

# Combinatorial Solving with Provably Correct Results

Jakob Nordström

University of Copenhagen and Lund University

Universidade Federal de Minas Gerais  
Belo Horizonte, Brazil  
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## Based on Joint Work With...

- Markus Anders
- Jeremias Berg
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# The Success Story of Combinatorial Solving and Optimization

- Rich field of mathematics and computer science
- Impact in other areas of science and also industry, e.g.:
  - airline scheduling
  - hardware verification
  - donor-recipients matching for kidney transplants [MO12, BvdKM<sup>+</sup>21]
- Discrete problems — computationally very challenging (NP-complete or worse)
- Lots of effort last couple of decades spent on developing sophisticated so-called **combinatorial solvers** that often work surprisingly well in practice for, e.g.,
  - Boolean satisfiability (SAT) solving [BHvMW21]
  - Constraint programming [RvBW06]
  - Mixed integer linear programming [AW13, BR07]
  - Satisfiability modulo theories (SMT) solving [BHvMW21]

## And the Dirty Little Secret. . .

- Solvers very fast, but sometimes wrong (even best commercial ones) [BLB10, CKSW13, AGJ<sup>+</sup>18, GSD19, BMN22, GCS23]
- Even worse: No way of knowing for sure when errors happen
- Solvers can propose infeasible “solutions” (but erroneous claims can in principle be checked)
- More challenging: How to achieve reliable claims of infeasibility?
- Or of optimality?
- Even off-by-one mistakes can snowball into large errors if solver used as subroutine

# What Can Be Done About Solver Bugs?

- **Software testing**

Very useful, but bugs slip through even with careful domain-specific testing

Progress using fuzzing and delta debugging [BB09, BLB10, KB22, NPB22, PB23]

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Prove that solver implementation adheres to formal specification

Current techniques cannot scale to level of complexity in modern solvers

(Despite valiant efforts in, e.g., [Fle20])

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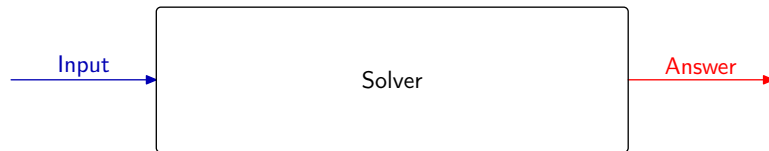
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- **Proof logging**

Make solver **certifying** [ABM<sup>+</sup>11, MMNS11] by adding code so that it outputs

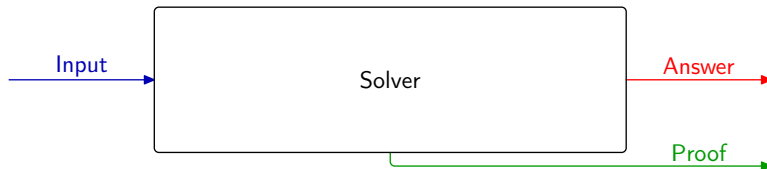
- ① not only **answer** but also
- ② simple, machine-verifiable **proof** that answer is correct

# Proof Logging with Certifying Solvers: Workflow



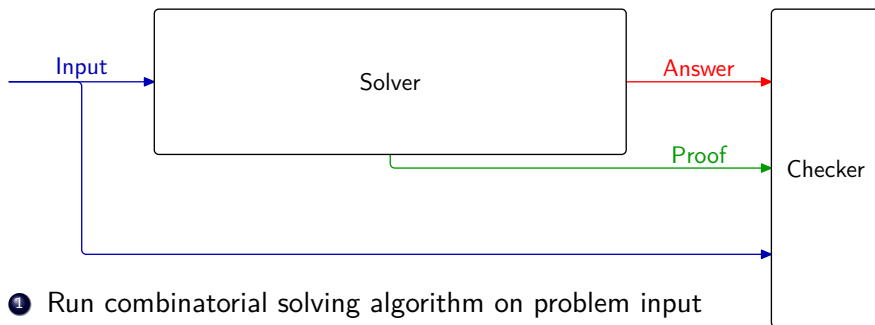
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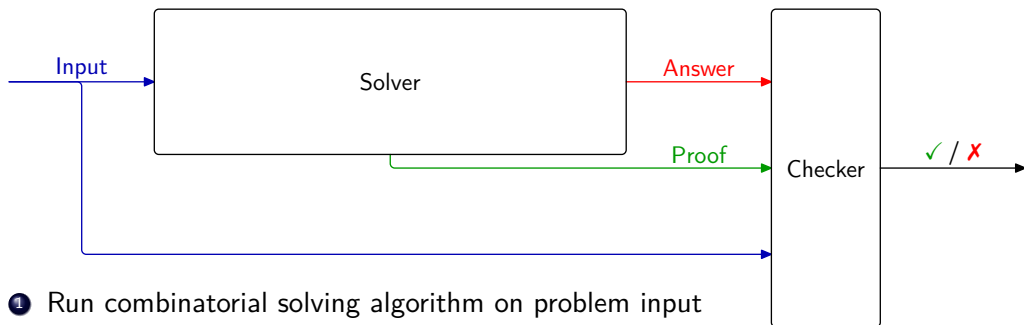
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- ③ Feed input + answer + proof to proof checker

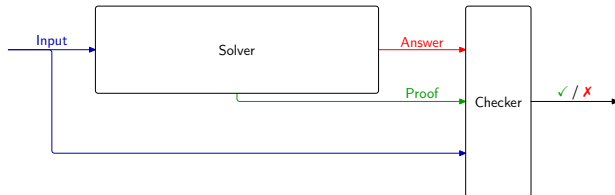
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- ① Run combinatorial solving algorithm on problem input
- ② Get as output not only answer but also proof
- ③ Feed input + answer + proof to proof checker
- ④ Verify that proof checker says answer is correct

# Proof Logging Desiderata

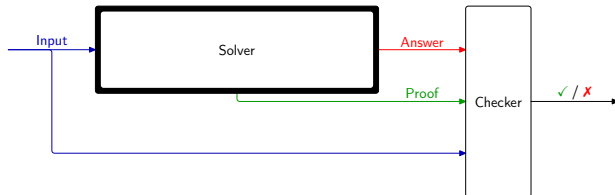
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- **very powerful:** minimal overhead for sophisticated reasoning

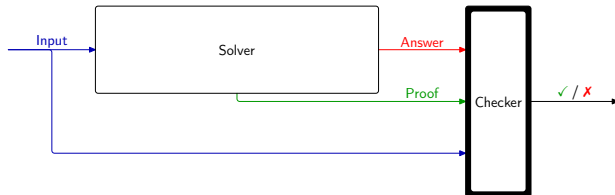




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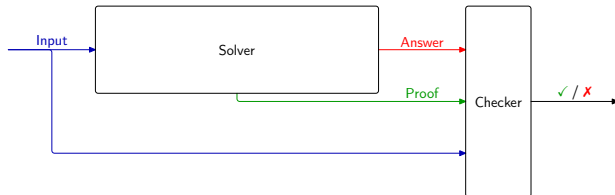


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Clear conflict expressivity vs. simplicity!



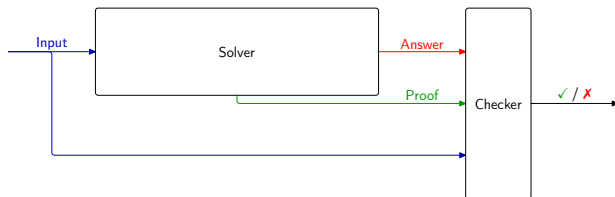
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- **dead simple:** checking correctness of proofs should be (almost) trivial

Clear conflict expressivity vs. simplicity!

Asking for both perhaps a little bit too good to be true?



# Some Previous Proof Logging Work

## Boolean satisfiability (SAT) solving

- Well established since over decade with several proof formats such as DRAT [HHW13a, HHW13b, WHH14], GRIT [CMS17], LRAT [CHH<sup>+</sup>17], ...
- But no efficient support for most advanced techniques

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## Satisfiability modulo theories (SMT) solving

- Immense amount of work (see, e.g., [BBC<sup>+</sup>23, HS22])
- Very complicated proof format, and not yet complete coverage of solver techniques

# Message of This Talk

Proof logging for wide range of combinatorial optimization problems is possible with  
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Proof logging for wide range of combinatorial optimization problems is possible with **single, unified method!**

- Build on successes in proof logging for SAT solving
- But represent constraints as **0–1 integer linear inequalities**
- Formalize reasoning using **cutting planes** [CCT87] proof system
- Add well-chosen **strengthening rules** [Goc22, GN21, BGMN23]
- Implemented in **VERiPB** (<https://gitlab.com/MIA0research/software/VeriPB>)

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- 2 Describe foundations of proof logging method
- 3 Discuss future challenges and directions

# Outline of This Talk

- 1 Combinatorial Solving
  - The Challenge of Ensuring Correctness
  - Can Proof Logging Solve This Problem?
  - This Talk
- 2 Pseudo-Boolean Proof Logging
  - Proof Logging Principles and Goals
  - Pseudo-Boolean Reasoning with the Cutting Planes Method
  - Strengthening Rules
- 3 Pseudo-Boolean Proof Logging Successes and Challenges
  - List of Successes
  - Three Concrete Showcases
  - Some Challenges

# Design Principles for Proof Logging

## Proof logging implementation

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- Proof logging overhead small constant fraction of running time ( $\lesssim 10\%$ )
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## Proof system

- Keep language simple — no XOR constraints, CP propagators, symmetries, ...
- But reason efficiently about such notions using power of proof system
- Combine proof logging with formally verified proof checker



# The Sales Pitch For Proof Logging

- ① Certifies correctness of computed results
- ② Detects errors even if due to compiler bugs, hardware failures, or cosmic rays
- ③ Provides debugging support during software development  
[GMM<sup>+</sup>20, KM21, BBN<sup>+</sup>23, EG23, KLM<sup>+</sup>25]
- ④ Facilitates performance analysis
- ⑤ Helps identify potential for further improvements
- ⑥ Enables auditability
- ⑦ Serves as stepping stone towards explainability

# Proof Language: Pseudo-Boolean Constraints

Proof consists of **0–1 integer linear inequalities** or **pseudo-Boolean constraints**:

$$\sum_i a_i \ell_i \geq A$$

- $a_i, A \in \mathbb{Z}$
- **literals**  $\ell_i$ :  $x_i$  or  $\bar{x}_i$  (where  $x_i + \bar{x}_i = 1$ )
- variables  $x_i$  take values **0 = false** or **1 = true**

Sometimes convenient to use **normalized form** [Bar95] with **all  $a_i, A$  positive** (without loss of generality)

# Some Types of Pseudo-Boolean Constraints

## ① Disjunctive clauses

$$x \vee \bar{y} \vee z \quad \Leftrightarrow \quad x + \bar{y} + z \geq 1$$

## ② Cardinality constraints

$$x_1 + x_2 + x_3 + x_4 \geq 2$$

## ③ General pseudo-Boolean constraints

$$x_1 + 2\bar{x}_2 + 3x_3 + 4\bar{x}_4 + 5x_5 \geq 7$$

# Pseudo-Boolean Proof Logging Wishlist

## Paradigms

- Boolean satisfiability (SAT) solving
- (linear) pseudo-Boolean solving
- subgraph solving
- constraint programming
- automated planning
- mixed integer linear programming
- SMT solving

## Problem types

- decision / feasibility
- optimization
- multi-objective optimization
- projected model enumeration
- projected model counting
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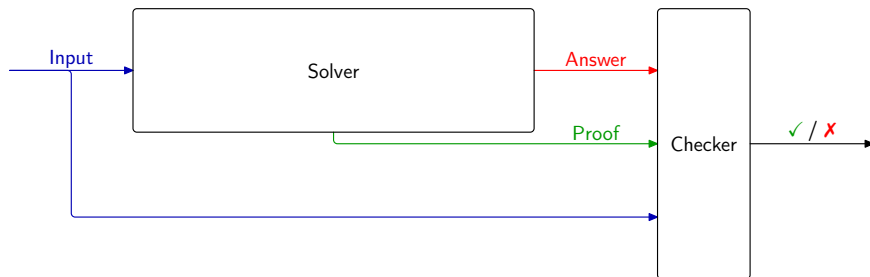
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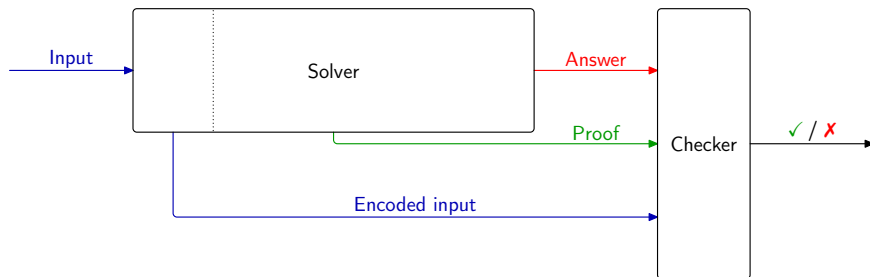
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$$9r + \bar{x}_1 + 2x_2 + 3\bar{x}_3 + 4x_4 + 5\bar{x}_5 \geq 9$$

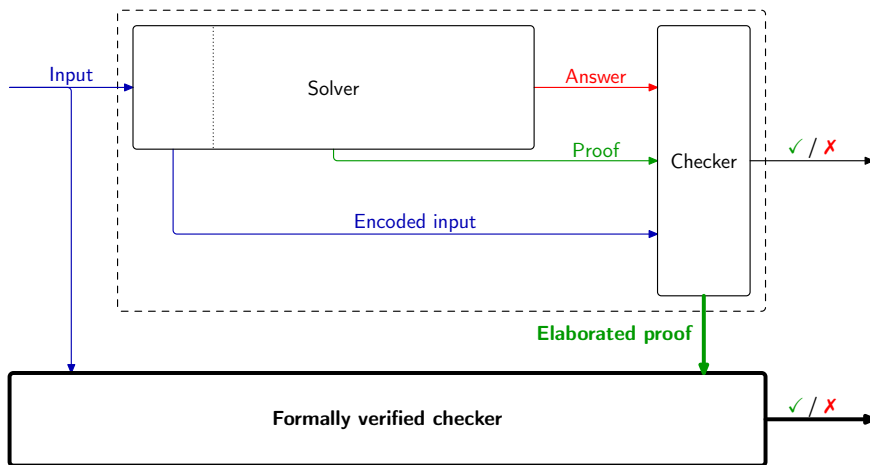
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Load input formula

Specify settings



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Listing of constraints currently in database

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Load input formula  
Specify settings

## ❷ Derivation section

Derivations of new constraints  
Logging of solutions

## ❸ Output section

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## ❹ Conclusions section

Specification of what was established

- satisfiability / unsatisfiability
- optimality (or upper and lower bounds)
- other types of conclusions

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## Core set $\mathcal{C}$

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**Objective**  $f = \sum_i w_i \ell_i + k$

- 0–1 linear function to minimize
- Or  $f = 0$  for decision problem
- Keep track of best known bound; initialize to  $\infty$

# Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

**Input axioms**

From the input

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**Input axioms**

**Literal axioms**

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$$\overline{l_i \geq 0}$$



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**Addition**

From the input

$$\frac{\overline{\ell_i \geq 0} \quad \sum_i a_i \ell_i \geq A \quad \sum_i b_i \ell_i \geq B}{\sum_i (a_i + b_i) \ell_i \geq A + B}$$

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$$\frac{\sum_i a_i l_i \geq A}{\sum_i c a_i l_i \geq cA}$$

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From the input

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 \hline
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 \hline
 \sum_i (a_i + b_i) \ell_i \geq A + B \\
 \\
 \frac{\sum_i a_i \ell_i \geq A}{\sum_i c a_i \ell_i \geq cA} \\
 \\
 \frac{\sum_i a_i \ell_i \geq A}{\sum_i \lceil \frac{a_i}{c} \rceil \ell_i \geq \lceil \frac{A}{c} \rceil}
 \end{array}$$

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**Saturation**

(constraint in normalized form)

From the input

$$\frac{\overline{\ell_i \geq 0} \quad \sum_i a_i \ell_i \geq A \quad \sum_i b_i \ell_i \geq B}{\sum_i (a_i + b_i) \ell_i \geq A + B}$$

$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i c a_i \ell_i \geq c A}$$

$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i \lceil \frac{a_i}{c} \rceil \ell_i \geq \lceil \frac{A}{c} \rceil}$$

$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i \min(a_i, A) \cdot \ell_i \geq A}$$

# Cutting Planes Toy Example

$$w + 2x + y \geq 2$$

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Multiply by 2  $\frac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}$

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Multiply by 2  $\frac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} \quad w + 2x + 4y + 2z \geq 5$

# Cutting Planes Toy Example

$$\begin{array}{rcl}
 \text{Multiply by 2} & \frac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} & \\
 \text{Add} & \frac{2w + 4x + 2y \geq 4 \quad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9} & 
 \end{array}$$



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By referring to constraints by labels and to literal axioms by the literal involved as

$$\text{@C1} \doteq 2x + y + w \geq 2$$

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such a calculation is written in the proof log in reverse Polish notation as

pol @C1 2 \* @C2 + ~z 2 \* + 3 d ;

# Deriving Non-implied Constraints by Redundance-Based Strengthening

$C$  is said to be “**redundant**” with respect to  $F$  if  $F$  and  $F \cup \{C\}$  are **equisatisfiable**  
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Redundance-based strengthening ([BT19, GN21], inspired by [JHB12])

$C$  is redundant with respect to  $F$  if and only if there is a **substitution**  $\omega$  (mapping variables to truth values or literals), called a **witness**, for which

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- In a proof, the implication needs to be **efficiently verifiable** — every  $D \in (F \cup \{C\})|_{\omega}$  should follow from  $F \cup \{\neg C\}$  either
  - ① “obviously” or
  - ② by explicitly presented derivation

## Example: Deriving $r \leftrightarrow (x \wedge y)$ Using the Redundance Rule

Want to derive

$$2\bar{r} + x + y \geq 2$$

$$r + \bar{x} + \bar{y} \geq 1$$

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red 2 ~r 1 x 1 y >= 2 : r -> 0 ;

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# Redundance and Dominance Rules in VERIPB (Slightly Simplified)

Redundance-based strengthening, optimization version [BGMN23]

Add constraint  $C$  to derived set  $\mathcal{D}$  if exists witness substitution  $\omega$  such that

$$C \cup \mathcal{D} \cup \{\neg C\} \models (C \cup \mathcal{D} \cup \{C\}) \upharpoonright_{\omega} \cup \{f \upharpoonright_{\omega} \leq f\}$$

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- Applying  $\omega$  should **strictly decrease**  $f$
- If so, don't need to show that  $(\mathcal{D} \cup \{C\})|_{\omega}$  implied!

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- ⑦ ...
- ⑧ Can't go on forever, so finally reach  $\alpha'$  satisfying  $\mathcal{C} \cup \{C\}$



# Strengthening Rules: Proof Format

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*subproofs for proof goals*  
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- Witness  $\omega$  should be explicitly specified in proof log
- Subproofs of proof goals should also be explicit
- But can be skipped for proof goals “obvious” to proof checker

# Successful Applications of VERIPB Proof Logging

Surprisingly, pseudo-Boolean reasoning with strengthening rules sufficient to efficiently certify wide range of combinatorial solving techniques:

# Successful Applications of VERIPB Proof Logging

Surprisingly, pseudo-Boolean reasoning with strengthening rules sufficient to efficiently certify wide range of combinatorial solving techniques:

- ① **Boolean satisfiability (SAT) solving** including advanced techniques such as
  - Gaussian elimination [GN21]
  - symmetry breaking [BGMN23]
- ② **SAT-based optimization (MaxSAT)** [VDB22, BBN<sup>+</sup>23, BBN<sup>+</sup>24, IOT<sup>+</sup>24]
- ③ (Linear) **Pseudo-Boolean solving** [GMNO22, KLM<sup>+</sup>25]
- ④ **Subgraph solving** (max clique, subgraph isomorphism, max common connected subgraph) [GMN20, GMM<sup>+</sup>20, GMM<sup>+</sup>24]
- ⑤ **Dynamic programming** and **decision diagrams** [DMM<sup>+</sup>24]
- ⑥ **Presolving** in 0–1 integer linear programming [HOGN24]
- ⑦ **Constraint programming** [EGMN20, GMN22, MM23, MMN24, MM25]
- ⑧ **Automated planning** [DHN<sup>+</sup>25]

# Three Pseudo-Boolean Proof Logging Vignettes

- ① Symmetry breaking [BGMN23]
- ② Graph solving (subgraph isomorphism) [GMN20, GMM<sup>+</sup>20, GMM<sup>+</sup>24]
- ③ Constraint programming [EGMN20, GMN22, MM23, MMN24, MM25]

# Symmetry Breaking in SAT Solving

- 1 Pretend to **solve optimisation problem** minimizing  $f \doteq \sum_{i=1}^n 2^{n-i} \cdot x_i$   
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$$f \leq f|_{\sigma} \quad \doteq \quad \sum_{i=1}^n 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0$$

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$y_0$	$\bar{y}_j \vee \overline{\sigma(x_j)} \vee x_j$
$\bar{y}_{j-1} \vee \bar{x}_j \vee \sigma(x_j)$	$y_j \vee \bar{y}_{j-1} \vee \bar{x}_j$
$\bar{y}_j \vee y_{j-1}$	$y_j \vee \bar{y}_{j-1} \vee \sigma(x_j)$



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$$\begin{array}{ll} y_0 \geq 1 & \bar{y}_j + \overline{\sigma(x_j)} + x_j \geq 1 \\ \bar{y}_{j-1} + \bar{x}_j + \sigma(x_j) \geq 1 & y_j + \bar{y}_{j-1} + \bar{x}_j \geq 1 \\ \bar{y}_j + y_{j-1} \geq 1 & y_j + \bar{y}_{j-1} + \sigma(x_j) \geq 1 \end{array}$$

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 \end{array}$$

VERIPB can certify fully general **SAT symmetry breaking** [BGMN23]

# The Subgraph Isomorphism Problem

## Input

- **Pattern** graph  $\mathcal{P}$  with vertices  $V(\mathcal{P}) = \{a, b, c, \dots\}$
- **Target** graph  $\mathcal{T}$  with vertices  $V(\mathcal{T}) = \{u, v, w, \dots\}$

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## Task

- Find all **subgraph isomorphisms**  $\varphi : V(\mathcal{P}) \rightarrow V(\mathcal{T})$
- I.e., one-to-one mappings  $\varphi$  such that if

- ①  $\varphi(a) = u$
- ②  $\varphi(b) = v$
- ③  $(a, b) \in E(\mathcal{P})$

then must have  $(u, v) \in E(\mathcal{T})$

# Pseudo-Boolean Proof Logging for Subgraph Isomorphism Solving

All reasoning steps in Glasgow Subgraph Solver [ADH<sup>+</sup>19, GSS] can be formalized efficiently in the cutting planes proof system [GMN20]

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All reasoning steps in Glasgow Subgraph Solver [ADH<sup>+</sup>19, GSS] can be formalized efficiently in the cutting planes proof system [GMN20]

Means that

- 1 Solver can justify each step by writing local formal derivation
- 2 Local derivations can be chained into global correctness proof
- 3 Proof checkable by stand-alone verifier that knows nothing about graphs
- 4 With end-to-end fully formally verified result [GMM<sup>+</sup>24]

# Subgraph Isomorphism as a Pseudo-Boolean Formula

- **Pattern** graph  $\mathcal{P}$  with  $V(\mathcal{P}) = \{a, b, c, \dots\}$
- **Target** graph  $\mathcal{T}$  with  $V(\mathcal{T}) = \{u, v, w, \dots\}$
- No loops (for simplicity)

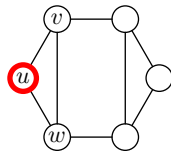
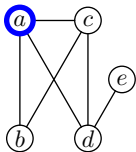
## Pseudo-Boolean encoding

$$\sum_{v \in V(\mathcal{T})} x_{a,v} = 1 \quad [\text{every } a \text{ maps somewhere}]$$

$$\sum_{b \in V(\mathcal{P})} \bar{x}_{b,u} \geq |V(\mathcal{P})| - 1 \quad [\text{mapping is one-to-one}]$$

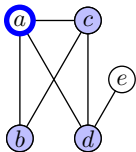
$$\bar{x}_{a,u} + \sum_{v \in N(u)} x_{b,v} \geq 1 \quad [\text{edge } (a, b) \text{ maps to edge } (u, v)]$$

# Pseudo-Boolean Proof Logging Example: Degree Preprocessing





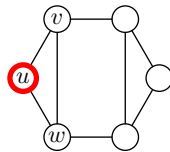
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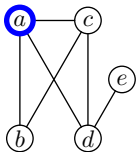
$$\bar{x}_{a,u} + x_{b,v} + x_{b,w} \geq 1$$

$$\bar{x}_{a,u} + x_{c,v} + x_{c,w} \geq 1$$

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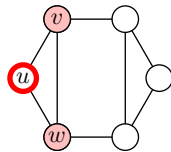
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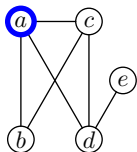
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$$\bar{x}_{a,v} + \bar{x}_{b,v} + \bar{x}_{c,v} + \bar{x}_{d,v} + \bar{x}_{e,v} \geq 4$$

$$\bar{x}_{a,w} + \bar{x}_{b,w} + \bar{x}_{c,w} + \bar{x}_{d,w} + \bar{x}_{e,w} \geq 4$$



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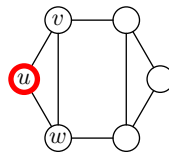
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$$x_{a,v} \geq 0$$

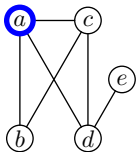
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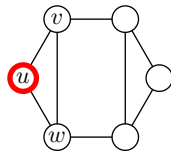
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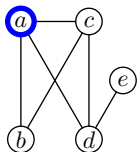
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Sum up all constraints & divide by 3 to obtain

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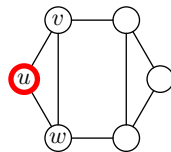
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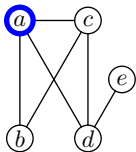
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$$3\bar{x}_{a,u} + 10 \geq 11$$

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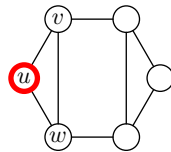
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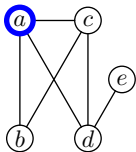
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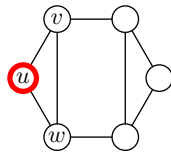
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# Constraint Programming: Integer Variables (1/2)

How to deal with integer variables in constraint programming?

Given  $A \in \{-3 \dots 9\}$ , the direct encoding is:

$$\begin{aligned} a_{=-3} + a_{=-2} + a_{=-1} + a_{=0} + a_{=1} + a_{=2} + a_{=3} \\ + a_{=4} + a_{=5} + a_{=6} + a_{=7} + a_{=8} + a_{=9} = 1 \end{aligned}$$



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This doesn't work for large domains. . .

We can instead use a binary encoding:

$$\begin{aligned} -16a_{\text{neg}} + 1a_{\text{b0}} + 2a_{\text{b1}} + 4a_{\text{b2}} + 8a_{\text{b3}} &\geq -3 && \text{and} \\ 16a_{\text{neg}} + -1a_{\text{b0}} + -2a_{\text{b1}} + -4a_{\text{b2}} + -8a_{\text{b3}} &\geq -9 \end{aligned}$$

Bad properties for solver propagation, but that isn't a problem for proof logging

## Constraint Programming: Integer Variables (2/2)

We can mix binary and order encodings! Define big-M linear inequalities encoding

$$a_{\geq 4} \Leftrightarrow -16a_{\text{neg}} + 1a_{\text{b0}} + 2a_{\text{b1}} + 4a_{\text{b2}} + 8a_{\text{b3}} \geq 4$$

$$a_{\geq 5} \Leftrightarrow -16a_{\text{neg}} + 1a_{\text{b0}} + 2a_{\text{b1}} + 4a_{\text{b2}} + 8a_{\text{b3}} \geq 5$$

$$a_{=4} \Leftrightarrow a_{\geq 4} \wedge \bar{a}_{\geq 5}$$

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When creating  $a_{\geq i}$ , also introduce pseudo-Boolean constraints encoding

$$a_{\geq i} \Rightarrow a_{\geq j} \quad \text{and} \quad a_{\geq h} \Rightarrow a_{\geq i}$$

for the closest values  $j < i < h$  that already exist

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We can do this:

- Inside the pseudo-Boolean model where needed
- Otherwise lazily during proof logging

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Constraints can be specified **extensionally** as list of feasible tuples, called a **table**

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Given table constraint

$$(A, B, C) \in [(1, 2, 3), (1, 3, 4), (2, 2, 5)]$$

define

$$3\bar{t}_1 + a_{=1} + b_{=2} + c_{=3} \geq 3 \quad \text{i.e., } t_1 \Rightarrow (a_{=1} \wedge b_{=2} \wedge c_{=3})$$

$$3\bar{t}_2 + a_{=1} + b_{=4} + c_{=4} \geq 3 \quad \text{i.e., } t_2 \Rightarrow (a_{=1} \wedge b_{=4} \wedge c_{=4})$$

$$3\bar{t}_3 + a_{=2} + b_{=2} + c_{=5} \geq 3 \quad \text{i.e., } t_3 \Rightarrow (a_{=2} \wedge b_{=2} \wedge c_{=5})$$

using tuple selector variables

$$t_1 + t_2 + t_3 = 1$$

# A Constraint Programming Solver with Pseudo-Boolean Proof Logging

Proof-of-concept CP solver at [github.com/ciaranm/glasgow-constraint-solver](https://github.com/ciaranm/glasgow-constraint-solver)  
supports proof logging for global constraints:

- All-different
- Integer linear inequality (including for very large domains)
- Smart table and regular
- Minimum / maximum of an array
- Element (kind of array indexing)
- Absolute value
- (Hamiltonian) Circuit
- and more...

Details in [EGMN20, GMN22, MM23, MMN24, MM25]



## Future Research Directions

### **Performance and reliability of pseudo-Boolean proof logging and checking**

- Trim proof while verifying (as in DRAT-TRIM [HHW13a])
- Compress proof file using binary format
- Formally verified end-to-end checking (as in [GMM<sup>+</sup>24, IOT<sup>+</sup>24, KLM<sup>+</sup>25])
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### Proof logging for other combinatorial problems and techniques

- Model enumeration and counting
- Mixed integer linear programming (*suggested extension of VERIPB in [DEGH23]*)
- SMT solving (*work on solvers CVC5, SMTINTERPOL, Z3, ... [BBC<sup>+</sup>23, HS22]*)

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- **We're hiring!** Talk to me to join the pseudo-Boolean proof logging revolution! ☺

# VERIPB Resources

## VERIPB tutorials

- Slides from tutorials at *CP '22* [BMN22] and *IJCAI '23* [BMN23]
- Video tutorial at [https://youtu.be/s\\_5BIi4I22w](https://youtu.be/s_5BIi4I22w)
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## Technical documentation [ABB<sup>+</sup>25] for SAT 2025 competition

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## VERIPB tutorials

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Lots of concrete example files at [gitlab.com/MIA0research/software/VeriPB](https://gitlab.com/MIA0research/software/VeriPB)



# Summing up

- Combinatorial solving and optimization is a true success story
- But ensuring correctness is a crucial, and not yet satisfactorily addressed, concern
- Certifying solvers producing machine-verifiable proofs of correctness seems like most promising approach
- Cutting planes reasoning with pseudo-Boolean constraints seems to hit a sweet spot between simplicity and expressivity
- **Action point:** What problems can VERIPB solve for you? 😊



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*Thank you for your attention!*



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