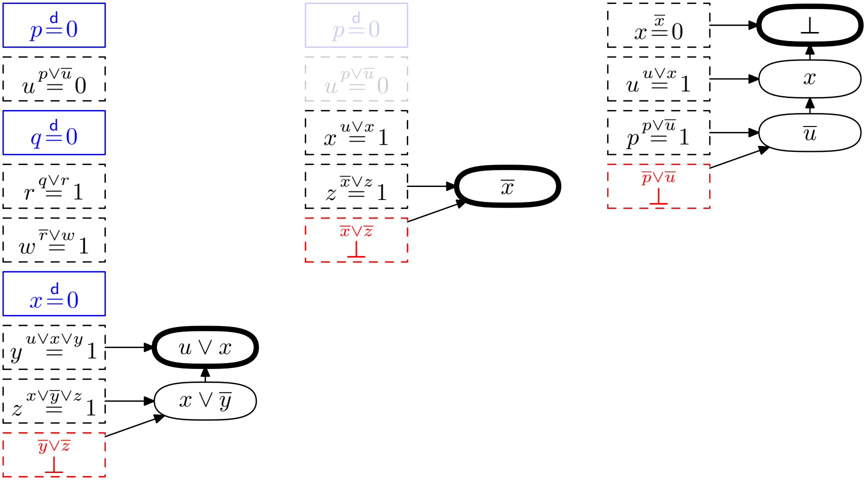
Obtain resolution proof from our example CDCL execution...



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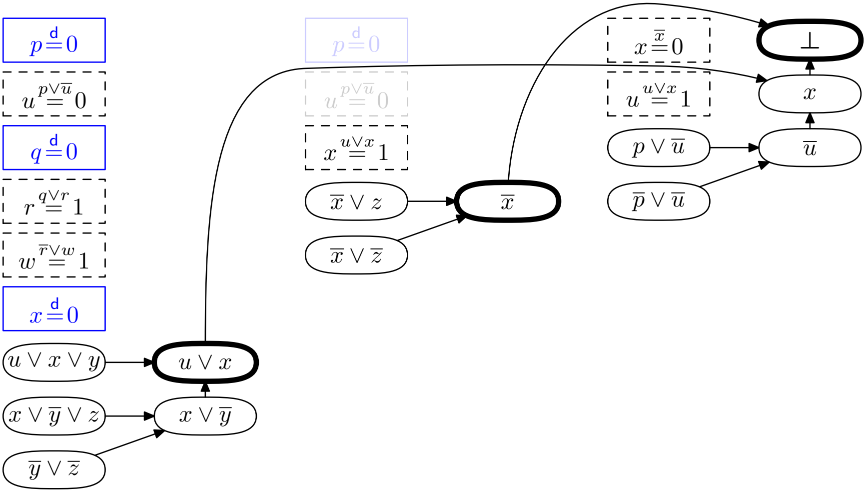
Obtain resolution proof from our example CDCL execution by stringing together conflict analyses:



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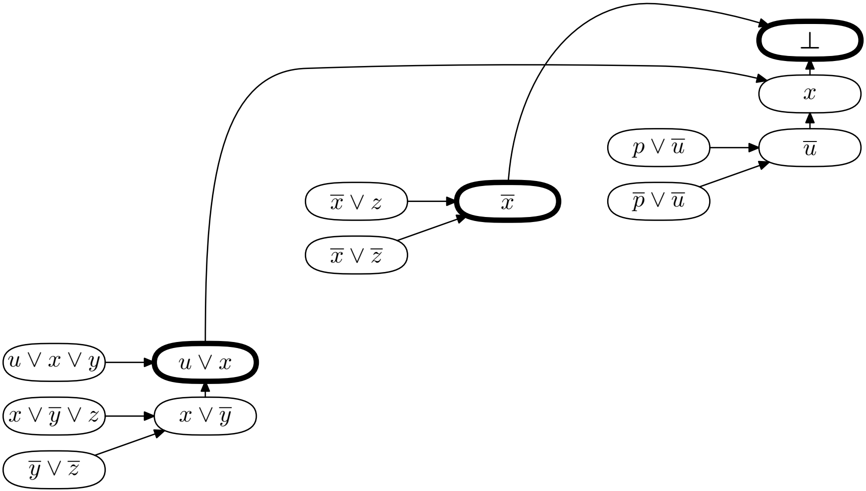

Reverse Unit Propagation (RUP)

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Fact

All clauses learned by CDCL solver are RUP clauses

RUP Proofs

So shorter proof of unsatisfiability for

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

is sequence of reverse unit propagation (RUP) clauses

1. $u \vee x$
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More concise, but requires a little bit more trust
Namely in correct unit propagation

Extension Variables and Redundant Clauses

Say we want new, fresh variable a encoding

$$a \leftrightarrow (x \land y)$$

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CDCL pre- and inprocessing techniques can do steps like this

But resolution proof system cannot certify such derivations (by definition)

Redundance-Based Strengthening

- C is redundant with respect to F if F and $F \wedge C$ are equisatisfiable
- Adding redundant clauses should be OK
- Notions such as RAT [JHB12] and propagation redundancy [HKB17]

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- Proof sketch for interesting direction: If α satisfies F but falsifies C, then $\alpha \circ \omega$ satisfies $F \wedge C$
- Implication should be efficiently verifiable (which is the case, e.g., if all clauses in $(F \wedge C)\restriction_\omega$ are RUP)

Deriving $a \leftrightarrow (x \wedge y)$ Using the Redundance Rule

Want to derive

$$a \vee \overline{x} \vee \overline{y} \qquad \overline{a} \vee x \qquad \overline{a} \vee y$$

using redundance-based strengthening condition $F \wedge \neg C \models (F \wedge C)\restriction_\omega$

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Proof Logging for State-of-the-Art SAT Solving

Resolution + redundance rule is as strong as extended Frege proof system

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Except we really care about efficiency, and for some important advanced techniques don't know of efficient enough proof logging methods

Reasoning with Cardinality Constraints

Given clauses

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$$x_1 \vee x_2 \vee x_4$$
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How provide proof logging for reasoning with such cardinality constraints?

Can solve pigeonhole principle efficiently, which is exponentially hard for basic CDCL [Hak85, BKS04]

Implemented in LINGELING [Lin], but not with DRAT proof logging
Resolution + extension rule can do it in theory, but efficiently in practice?!

Pseudo-Boolean Constraints

Pseudo-Boolean constraints are 0-1 integer linear constraints

$$\sum_i a_i \ell_i \geq A$$

- $a_i, A \in \mathbb{Z}$
- literals ℓ_i: x_i or $\overline{x}_i$ (where $x_i + \overline{x}_i = 1$)
- as before, variables x_i take values $0 = \mathit{false}$ or $1 = \mathit{true}$

Some Types of Pseudo-Boolean Constraints

1. Clauses

$$x \lor \overline{y} \lor z \quad \Leftrightarrow \quad x + \overline{y} + z \geq 1$$

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3. General pseudo-Boolean constraints

$$x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$$

Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

Literal axioms $$\frac{}{\ell_i \geq 0}$$

Linear combination $$\frac{\sum_i a_i \ell_i \geq A \qquad \sum_i b_i \ell_i \geq B}{\sum_i (c_A a_i + c_B b_i) \ell_i \geq c_A A + c_B B} \quad [c_A, c_B \in \mathbb{N}]$$

Division $$\frac{\sum_i c a_i \ell_i \geq A}{\sum_i a_i \ell_i \geq \lceil A/c \rceil} \quad [c \in \mathbb{N}^+]$$

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Lin comb $\dfrac{2x + 4y + 2z + w \geq 5 \qquad 2x + y + w \geq 2}{}$

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Recovering cardinality constraints from CNF

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Pseudo-Boolean constraints

$$x_1 + x_2 + x_3 \geq 1$$
$$x_1 + x_2 + x_4 \geq 1$$
$$x_1 + x_3 + x_4 \geq 1$$
$$x_2 + x_3 + x_4 \geq 1$$

Add all up

$$3x_1 + 3x_2 + 3x_3 + 3x_4 \geq 4$$

and divide by 3 to get

$$x_1 + x_2 + x_3 + x_4 \geq 2$$

Extended Cutting Planes

Combine cutting planes method with **redundance** rule

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Combine cutting planes method with **redundance** rule

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Add constraint C to formula F if exists witness substitution ω such that

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- Cutting planes can do efficiently anything that resolution can do
- Reverse unit propagation works also for 0-1 linear inequalities
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$\Rightarrow$ Strict extension of DRAT

- Lifts reasoning from clauses to 0-1 inequalities (still simple objects)
- Implemented in proof checker VeriPB [Ver]
- Yields surprisingly expressive proof logging system

CDCL Solvers on Pseudo-Boolean Inputs

Can re-encode to CNF and run CDCL:

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to clauses with extension variables

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VERIPB can certify pseudo-Boolean-to-CNF rewriting [GMNO22]

Parity (XOR) Reasoning

Given clauses

$$x \lor y \lor z$$
$$x \lor \overline{y} \lor \overline{z}$$
$$\overline{x} \lor y \lor \overline{z}$$
$$\overline{x} \lor \overline{y} \lor z$$

and

$$y \lor z \lor w$$
$$y \lor \overline{z} \lor \overline{w}$$
$$\overline{y} \lor z \lor \overline{w}$$
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want to derive

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This is just parity reasoning:

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want to derive

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This is just parity reasoning:

$$x + y + z = 1 \pmod 2$$
$$y + z + w = 1 \pmod 2$$

imply

$$x + w = 0 \pmod 2$$

Parity (XOR) Reasoning

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Could add XORs to language, but prefer to keep things super-simple and verifiable...

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("$=$" syntactic sugar for "$\geq$" plus "$\leq$")

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VeriPB can certify XOR reasoning [GN21]

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Symmetric learning

- Allow to add all symmetric versions of learned clause [DBB17]
- Recently proposed proof logging in [TD20]
  1. Special-purpose, specific approach
  2. Requires adding explicit concept of symmetries
  3. Not compatible with preprocessing techniques

  Better to keep proof system super-simple and verifiable...

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Deal with symmetries by switching focus to optimization
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Spoiler alert:
For decision problem, nothing stops us from inventing objective function
(like lexicographic order $\sum_{i=1}^{n} 2^i \cdot x_i$)

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Add constraint C to formula F if exists witness substitution ω such that

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8. Can't go on forever, so finally reach α' satisfying $F \land D$

Strength of Dominance Rule

Dominance-based strengthening (stronger, still simplified) [BGMN22]

If $D_1, D_2, \ldots, D_{m-1}$ have been derived from F (maybe using dominance), then can derive also D_m if exists witness substitution ω such that

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- More in Stephan Gocht's talk (after lunch)

Proof Logging for Constraint Programming

Pseudo-Boolean proof logging can also certify reasoning in

- subgraph solvers [GMN20, GMM$^+$20]
- general constraint programming solvers [EGMN20, GMN22]

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Further Challenges Beyond SAT

Proof logging for combinatorial optimization

- Symmetric learning and recycling (substitution) of subproofs
- Maximum satisfiability (MaxSAT) and pseudo-Boolean optimization
- Mixed integer linear programming (some work in [CGS17, EG21])
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- Certified explanations for decisions or classifications?
- Proofs of neural network robustness to limited input perturbations?
- Rigorous symbolic reasoning about fairness?

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- Lots of challenging problems and interesting ideas

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- We're hiring! Talk to me to join the proof logging revolution!

Summing up

- Combinatorial solving and optimization is a true success story
- But ensuring correctness is a crucial, and not yet satisfactorily addressed, concern
- Certifying solvers producing machine-verifiable proofs of correctness seems like a promising approach
- Well established for Boolean satisfiability (SAT) solving, but even there advanced techniques have remained out of reach
- Cutting planes reasoning with pseudo-Boolean constraints seems to hit a sweet spot between simplicity and expressivity

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Thank you for your attention!

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