Truly Supercritical Trade-offs for Resolution, Cutting Planes, Monotone Circuits and Weisfeiler–Leman

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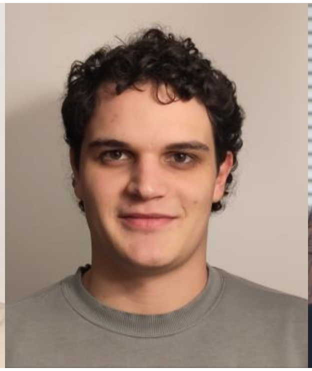
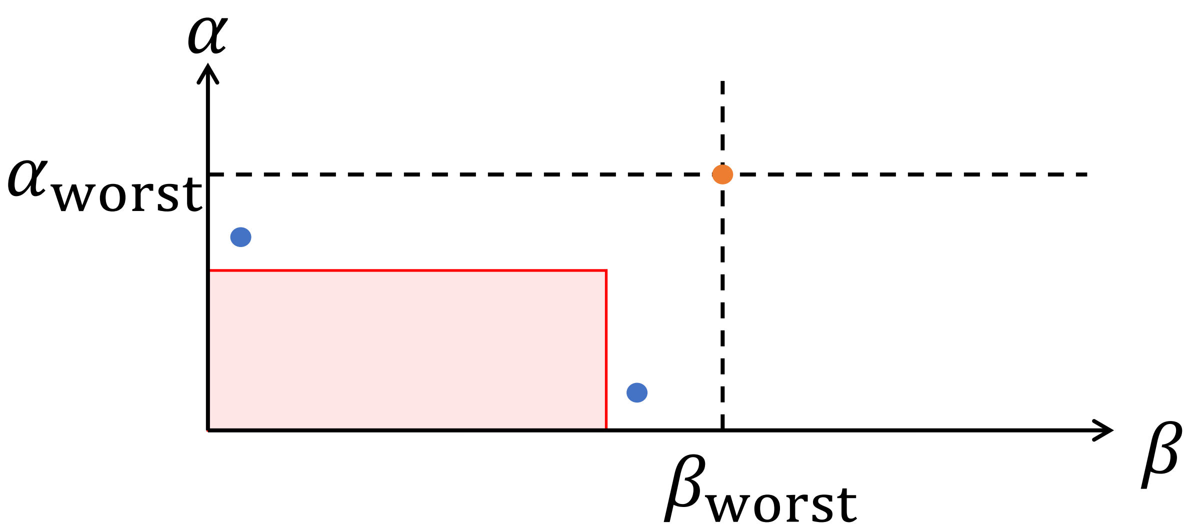



joint with Susanna F. de Rezende, Noah Fleming, Duri Andrea Janett, Shuo Pang

MATHEMATICAL INSIGHTS INTO ALGORITHMS FOR OPTIMIZATION
MIAO

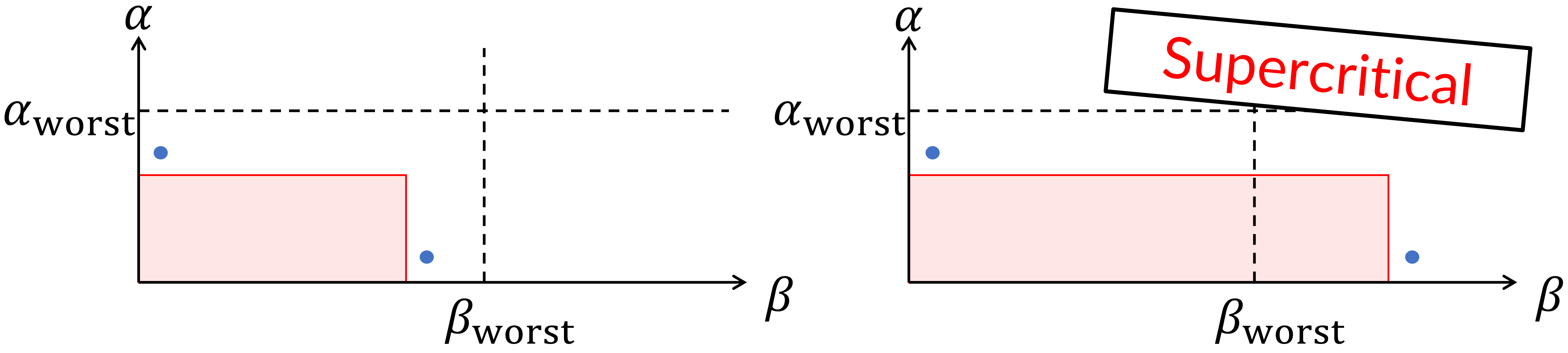
What Is a Trade-off Result?



Computational model with two complexity measures α, β
(e.g. α = time and β = space)

- brute force algorithm can achieve worst case
- can optimize β, but then α bad
- can optimize α, but then β bad
- impossible to optimize both

A New Kind of Trade-off [Razborov '16]

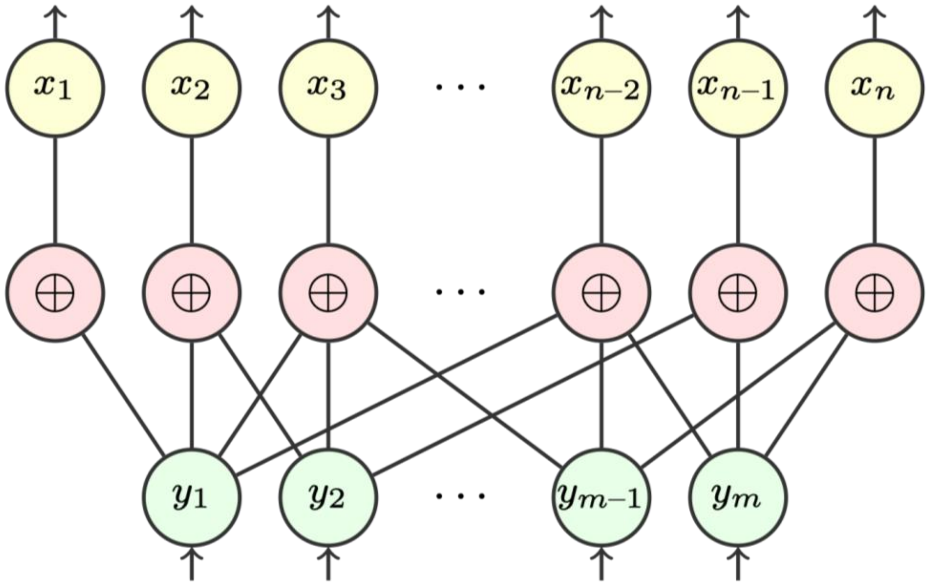


→ Optimizing α pushes β way beyond brute-force worst case

Follow-up work in [Raz'17, Raz'18, BN'20, FPR'22, BN'23, GLN'23, BT'24, CD'24]

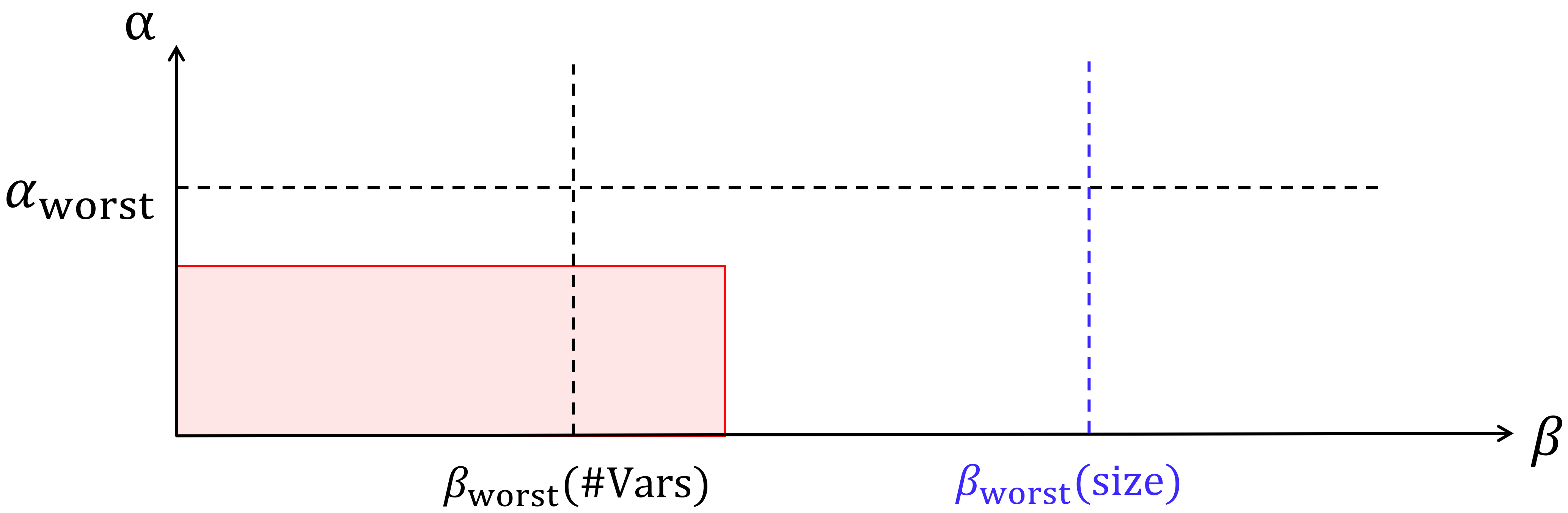
Supercritical Trade-offs Through Hardness Condensation

- Take medium-hard input in variables $x_1, \dots, x_n$
- «Condense» by substituting x_i by XORs over subsets of variables $y_1, \dots, y_m$
- Show hardness is nearly preserved
- But measured in $m \ll n$: supercritical



Used in [Razborov '16, BN'20, FPR'22, BN'23, GLN'23, CD'24, ...]

Supercritical in What?



All trade-offs supercritical in # variables only, except [Berkholz '12, BBI '12/'16, BNT '13]

→ Are there trade-offs *truly* supercritical in input size?

Overview of Our Results

Truly supercritical trade-offs for
- width vs depth in resolution
- width vs size in tree-like resolution
- size vs depth in resolution and cutting planes
- size vs depth for monotone circuits
- dimension vs iteration number for the Weisfeiler–Leman algorithm

Answering open questions in [Razborov '16, GGKS '18, FGIPRTW '21, FPR '22, GLNS '23]

See also concurrent work by [Göös-Maystre-Risse-Sokolov '25]

Proof Structure

Proof in two steps:

1. Establish width vs depth trade-off for resolution

2. Derive other proof complexity trade-off results with lifting

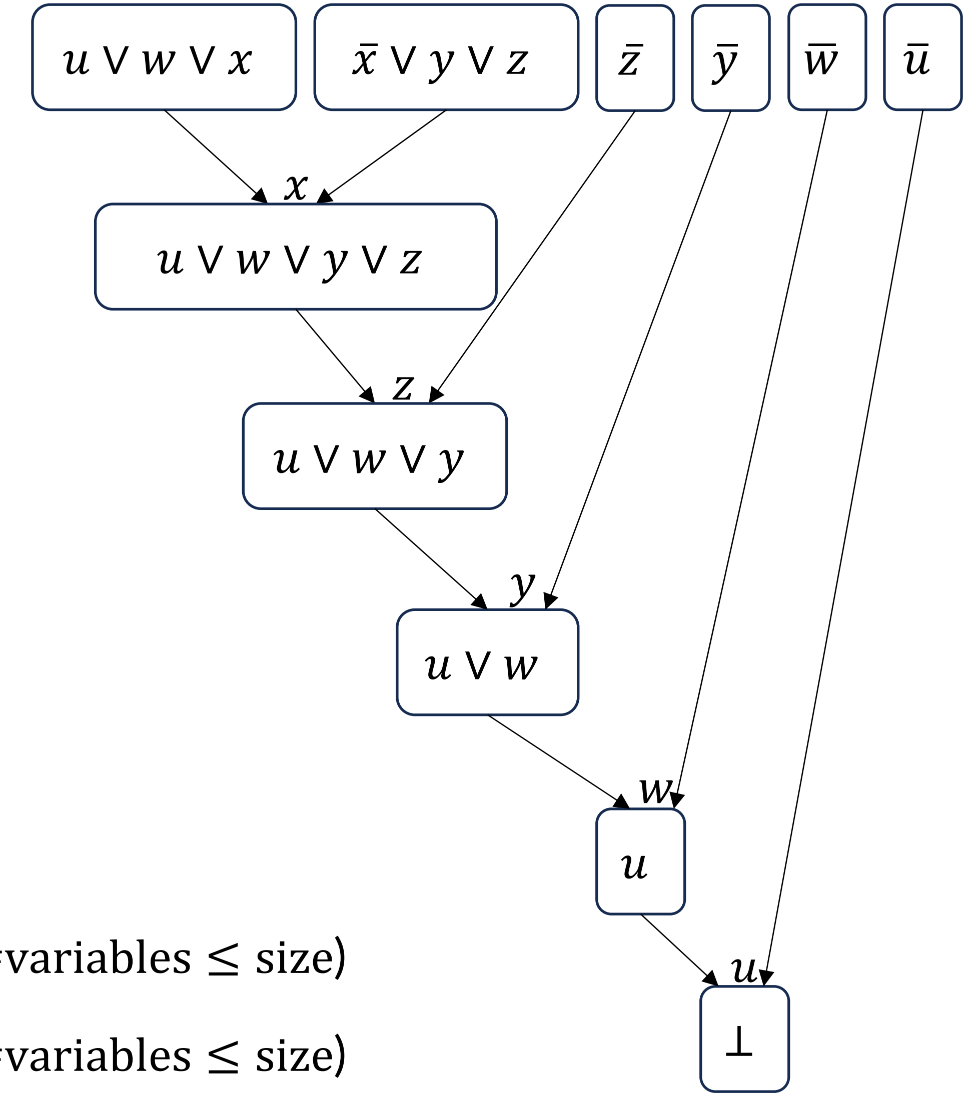
Resolution Proof System

Goal: prove CNF formula unsatisfiable
Proof of unsatisfiability: Refutation

Resolution rule:

$$\frac{C \vee x \qquad D \vee \overline{x}}{C \vee D}$$

width = max clause size = 4 (worst case ≤ #variables ≤ size)
depth = max path length = 5 (worst case ≤ #variables ≤ size)

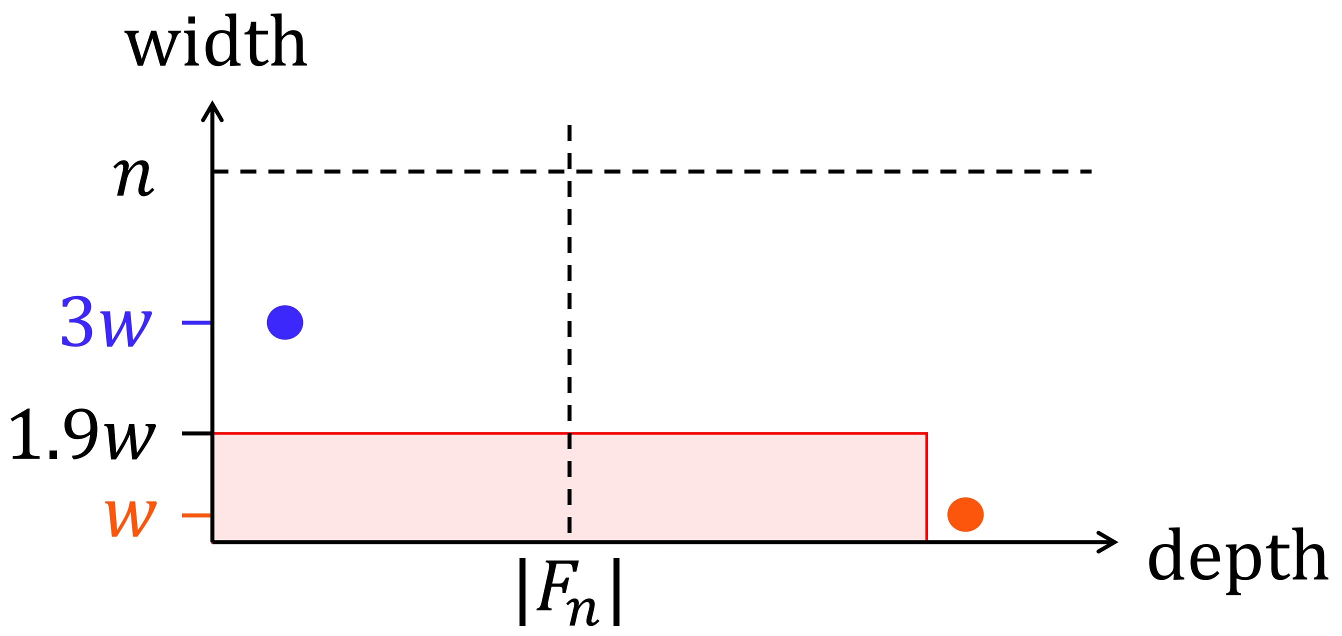


Resolution Width-Depth Trade-off

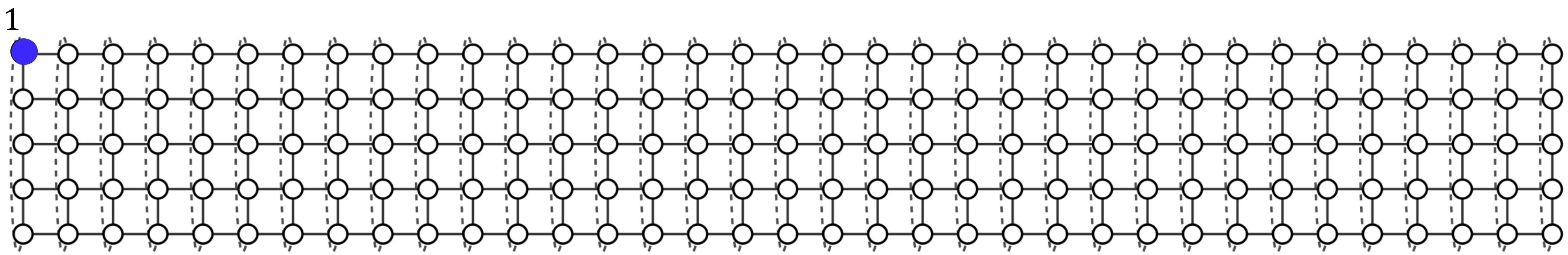
Theorem (width-depth trade-off for resolution)

$\exists$ CNF formulas F_n on n variables s.t.

- Refutable by resolution in width w
- But width $\leq 1.9w \implies$ supercritical depth $\mathrm{superlinear}(|F|)$



Starting Point: Tseitin Formula on Cylinder Graph

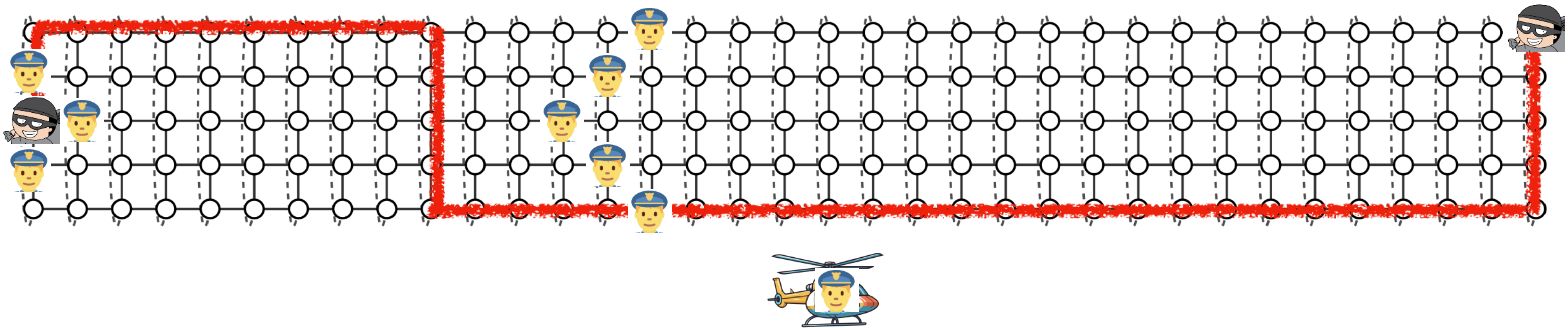


- Long, skinny cylinder $G = (V, E)$
  (wrap-around vertically, but not horizontally)
- Vertex at top left labelled 1, all other vertices labelled 0
- Edges $e \in E$ $\Leftrightarrow$ variables x_e
- Constraints: $\sum_{e \ni v} x_e \equiv \text{label}(v) \pmod 2$

  $\Rightarrow$ contradictory due to Handshake Lemma

$x_{\text{up}} \vee x_{\text{down}} \vee x_{\text{right}}$
$x_{\text{up}} \vee \overline{x_{\text{down}}} \vee \overline{x_{\text{right}}}$
$\overline{x_{\text{up}}} \vee x_{\text{down}} \vee \overline{x_{\text{right}}}$
$\overline{x_{\text{up}}} \vee \overline{x_{\text{down}}} \vee x_{\text{right}}$

Proof: By Analyzing the Cop-Robber Game [Seymour-Thomas '93]

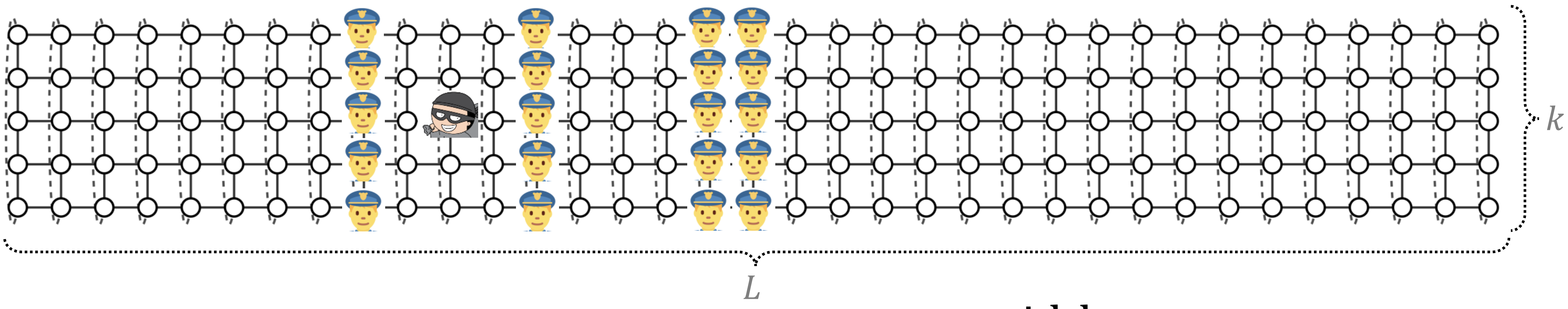


- Start: K cops, one robber at v_0
- In each round:
  – One cop enters helicopter and signal a vertex v
  – Robber moves
  – Cop lands at v
- Ends when robber is caught (by cop at same vertex)

width ≈ # cops
depth ≈ # rounds

[GTT '18]

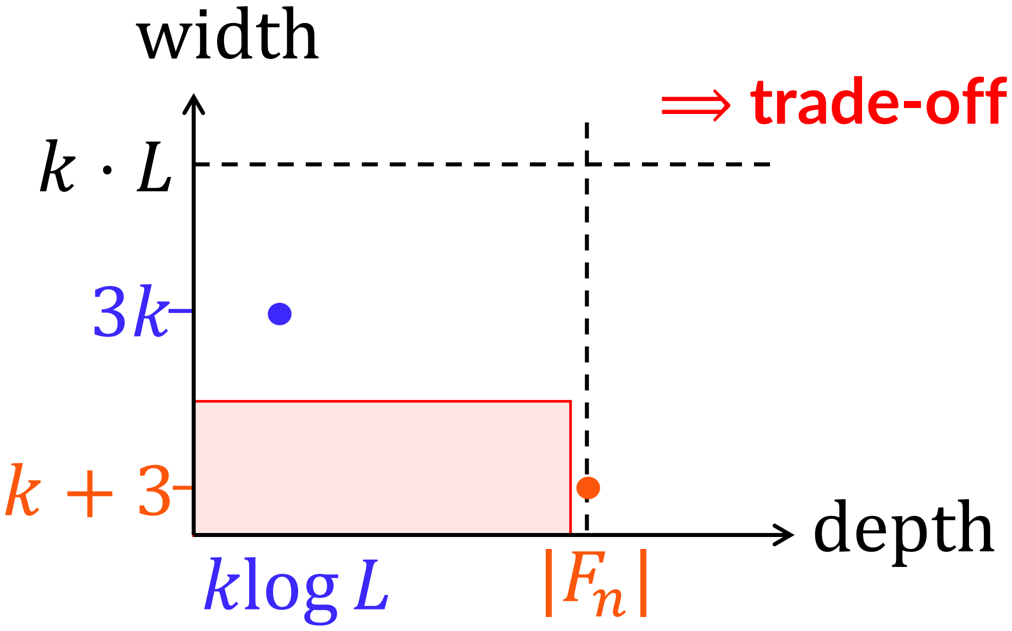
Upper Bounds: Simple Cop Strategy



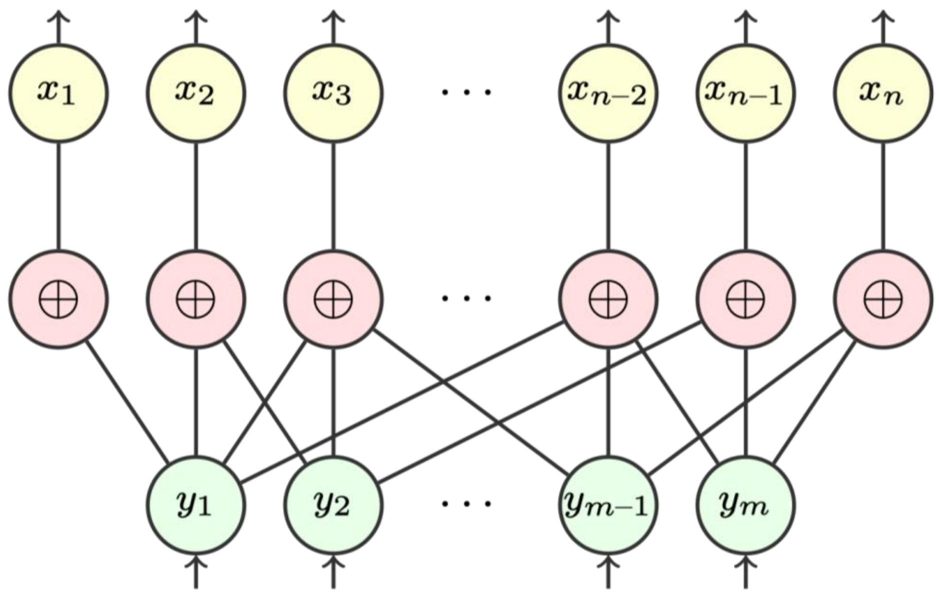
- $k+1$ cops:
  - Place cops on middle column
  - March towards robber in $k \cdot L$ rounds

  ⇒ translates to resolution proof of width $k+3$, but depth $k \cdot L$
- $3k$ cops:
  - Binary search

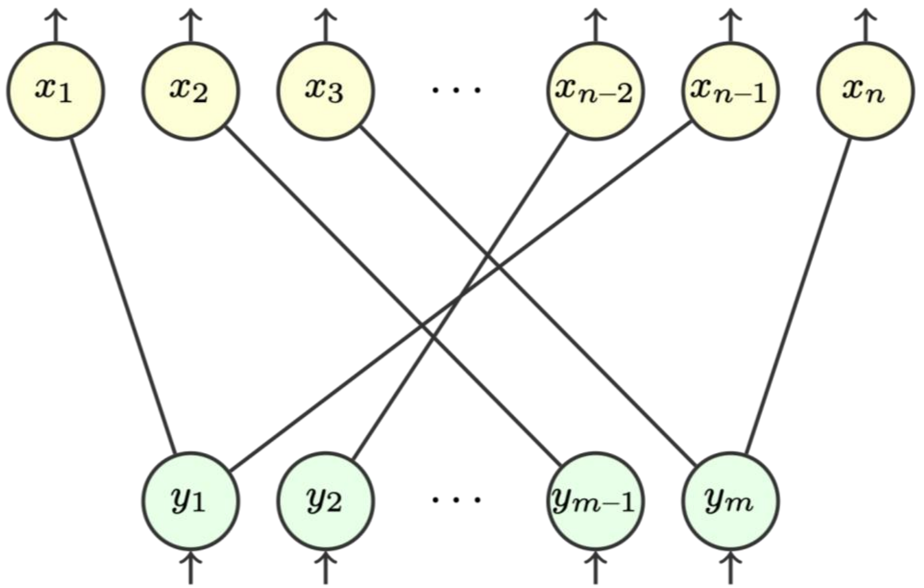
  ⇒ translates to resolution proof of width $3k$ and depth $k \cdot \log L$



Technique: Variable Compression [Grohe-Lichter-Neuen-Schweitzer '23/'25]

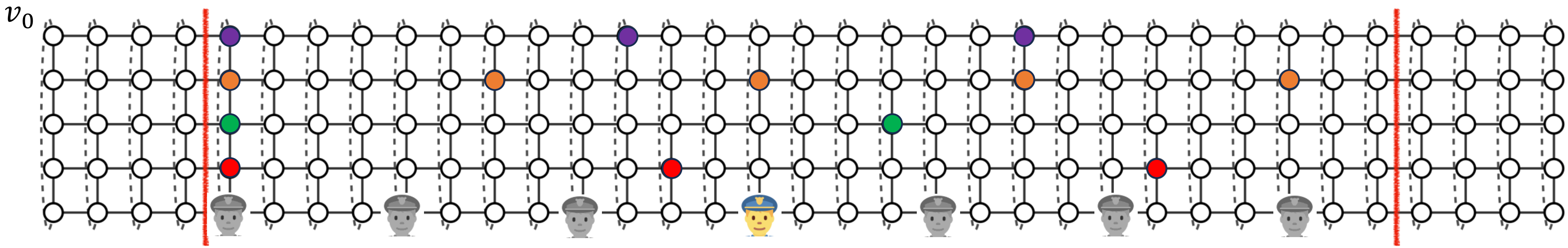


Substitution with XOR gadgets



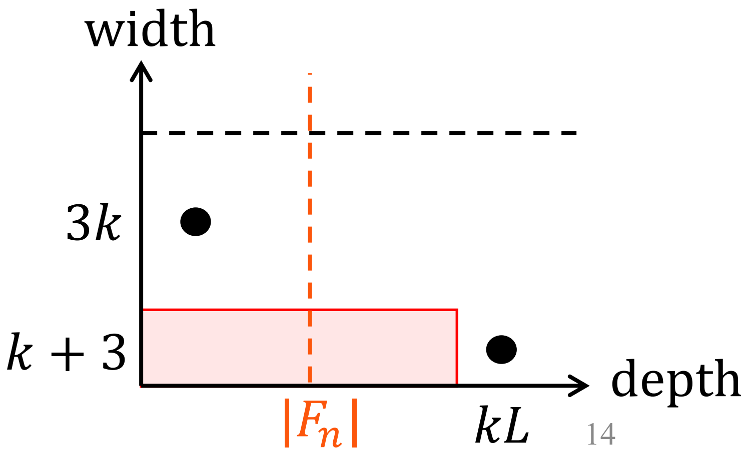
Variable substitution
(with lots of collisions)

Supercritical Bounds via Compression [Grohe-Lichter-Neuen-Schweitzer ’23/’25]

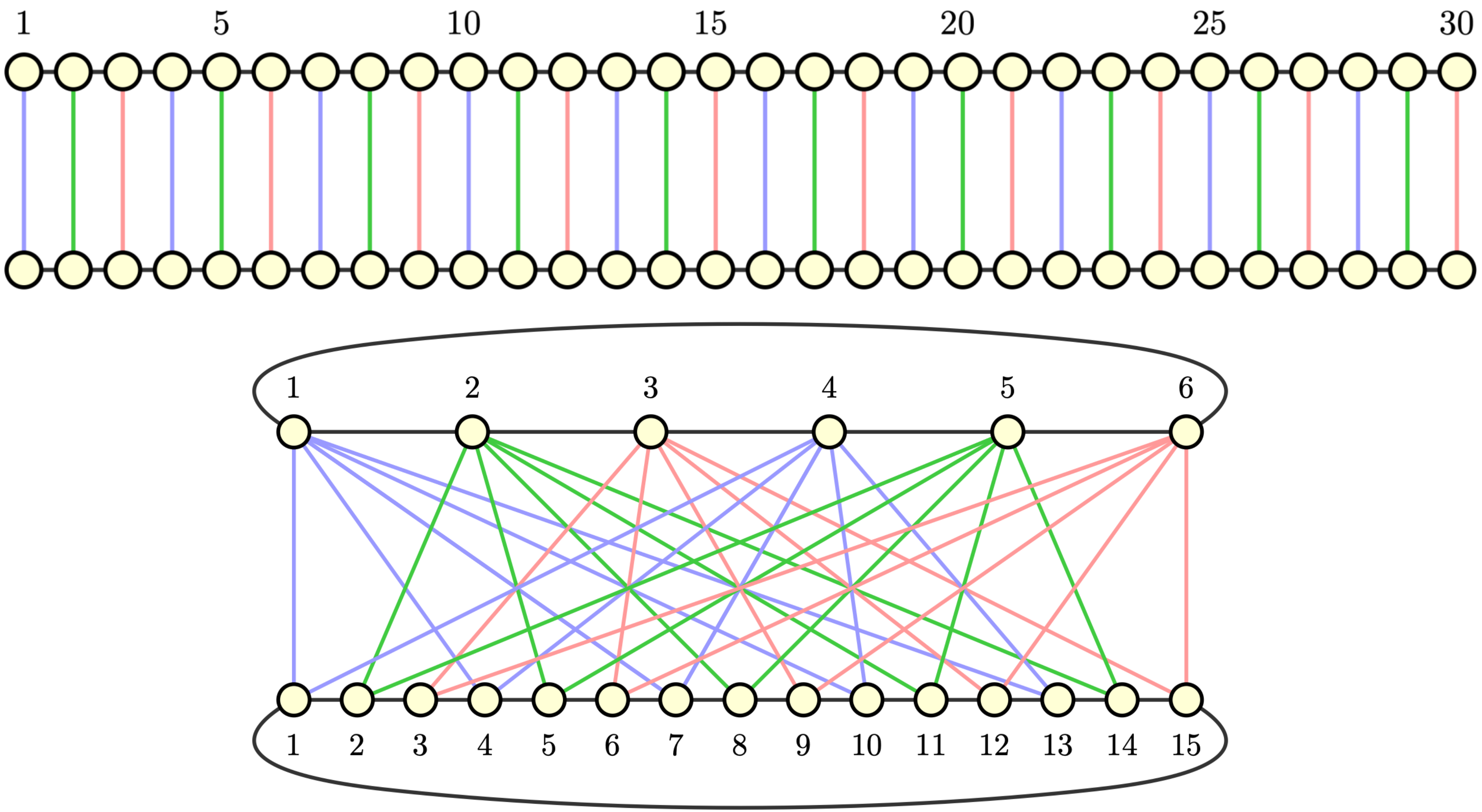


- Compress graph by identifying vertices on the same row
- Induces edge equivalence classes $[e]$
- Compressed formula: $\sum_{[e]\ni[v]} y_{[e]} \equiv 1 \pmod{2}$ iff $[v] = [v_0]$

- Compressed game: cops have **clones**
  - Cop strategies for uncompressed game still work
    $\Longrightarrow$ same upper bounds
  - Harder to get lower bounds (due to clones)
    $\Longrightarrow$ but they are now **supercritical**



Edge Equivalence Between Two Rows ($m_1 = 6, m_2 = 15$)



More Results: Resolution and Cutting Planes

Theorem (size-width trade-off for treelike resolution)

$\exists$ CNF formulas F_n on n variables s.t.

- Refuted by treelike resolution in width $\mathrm{w} = \mathrm{poly}(\log(n))$
- But width $\leq w + \sqrt{w} \Longrightarrow$ supercritical size $\exp(\mathrm{superpoly}(|F|))$

Theorem (size-depth trade-off for resolution)

$\exists$ CNF formulas F_n on n variables s.t.

- Refuted by resolution in size $s = \mathrm{quasipoly}(|F|)$
- But size $\leq ns \Longrightarrow$ supercritical depth $\mathrm{superpoly}(|F|)$

Theorem (size-depth trade-off for cutting planes)

$\exists$ CNF formulas F_n on n variables s.t.

- Refuted by cutting planes in size $s = \mathrm{quasipoly}(|F|)$
- But size $\leq ns \Longrightarrow$ supercritical depth $\mathrm{superpoly}(|F|)$

All results supercritical in input size

Derive other Results: Lifting

- Use composition to relate (different) models of computation:

  Complexity of f in (weak) model A corresponds to

  $\Rightarrow$ Complexity of composed problem $f \circ g$ in (strong) model B

- [Garg-Göös-Kamath-Sokolov '18]: For composition with Index function:

  Resolution width-depth trade-off

  $\Rightarrow$ monotone circuit size-depth trade-off

- Need tighter lifting theorem than [GGKS '18] with better constants

- We also prove essentially optimal lifting theorems for resolution

Conclusion

- We give truly supercritical (in terms of size) trade-offs for
  - proof complexity (resolution and cutting planes)
  - monotone circuits
  - Weisfeiler–Leman algorithm
- Proof in two steps:
  - Base trade-off width vs depth for **compressed** Tseitin formulas
  - Lifting theorems with **improved parameters** yield other results

Future Supercritical Research Directions

Thanks for your attention!

- Via **compression** of other formulas or graphs?
- Trade-offs for **other measures** (e.g. size vs space)?
- Size-depth trade-offs for **(uncompressed) Tseitin formulas**?
- Monotone circuit trade-offs for **perfect matching**?