Solving Logic Formulas in Linear Time

Jakob Nordström

KTH Royal Institute of Technology
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Datalogisk Institut på Københavns Universitet
April 13, 2018

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(At Least Surprisingly Often)

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This Is Me...

Jakob Nordström

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. . . And This Is What I Do for a Living

$(x_{1,1} \lor x_{1,2} \lor x_{1,3} \lor x_{1,4} \lor x_{1,5} \lor x_{1,6} \lor x_{1,7}) \land (x_{2,1} \lor x_{2,2} \lor x_{2,3} \lor x_{2,4} \lor x_{2,5} \lor x_{2,6} \lor x_{2,7}) \land (x_{3,1} \lor x_{3,2} \lor x_{3,3} \lor x_{3,4} \lor x_{3,5} \lor x_{3,6} \lor x_{3,7}) \land (x_{4,1} \lor x_{4,2} \lor x_{4,3} \lor x_{4,4} \lor x_{4,5} \lor x_{4,6} \lor x_{4,7}) \land (x_{5,1} \lor x_{5,2} \lor x_{5,3} \lor x_{5,4} \lor x_{5,5} \lor x_{5,6} \lor x_{5,7}) \land (x_{6,1} \lor x_{6,2} \lor x_{6,3} \lor x_{6,4} \lor x_{6,5} \lor x_{6,6} \lor x_{6,7}) \land (x_{7,1} \lor x_{7,2} \lor x_{7,3} \lor x_{7,4} \lor x_{7,5} \lor x_{7,6} \lor x_{7,7}) \land (x_{8,1} \lor x_{8,2} \lor x_{8,3} \lor x_{8,4} \lor x_{8,5} \lor x_{8,6} \lor x_{8,7}) \land (\overline{x}_{1,1} \lor \overline{x}_{2,1}) \land (\overline{x}_{1,1} \lor \overline{x}_{3,1}) \land (\overline{x}_{1,1} \lor \overline{x}_{4,1}) \land (\overline{x}_{1,1} \lor \overline{x}_{5,1}) \land (\overline{x}_{1,1} \lor 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$$(u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{u} \lor w) \land (\overline{u} \lor \overline{w})$$

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- Mentioned already in Gödel's famous letter in 1956 to von Neumann (the "father of computer science")
- Intense research in theoretical computer science ever since early 1970s
- Now one of Millennium Prize Problems in mathematics

. . . with Huge Practical Implications

- Many problems can be encoded as logic formulas, e.g.:
  - hardware verification
  - software testing
  - artificial intelligence
  - cryptography
  - bioinformatics
  - et cetera. . .
- Leads to **humongous** formulas (100,000s or 1,000,000s of variables)
- Dramatic progress last 15–20 years on so-called SAT solvers
  Today routinely used to solve large-scale real-world problems
- But. . . There are also small formulas (just ∼100 variables) that are completely beyond reach of even the very best SAT solvers

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(*) Obviously, can't give all details in 15 minutes, but aim to cover essentials

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To understand how large this number is, consider that even if every atom in the known universe was a modern supercomputer running at full speed ever since the beginning of time some 13.7 billion years ago, all of them together would only have covered a completely negligible fraction of these cases by now. So we really would not have time to wait for them to finish...

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Want more refined case analysis over variable assignments

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Two kinds of assignments — illustrate on our example formula:

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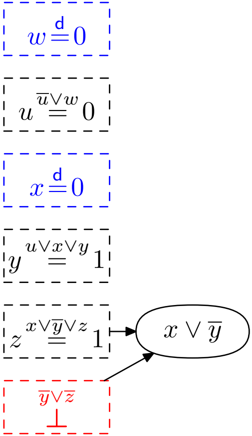

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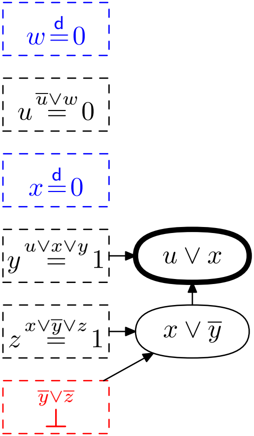
Case analysis over z for last two clauses:

- $x \lor \overline{y} \lor z$ wants $z = 1$
- $\overline{y} \lor \overline{z}$ wants $z = 0$
- Merge & remove z — must satisfy $x \lor \overline{y}$

Conflict-Driven Clause Learning

Time to analyse this conflict!

$(u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{u} \lor w) \land (\overline{u} \lor \overline{w})$



Could backtrack by flipping last decision

But want to learn from conflict and cut away as much of search space as possible

Case analysis over z for last two clauses:

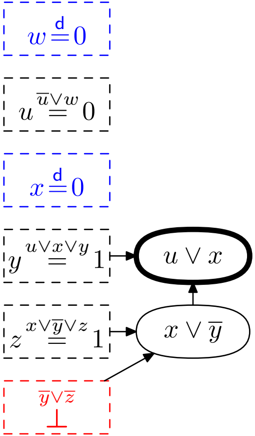
- $x \lor \overline{y} \lor z$ wants $z = 1$
- $\overline{y} \lor \overline{z}$ wants $z = 0$
- Merge & remove z — must satisfy $x \lor \overline{y}$

Repeat until only 1 variable after last decision — learn that clause and backjump

Complete Example of CDCL Execution

Backjump: roll back max # assignments so that last variable still flips

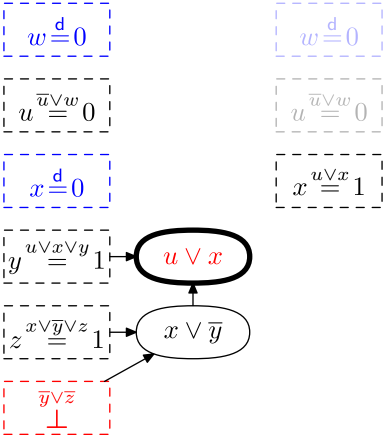
$(u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{u} \lor w) \land (\overline{u} \lor \overline{w})$



Complete Example of CDCL Execution

Backjump: roll back max # assignments so that last variable still flips

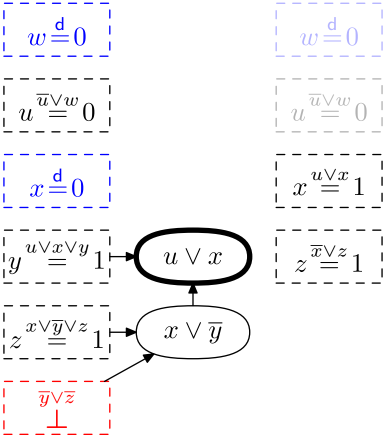
$(u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{u} \vee w) \wedge (\overline{u} \vee \overline{w})$



Complete Example of CDCL Execution

Backjump: roll back max # assignments so that last variable still flips

$(u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{u} \lor w) \land (\overline{u} \lor \overline{w})$



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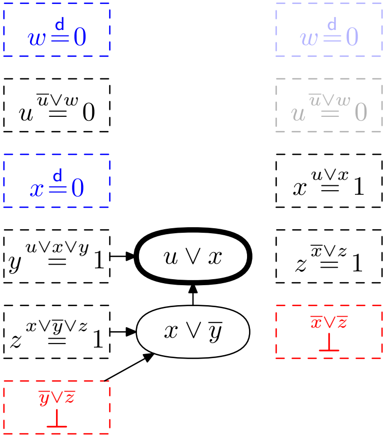

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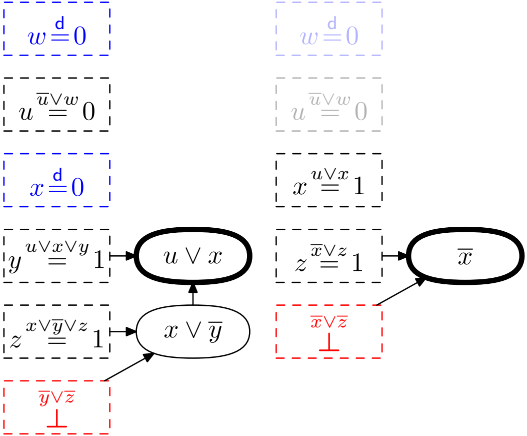
Backjump: roll back max # assignments so that last variable still flips

$(u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{u} \lor w) \land (\overline{u} \lor \overline{w})$



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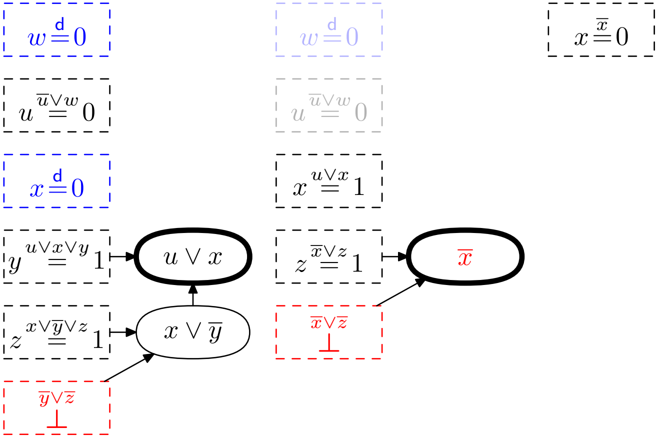
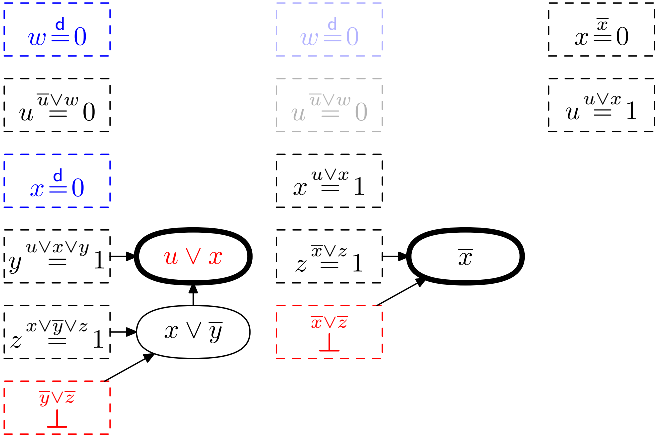


Complete Example of CDCL Execution

Backjump: roll back max # assignments so that last variable still flips

$$(u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{u} \lor w) \land (\overline{u} \lor \overline{w})$$



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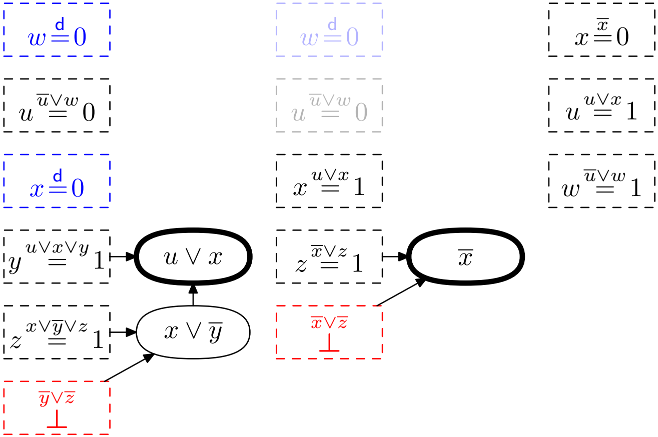

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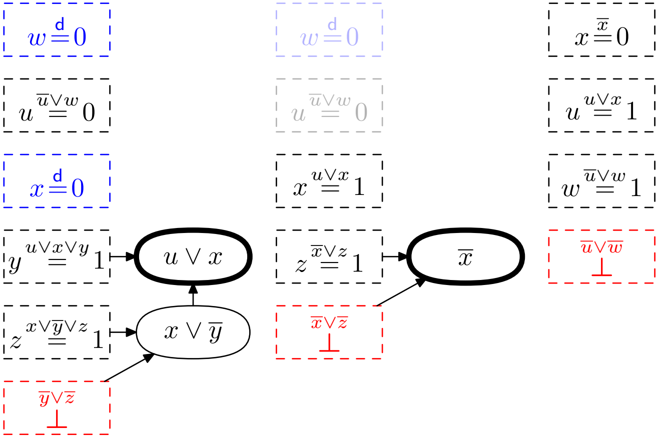

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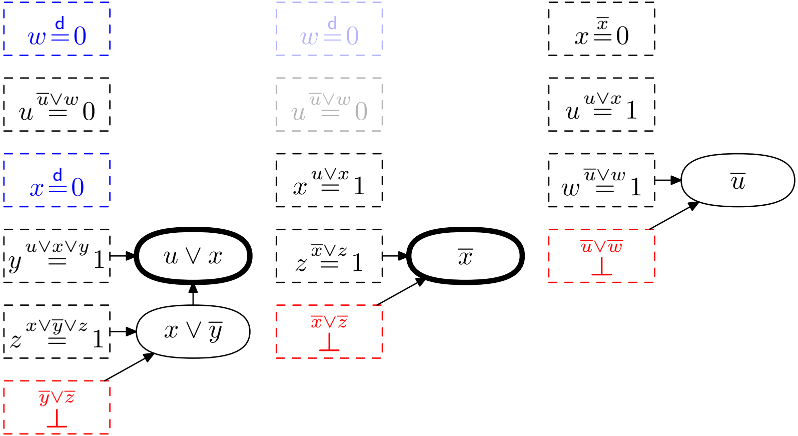

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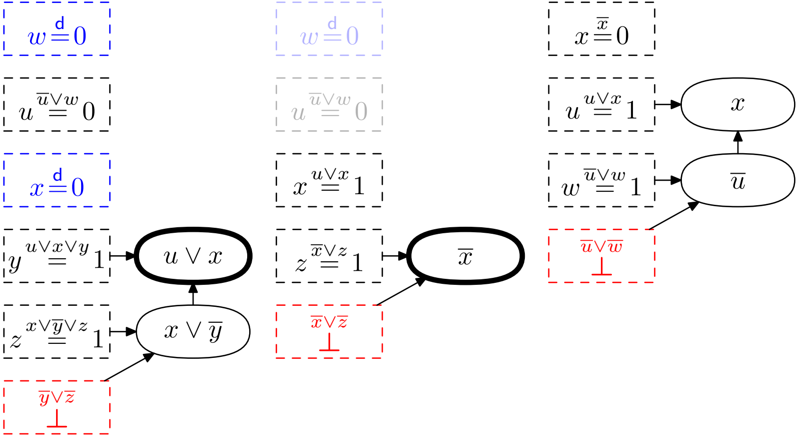

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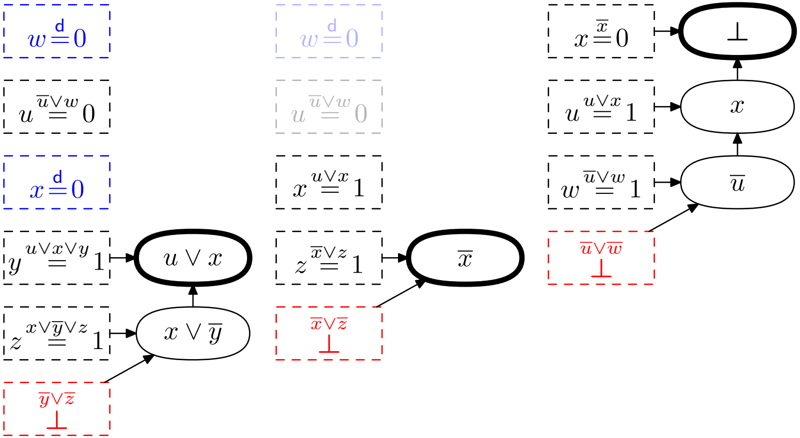

State-of-the-Art SAT Solving in One Slide

repeat
  if current assignment falsifies clause
    if no decisions made
      terminate with output `UNSATISFIABLE`
    apply learning scheme to add new clause & backjump
  else if all variables assigned
    terminate with output `SATISFIABLE`
  else if exists unit clause C propagating x to value $b \in \{0, 1\}$
    add propagated assignment $x \overset{C}{=} b$
  else if time to restart
    undo all variable assignments
  else
    if time for clause database reduction
      erase (roughly) half of learned clauses in memory
    use decision scheme to add assignment $x \overset{\mathsf{d}}{=} b$

CDCL Analysis and the Resolution Proof System

How to analyse CDCL performance?

Many intricate, hard-to-understand heuristics

Focus instead on underlying method of reasoning

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Resolution proof system

- Start with clauses of formula
- Derive new clauses by resolution rule

$$\frac{C \vee x \qquad D \vee \overline{x}}{C \vee D}$$

- Done when contradiction $\perp$ in form of empty clause derived

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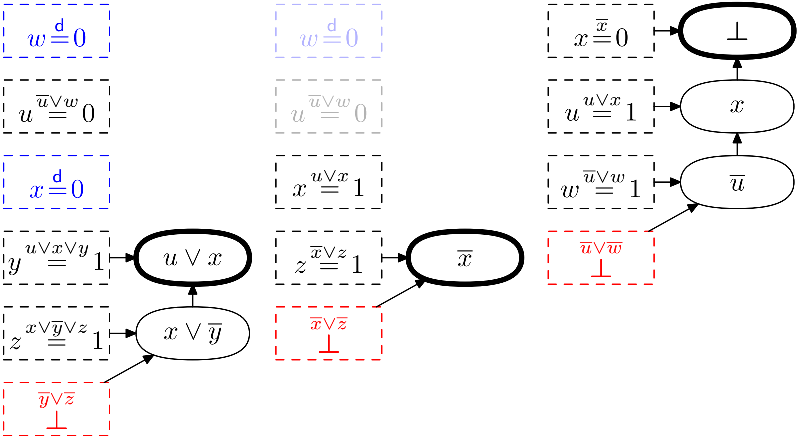
(*) Ignores preprocessing, but we don't have time to go into this

Resolution Proofs from CDCL Executions

Obtain resolution proof. . .

Resolution Proofs from CDCL Executions

Obtain resolution proof from our example CDCL execution. . .



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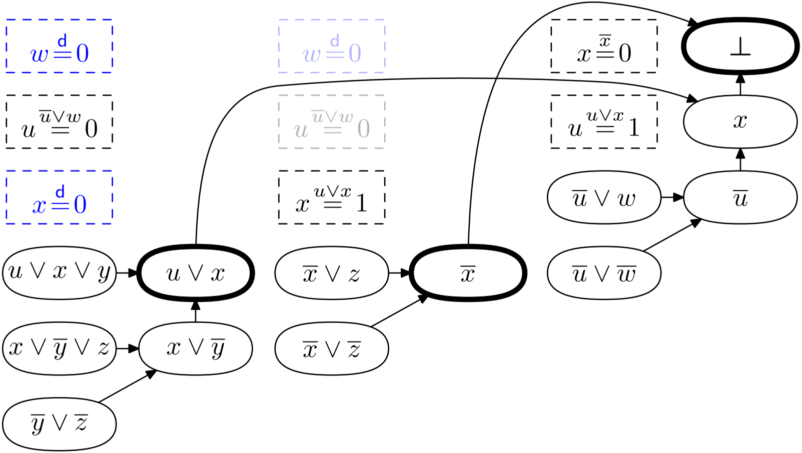
Obtain resolution proof from our example CDCL execution by stringing together conflict analyses:



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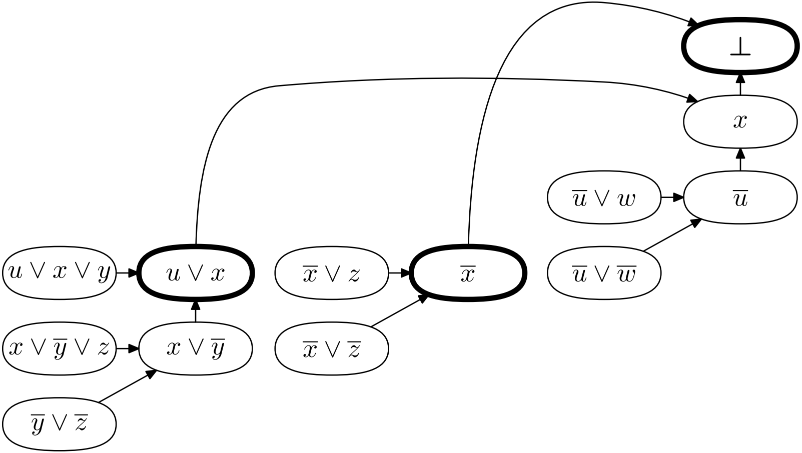

Conclusions and Open Problems

Current state of affairs

- Modern solvers perform amazingly well ("SAT is easy in practice")
- Very poor theoretical understanding:
  - Why do heuristics work?
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Directions for future work

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Thank you for your attention!

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