Supercritical and Robust Trade-offs for Resolution Depth Versus Width and Weisfeiler-Leman

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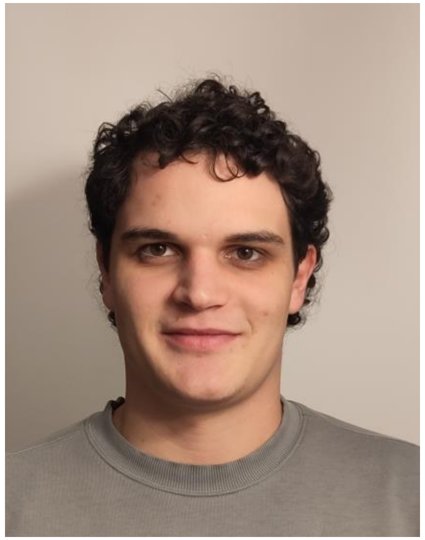
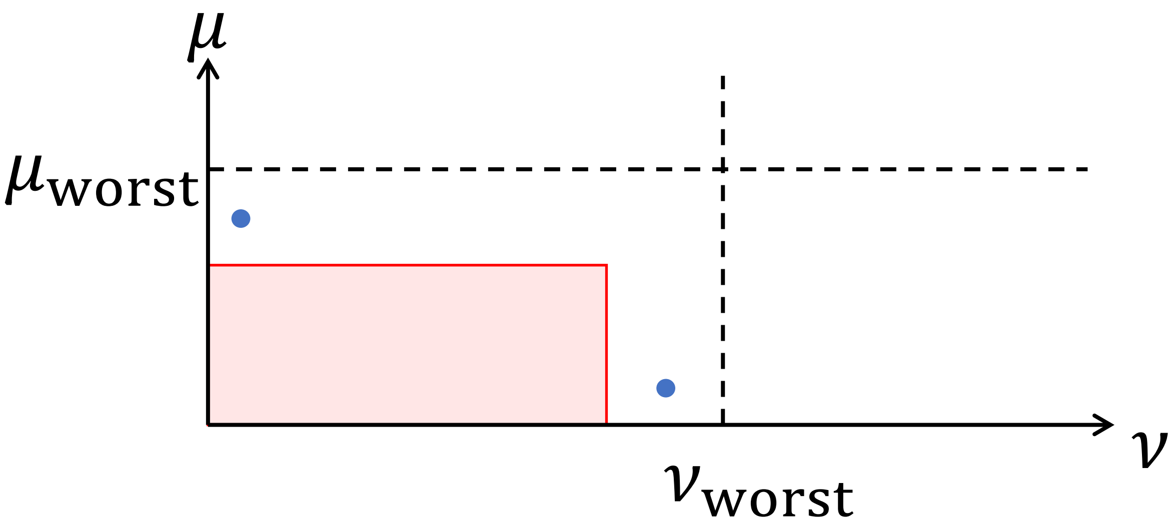
SAT and Interactions
Dagstuhl, October 13-18, 2024



Joint work with Duri Andrea Janett and Shuo Pang

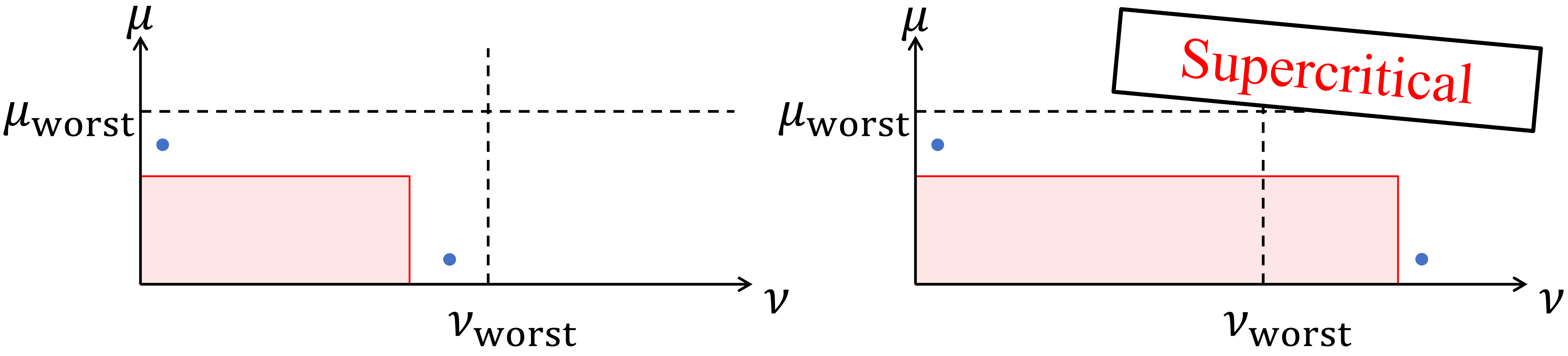
What Is a Trade-off Result?



Take a computational model with two complexity measures μ, ν (e.g. μ = time and ν = space)

Robust ≈ rectangle large

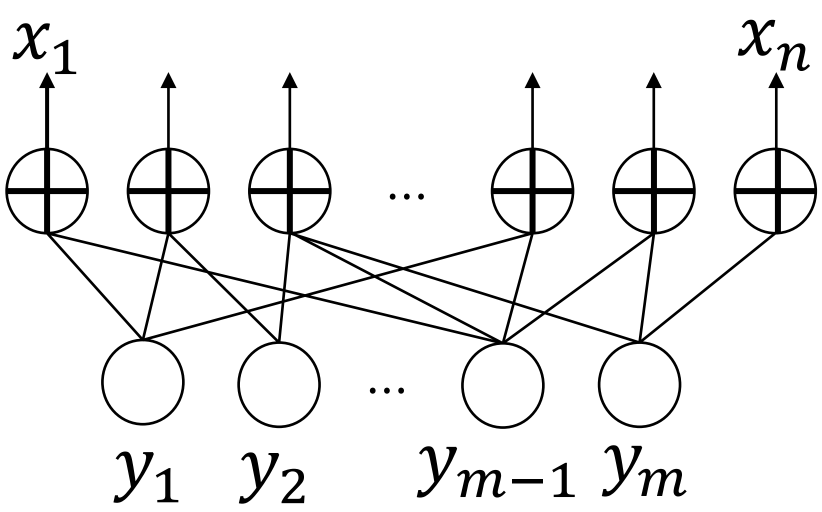
A New Kind of Trade-off [Razborov 2016]



Achieved through Hardness Condensation

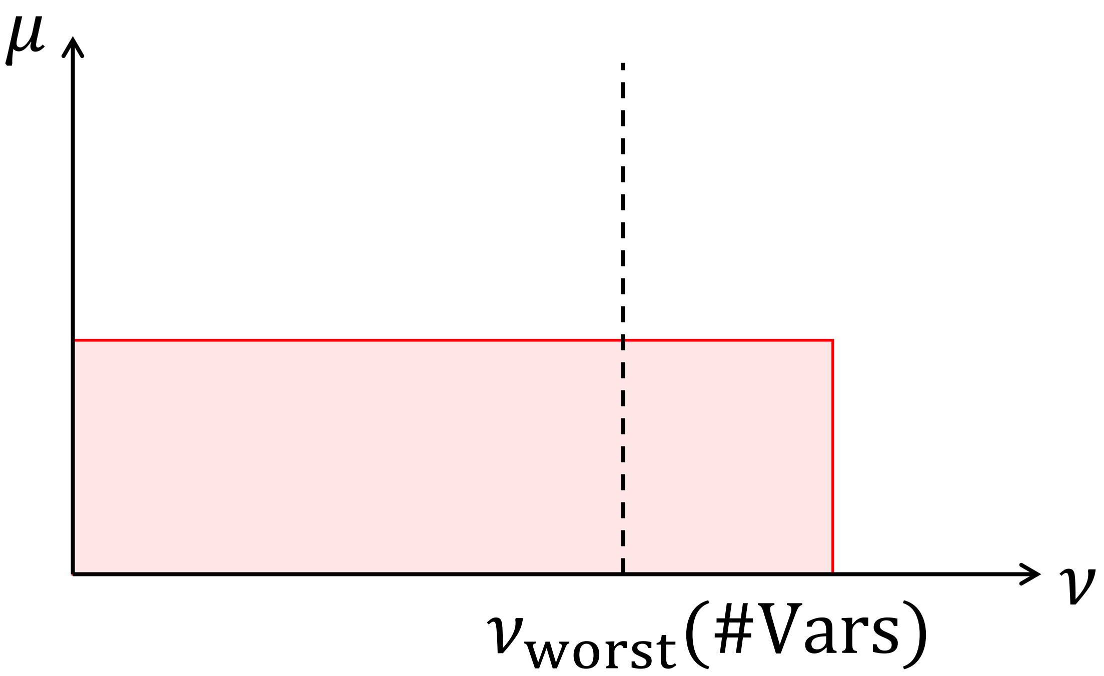
- Take medium-hard input in variables $x_1, \ldots, x_n$
- «Compress» by substituting with variables $y_1, \ldots, y_m$
- But so that most of original hardness preserved
- Now measured in $m \ll n$

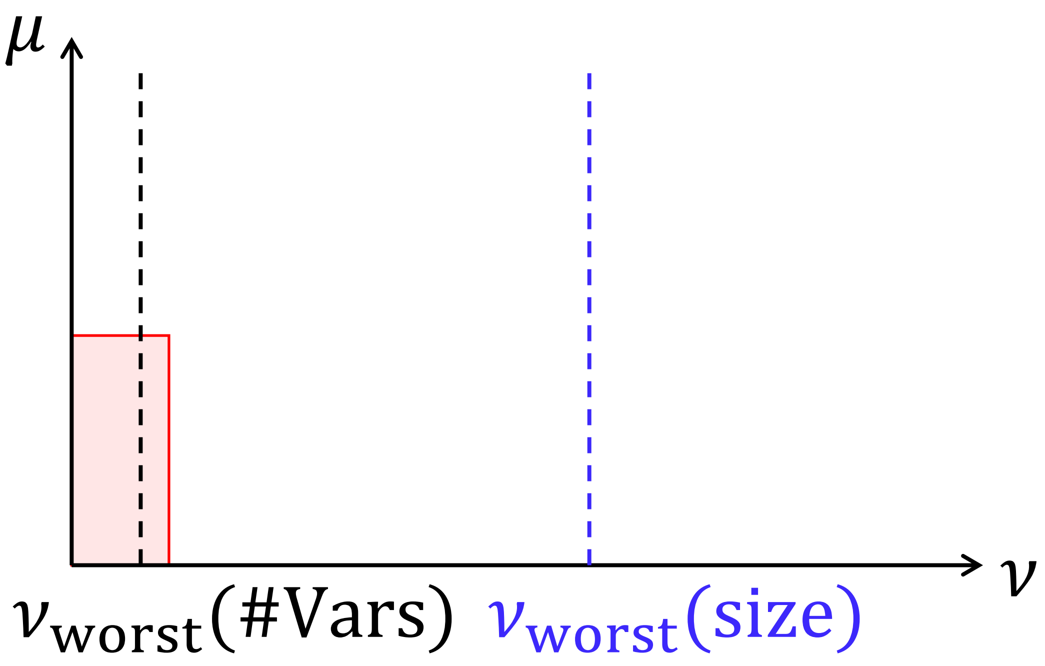
  $\Rightarrow$ Supercritical!



[Razborov '16, Razborov '17, Razborov '18, Berkholz-Nordström '20, Fleming-Pitassi-Robere '22, Berkholz-Nordström '23, ...]

But Supercritical *in What?*





All trade-offs supercritical in #variables only, except [Berkholz '12, Beck-Nordström-Tang '13, Beame-Beck-Impagliazzo '16]

Our Work

Computation model: Resolution proof system
Complexity measures: width and depth (worst case ≤ #variables ≤ formula size)

Theorem

For any large enough k and $c < k$ exist 4-CNF formulas such that

- formula size $s \approx n^c$
- exists proof in width $k + 3$
- but width $< k + c \Rightarrow$ depth $s^{k/c}$

Supercritical in **input size**

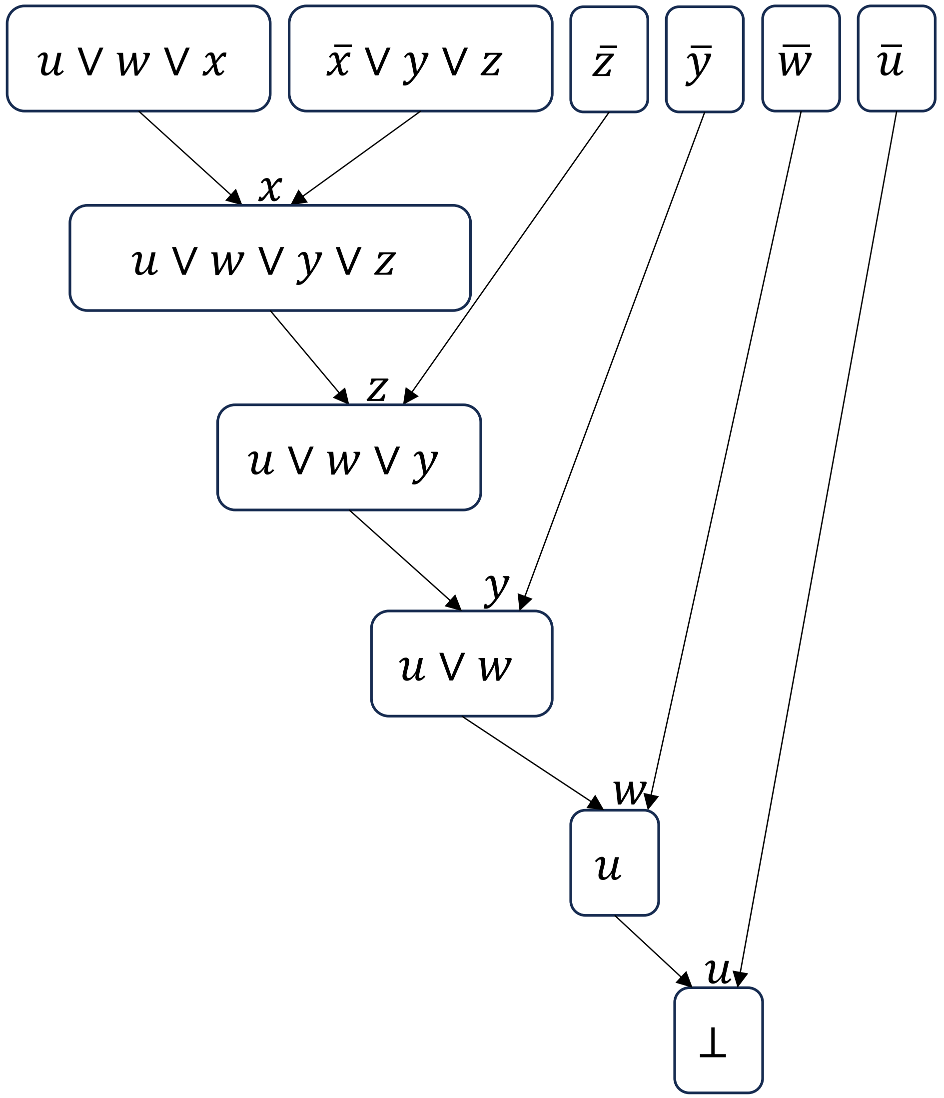
Resolution Proof System

Goal: prove CNF formula unsatisfiable

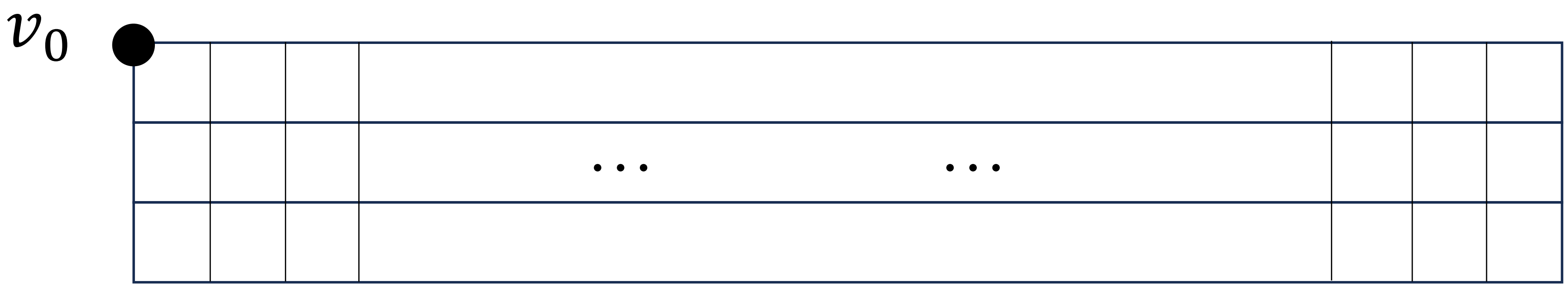
Resolution rule:

$$\frac{C \vee x \qquad D \vee \overline{x}}{C \vee D}$$

size	= #nodes	= 11
width	= max clause size	= 4
depth	= max path length	= 5

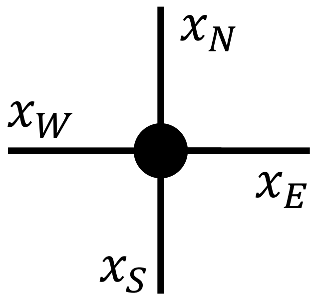


Tseitin Formula: Encoding Handshake Lemma



Cylinder Graph: every vertex has edges N E S W, wraps around vertically

Variables: x_e for edge e



$$\sum_{e \ni v} x_e = 1 \bmod 2 \quad \text{iff} \quad v = v_0$$

$$\overline{x_N} \vee x_E \vee x_S \vee x_W$$
$$x_N \vee \overline{x_E} \vee x_S \vee x_W$$
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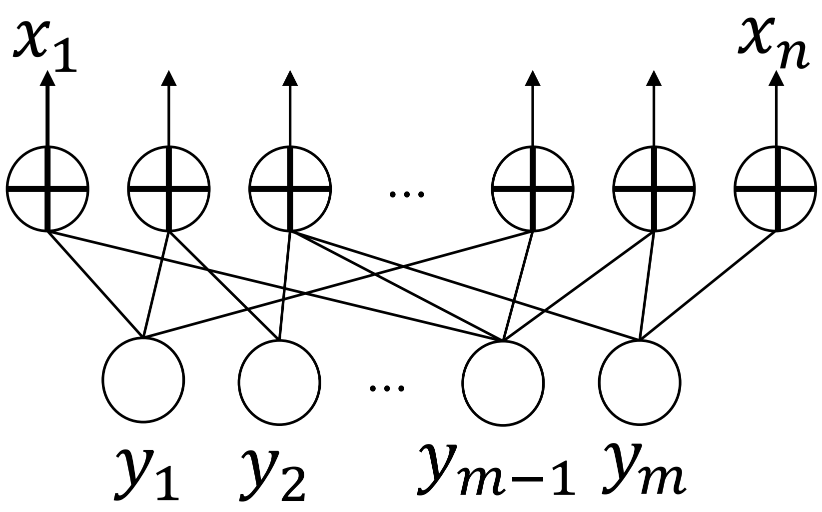
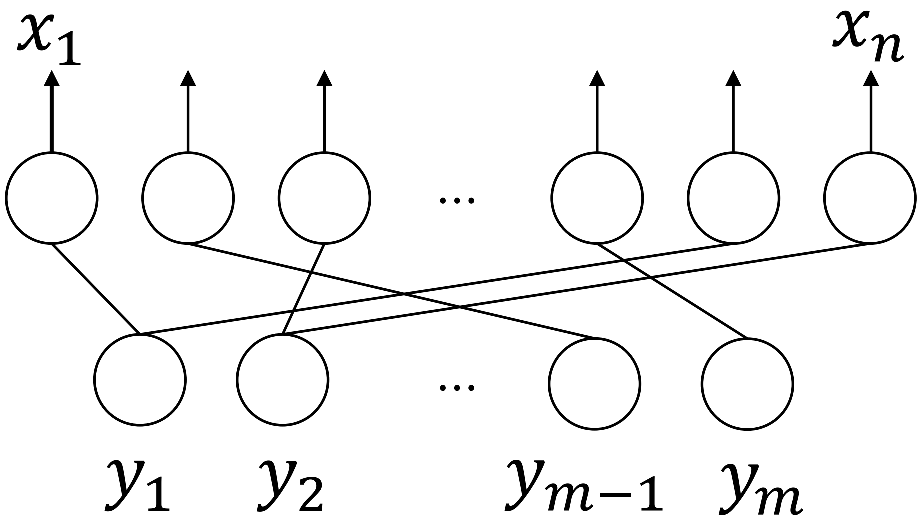
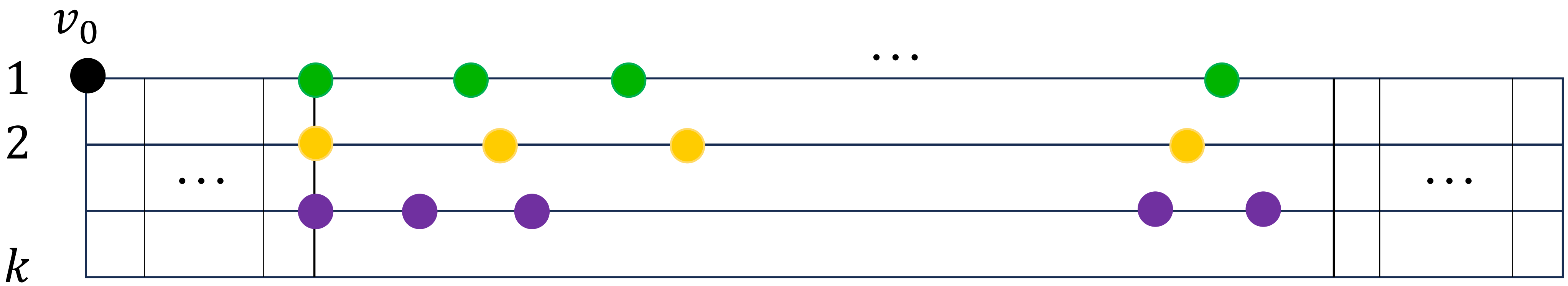
Substitution [Grohe-Lichter-Neuen-Schweitzer 2023]





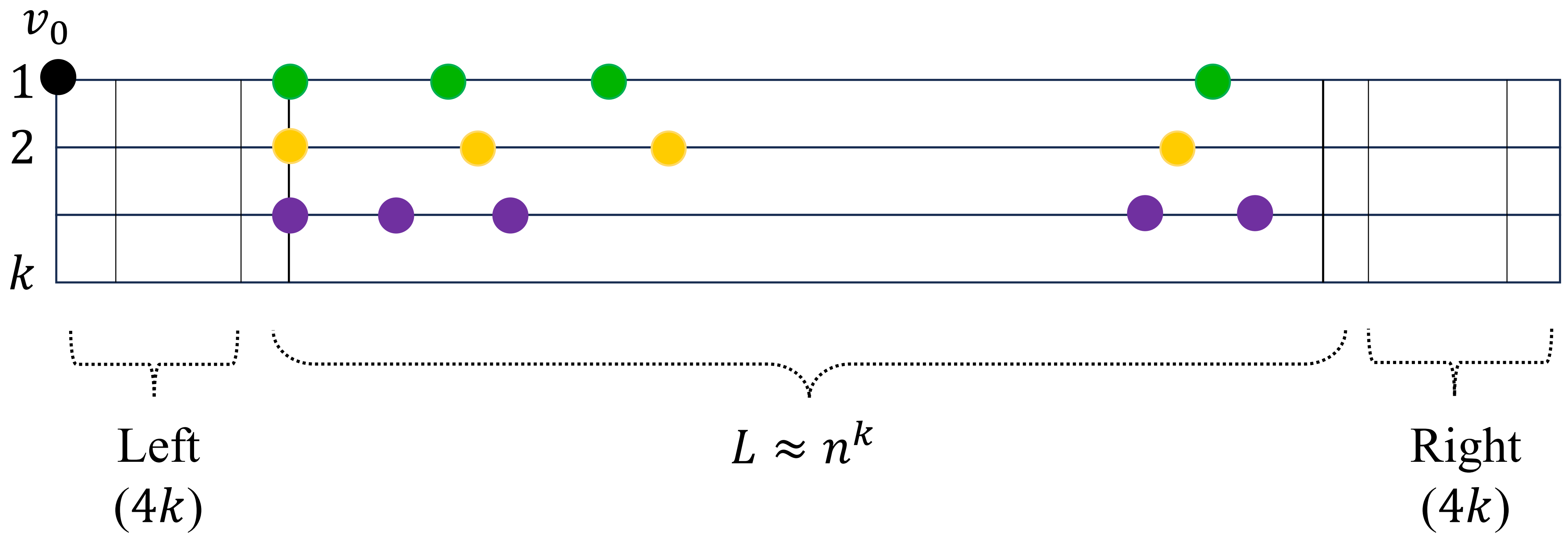
Structured identification

Substitution [Grohe-Lichter-Neuen-Schweitzer 2023]



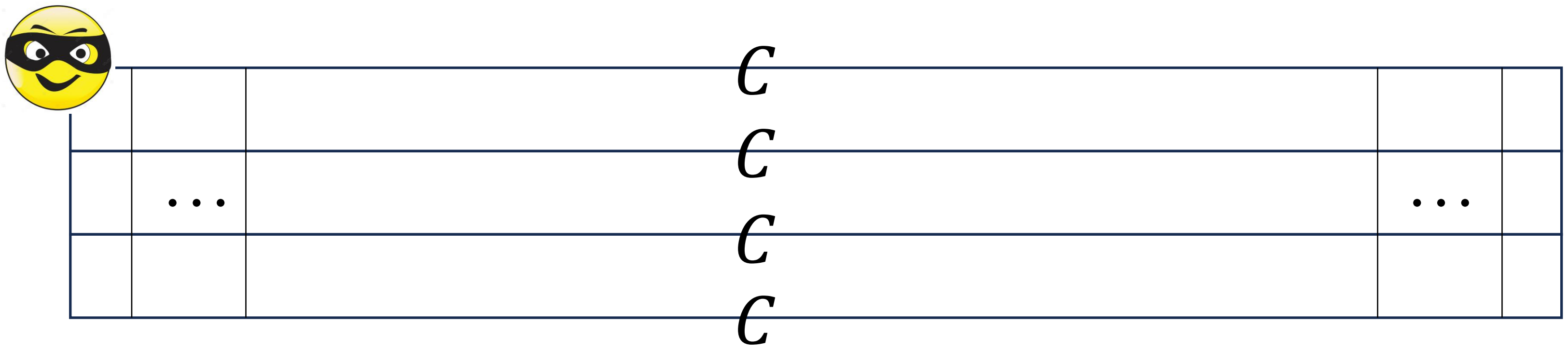
$\equiv_V$: row i mod m_i

Condensed Formula [Grohe-Lichter-Neuen-Schweitzer 2023]



$$\sum_{[e]\ni[v]} y_{[e]} = 1 \bmod 2 \quad \text{iff} \quad [v] = [v_0]$$

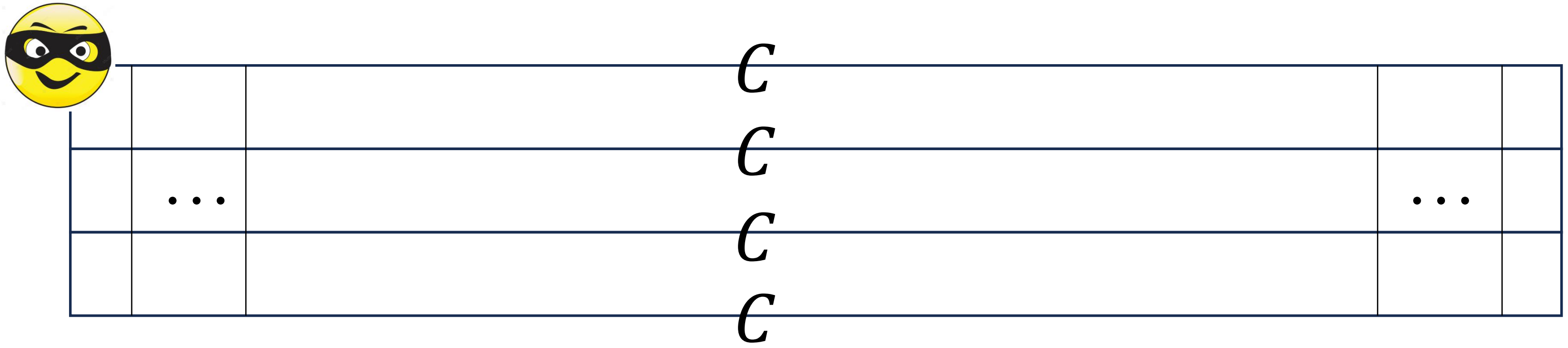
Proof: By Analyzing the Cop-Robber Game



- Start: $(k + c)$ cops, one robber at v_0
- In every round:
  - Lift a cop and signal a vertex v
  - Robber moves
  - Cop lands at v
- Ends when Robber is caught
- #cops ≈ resolution width; #rounds ≈ resolution depth

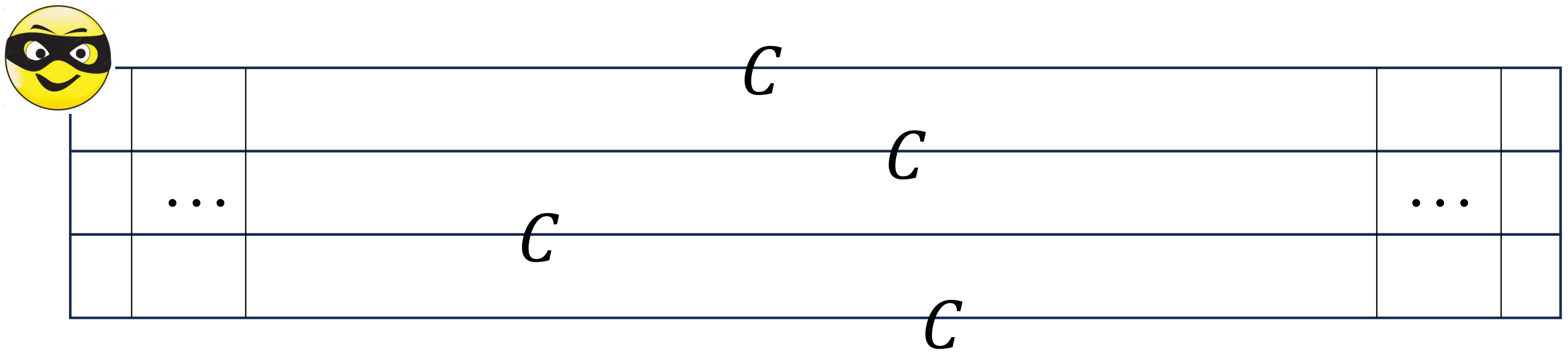
[Seymour-Thomas 93, Galesi-Talebanfard-Torán 18]

Cop Strategy



- With $(k + c)$ cops, c small:
  - Place cops on middle column
  - March slowly towards where robber is
  - # rounds ≈ width of cylinder
- With $3k$ cops:
  - Binary search
  - # rounds ≈ logarithm of width of graph

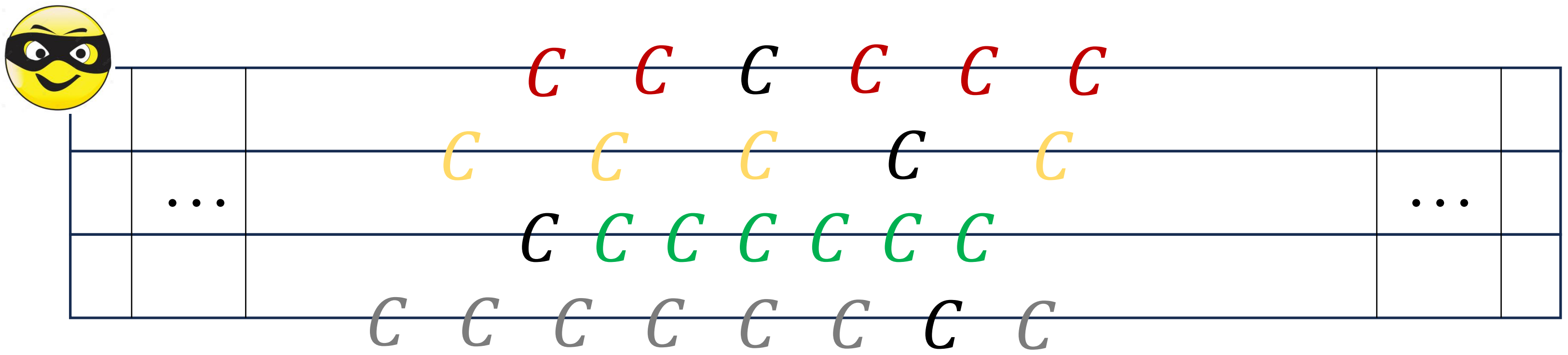
Compressed Cop-Robber Game



- $(k + c)$ cops, one robber at v_0
  - Lift a cop and signal a vertex v
  - Robber does a $\equiv$-compressible move
  - Cop lands at $[v]$

[Grohe-Lichter-Neuen-Schweitzer 23]

Compressed Cop-Robber Game



- $(k + c)$ cops, one robber at v_0
  - Lift a cop and signal a vertex v
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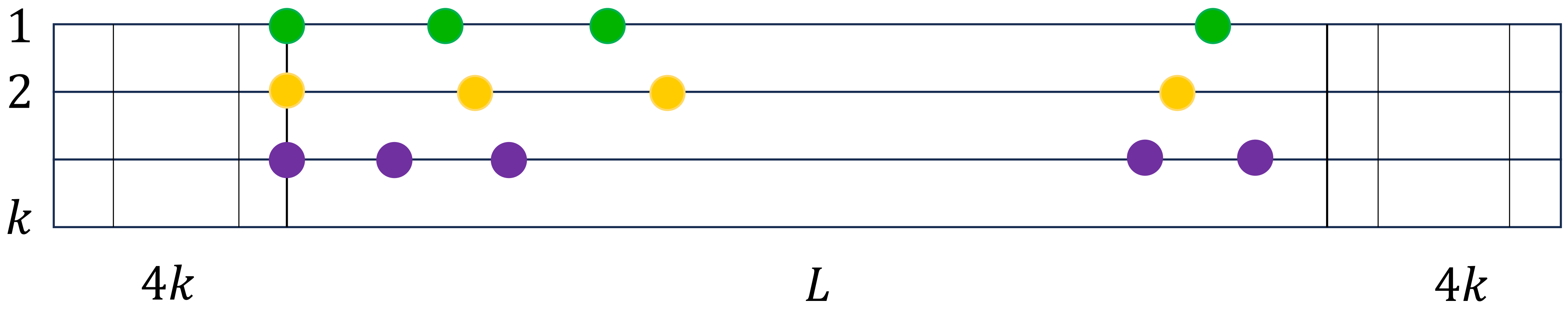
[Grohe-Lichter-Neuen-Schweitzer 23]

How to Compress the Graph: The Moduli

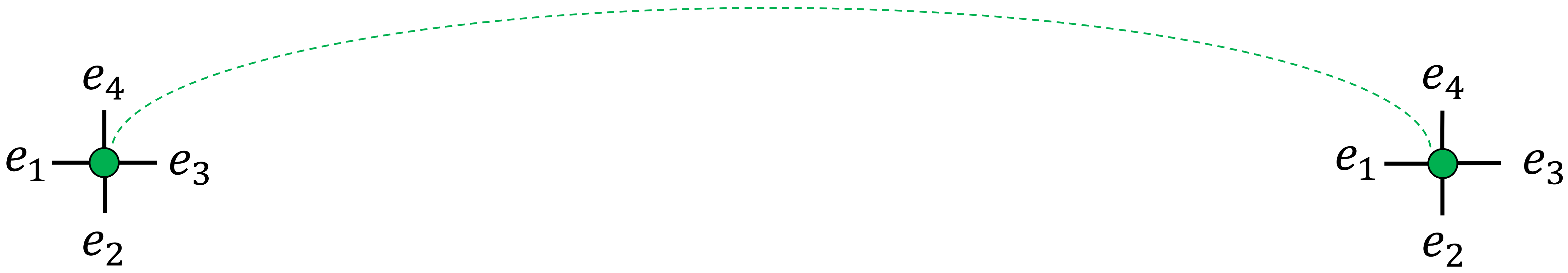


- Fix $1 \leq c \leq k-2$

- Pick k coprime numbers $P_1, \ldots, P_k$, $|P_i| \approx n$

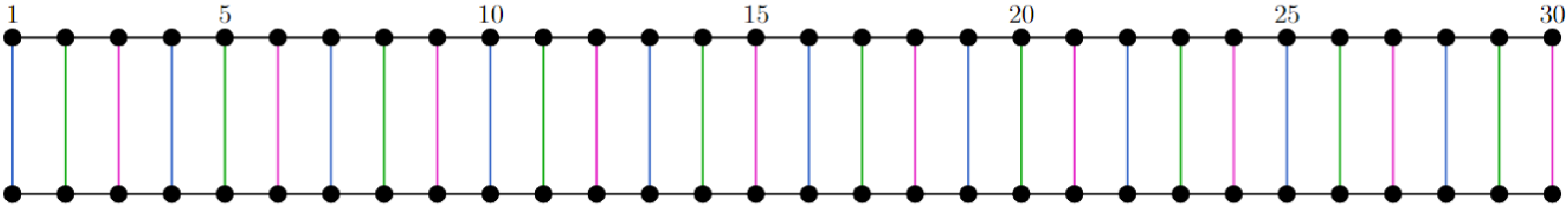
  $$m_i := (4k) \cdot P_i \cdots P_{i+c}$$
  $$L := \operatorname{lcm}\{m_i\} = (4k) \cdot P_1 \cdots P_k$$

- Compressed formula size $n^k \rightarrow n^{c+1}$

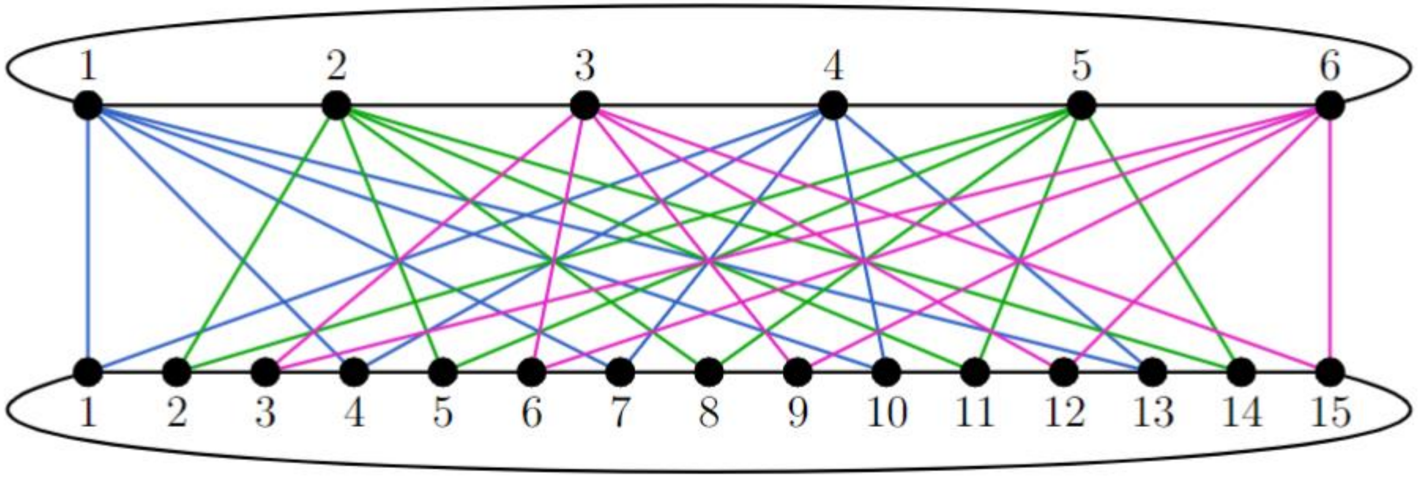


$\equiv_E$: via adjacency list

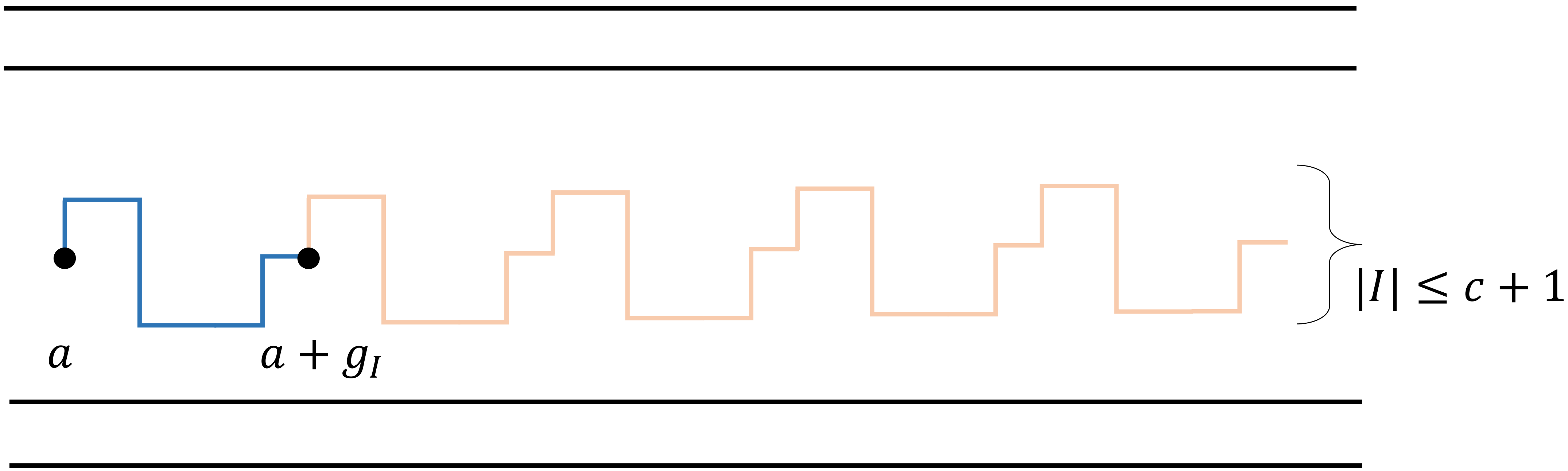
Edge Equivalence



$$m_1 = 6,\ m_2 = 15$$

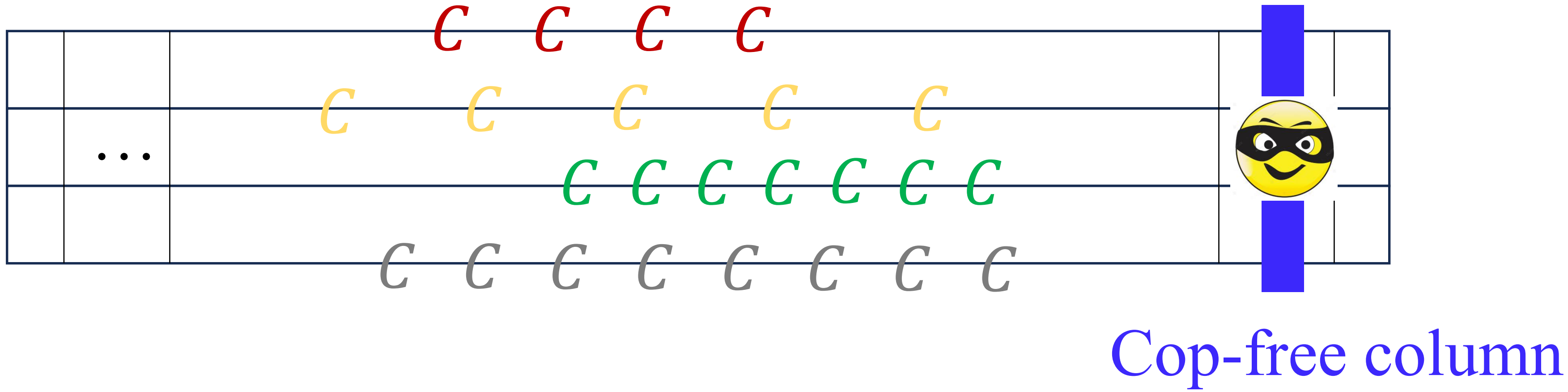


Moves Translatable to Compressed Setting



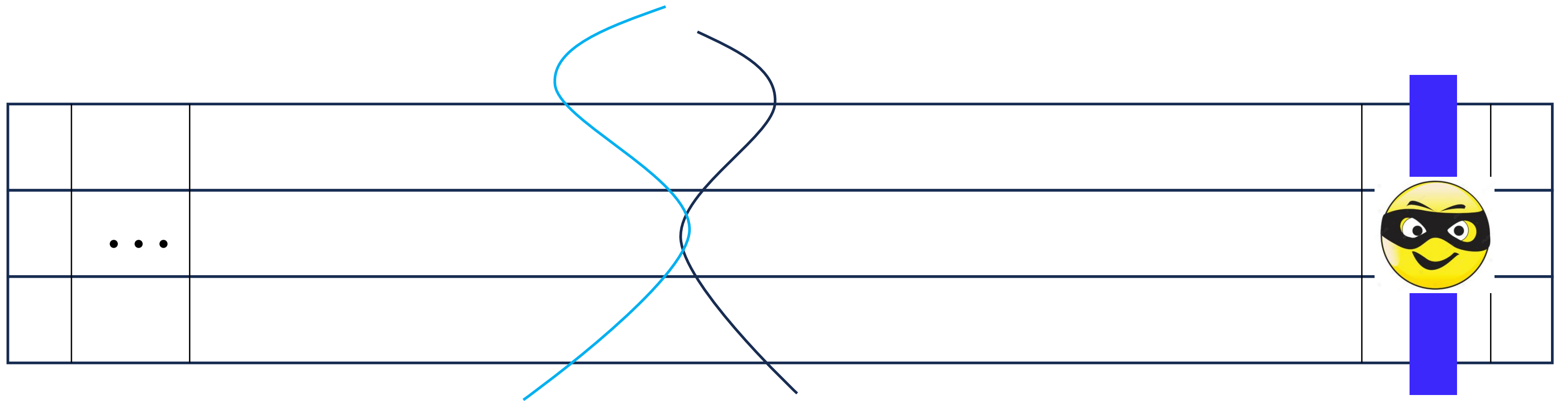
$$g_I \coloneqq \gcd(m_i\colon\ i \in I)$$

Idea for Robber Strategy



Slide between L, R using special moves translatable to compressed setting

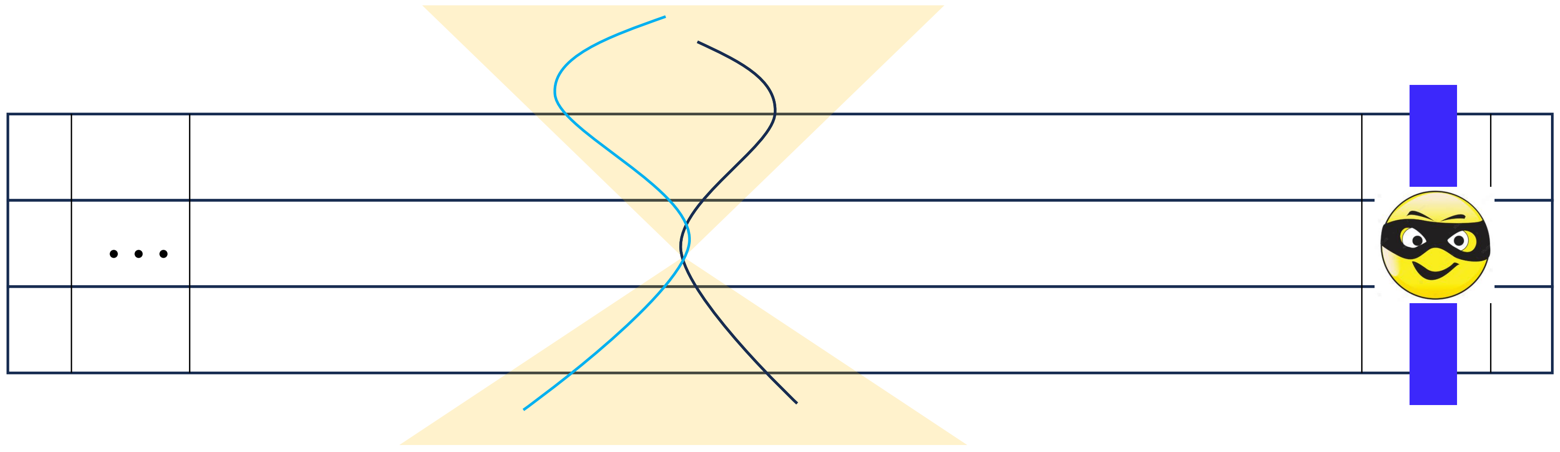
Dangers of Robber Life: Separators



Slide between L, R using special moves

SEPARATOR WARNING

Robber Strategy



Keep away from potential vertex separators S

Survive roughly as long as on original cylinder

Motivation for Grohe et al.: Weisfeiler-Leman algorithm

Theorem [Grohe-Lichter-Neuen-Schweitzer 2023]

∃ graph pairs such that $(k+1)$-dimensional Weisfeiler-Leman algorithm can distinguish them, but only after $N^{\frac{k}{2}}$ iterations.

- dimension ≈ resolution width
- iterations ≈ resolution depth
- graph pair ≈ Tseitin [Berkholz-Nordström '16/'23]

- But GLNS23 yields no proof complexity results (because of "≈")

Our Result for Weisfeiler-Leman algorithm

Corollary (Weisfeiler-Leman)

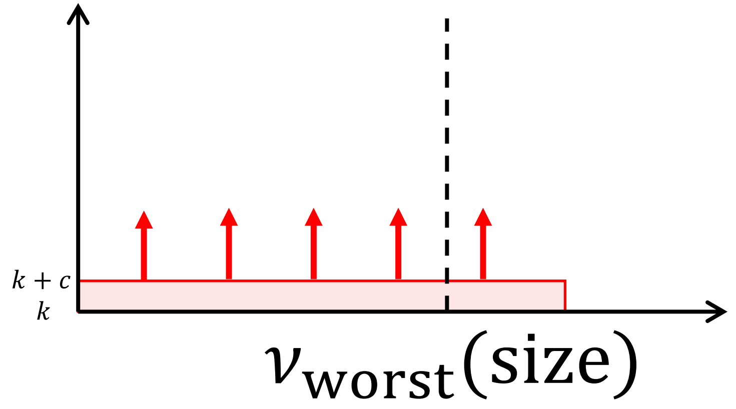
For any $c \leq k - 2$, $\exists$ graph pairs of size N such that:

- dimension-$(k + 1)$ WL can distinguish them
- dimension-$(k + c)$ WL requires $N^{\frac{k}{c+1}}$ iterations

- More robust trade-offs for Weisfeiler-Leman than GLNS23
- And thanks to robustness yields proof complexity consequences

Conclusion

- Depth-width tradeoff, supercritical in formula size
- Robust (somewhat): applies not only to minimal width
- Similar trade-offs obtained independently by Göös et al.
- Our results apply also to Weisfeiler-Leman algorithm



Open problems:

- Better robustness?
- Trade-offs size-depth, size-space? (stay tuned…)
- Can we compress other graphs than cylinders?

Thank you for your attention!