Sure, Your Algorithm Is Really Fast, But Is It Really Correct?

Jakob Nordström

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AI Lund Lunch Seminar
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This Is Me...

Jakob Nordström

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. . . And Here Are Three Problems I Get Paid for Thinking About

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land$$
$$(x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

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- Variables should be set to **true** or **false**
- Constraints like $(x \lor \overline{y} \lor z)$ means x or z should be true or y false
- $\land$ means all constraints should hold simultaneously
- Is there a truth value assignment satisfying all constraints?

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$$\begin{aligned}
(1-p)u &= 0 \\
(1-q)(1-r) &= 0 \\
r(1-w) &= 0 \\
(1-u)(1-x)(1-y) &= 0 \\
(1-x)y(1-z) &= 0 \\
x(1-z) &= 0 \\
yz &= 0 \\
xz &= 0 \\
pu &= 0
\end{aligned}$$

For **true** $= 1$ and **false** $= 0$, is there a $\{0,1\}$-valued solution?

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u - pu &= 0 \\
1 - q - r + qr &= 0 \\
r - rw &= 0 \\
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$u - pu = 0$	$p + (1-u) \geq 1$
$1 - q - r + qr = 0$	$q + r \geq 1$
$r - rw = 0$	$(1-r) + w \geq 1$
$1 - u - x - y + ux + uy + xy - uxy = 0$	$u + x + y \geq 1$
$y - xy - yz + xyz = 0$	$x + (1-y) + z \geq 1$
$x - xz = 0$	$(1-x) + z \geq 1$
$yz = 0$	$(1-y) + (1-z) \geq 1$
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$$\begin{aligned} p - u &\geq 0 \\ q + r &\geq 1 \\ -r + w &\geq 0 \\ u + x + y &\geq 1 \\ x - y + z &\geq 0 \\ -x + z &\geq 0 \\ -y - z &\geq -1 \\ -x - z &\geq -1 \\ -p - u &\geq -1 \end{aligned}$$

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Highly Concrete Applications of These Very Abstract Problems

- Software analysis, testing, and synthesis [DMB11]
- Hardware verification [Sha09]
- Air and train traffic control [ABFP12, FFH+16, ZR14]
- Smart crypto contracts [AGRS20, AGH+22]
- Gene regulatory network inference [PBD+22]
- Computational protein design [AAB+14, HD19]
- Assigning donated organs for transplants [MO12, BvdKM+21]
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- Proving theorems in pure mathematics [HK17]

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(Requires fleshing out quite a bit of details, but we don't have time for this in this brief talk. . .)

Bad News

- This type of problems discussed already in Gödel's famous letter in 1956 to von Neumann ("the father of computer science")
- Topic of intense research in computer science and artificial intelligence (AI) ever since early 1960s
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- Modern problems involve **humongous** formulas (100,000s or even 1,000,000s of variables)
- And machine learning approaches do not work

What's the Problem with Machine Learning?

Compare image recognition. . .

- Start with a picture of a cat
- Modify a few pixels
- Still picture of a cat
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- Start with logic formula encoding some technical system or specification
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- Automated reasoning is all about understanding such subtle differences
- Neurosymbolic AI is the idea of using automated reasoning to make ML more reliable

The Success of Combinatorial Solving (and the Dirty Little Secret)

- Revolution since the turn of the millennium in automated reasoning for so-called combinatorial solvers for, e.g.:
  - Boolean satisfiability (SAT) solving and optimization [BHvMW21]
  - Constraint programming [RvBW06]
  - Mixed integer linear programming [AW13, BR07]
  - Satisfiability modulo theories (SMT) solving [BHvMW21]
- Often solve these very hard problems extremely successfully in practice!
- Except the solvers are sometimes wrong... (even best commercial ones) [BLB10, CKSW13, AGJ+18, GSD19, BMN22, BBN+23, GCS23, Tin24]
- Even worse: No way of knowing for sure when errors happen

What Can Be Done About Solver Bugs?

- **Software testing**
  Very useful, but bugs slip through even with careful domain-specific testing
  Progress using fuzzing and delta debugging [BB09, BLB10, KB22, NPB22, PB23]
  But testing inherently can only detect presence of bugs, not absence

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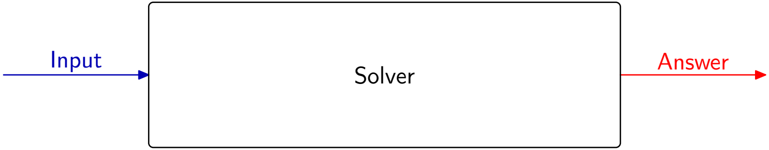
- **Proof logging**
  Make solver certifying [ABM+11, MMNS11] by adding code so that it outputs
  1. not only answer but also
  2. simple, machine-verifiable proof that answer is correct

Proof Logging with Certifying Solvers: Workflow



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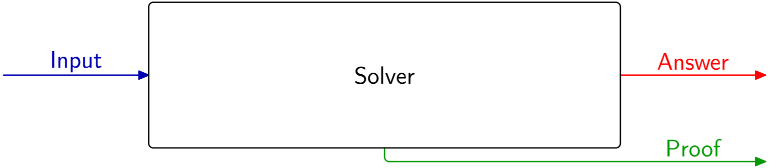

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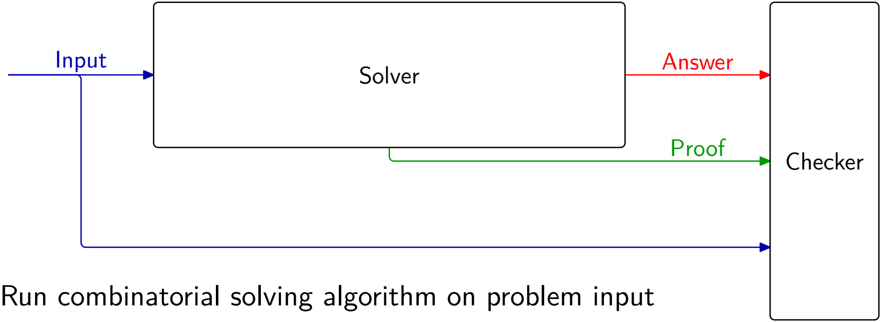

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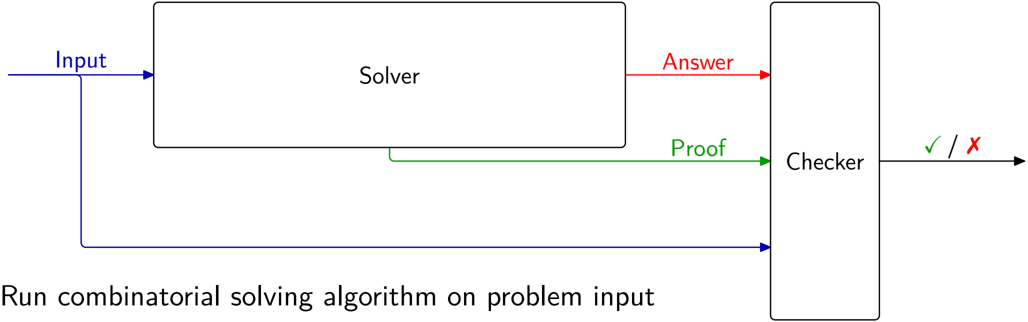

1. Run combinatorial solving algorithm on problem input
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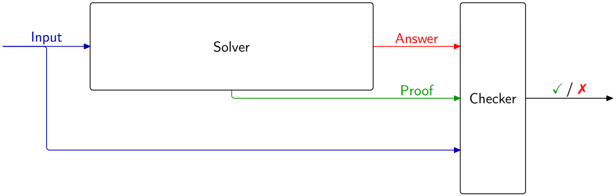
4. Verify that proof checker says answer is correct

Proof Logging Desiderata

Proof format for certifying solver should be



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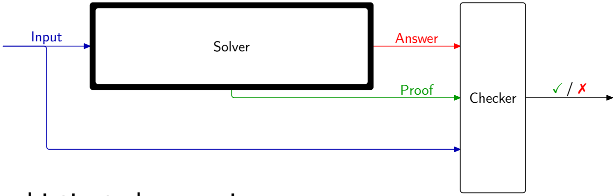

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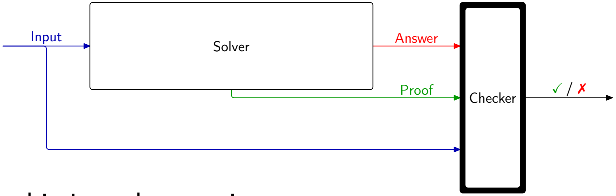

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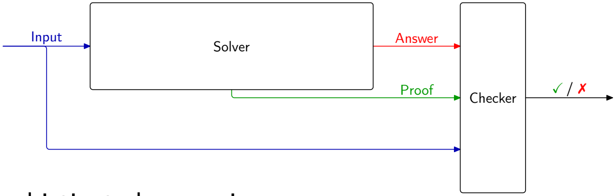

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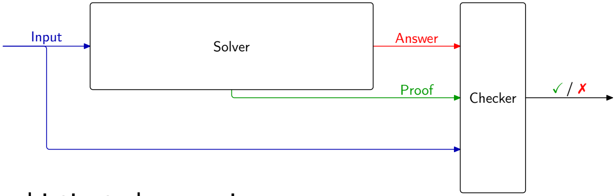

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Asking for both perhaps a little bit too good to be true?

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(These slides with references are online at `https://jakobnordstrom.se/presentations/`)

Design Principles for Proof Logging

Proof logging implementation

- Don't change solver
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Proof system

- Keep language simple — no XOR constraints, theory propagators, symmetries, . . .
- But reason efficiently about such notions using power of proof system
- Combine proof logging with formally verified proof checker

The Sales Pitch For Proof Logging

1. Certifies correctness of computed results
2. Detects errors even if due to compiler bugs, hardware failures, or cosmic rays
3. Provides debugging support during software development [GMM+20, KM21, BBN+23, EG23, Tin24, KLM+25]
4. Facilitates performance analysis
5. Helps identify potential for further improvements
6. Enables auditability
7. Serves as stepping stone towards explainability

The SAT Problem

- Variable x: takes value **true** $(=1)$ or **false** $(=0)$
- Literal ℓ: variable x or its negation $\overline{x}$
- Clause $C = \ell_1 \vee \cdots \vee \ell_k$: disjunction of literals
  (Consider as sets, so no repetitions and order irrelevant)
- Conjunctive normal form (CNF) formula $F = C_1 \wedge \cdots \wedge C_m$: conjunction of clauses

The SAT Problem

Given a CNF formula F, is it satisfiable?

For instance, what about:

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge$$
$$(x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

Proofs for SAT

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For unsatisfiability: a proof is a sequence of clauses

- Each clause follows "obviously" from everything we know so far
- Final clause is empty, meaning contradiction (written $\perp$)
- Means original formula must be inconsistent

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Clause C unit propagates ℓ under partial assignment ρ if ρ falsifies all literals in C except ℓ

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Proof checker should know how to unit propagate until saturation

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DPLL [DP60, DLL62]: Assign variables and propagate; backtrack when clause violated

“Proof trace”: when backtracking, write negation of decisions made

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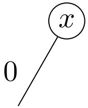

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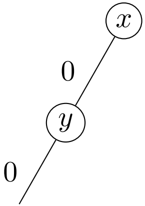

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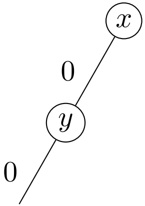

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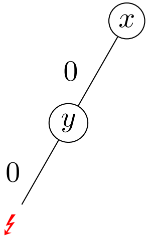

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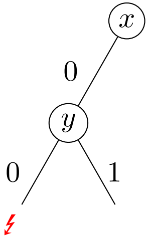

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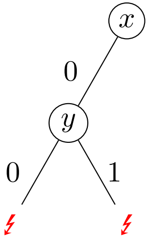

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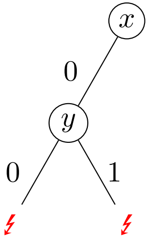
1. $x \lor y$
2. $x \lor \overline{y}$
3. x



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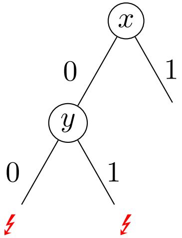

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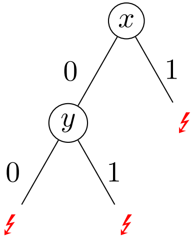
1. $x \lor y$
2. $x \lor \overline{y}$
3. x
4. $\overline{x}$



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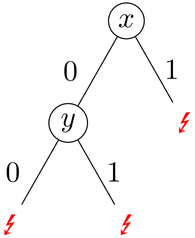
1. $x \vee y$
2. $x \vee \overline{y}$
3. x
4. $\overline{x}$
5. $\perp$



Reverse Unit Propagation (RUP)

- Backtrack clauses are valid constraints inferred by DPLL algorithm
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Reverse unit propagation (RUP) clause [GN03, Van08]

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Fact

Backtrack clauses from DPLL solver generate a RUP proof

What About Modern Conflict-Driven Clause Learning (CDCL)?

Run CDCL SAT solver [BS97, MS99, MMZ+01] on our favourite CNF formula:

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

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Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

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Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

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Always propagate if possible, otherwise decide

Add to assignment trail

Continue until satisfying assignment or conflict

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Trail:

- $p \stackrel{\mathsf{d}}{=} 0$
- $u \stackrel{p \lor \overline{u}}{=} 0$
- $q \stackrel{\mathsf{d}}{=} 0$
- $r \stackrel{q \lor r}{=} 1$

Decision

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$\overset{\overline{y} \lor \overline{z}}{\bot}$

Decision

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- $p \stackrel{\mathsf{d}}{=} 0$
- $u \stackrel{p \lor \overline{u}}{=} 0$

decision level 1

- $q \stackrel{\mathsf{d}}{=} 0$
- $r \stackrel{q \lor r}{=} 1$
- $w \stackrel{\overline{r} \lor w}{=} 1$

decision level 2

- $x \stackrel{\mathsf{d}}{=} 0$
- $y \stackrel{u \lor x \lor y}{=} 1$
- $z \stackrel{x \lor \overline{y} \lor z}{=} 1$
- $\stackrel{\overline{y} \lor \overline{z}}{\bot}$

decision level 3

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

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Notation $u \stackrel{p \lor \overline{u}}{=} 0$ ($p \lor \overline{u}$ is reason clause)

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Add to assignment trail

Continue until satisfying assignment or conflict

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$p \stackrel{\mathsf{d}}{=} 0$

$u \stackrel{p \lor \overline{u}}{=} 0$

decision level 1

$q \stackrel{\mathsf{d}}{=} 0$

$r \stackrel{q \lor r}{=} 1$

$w \stackrel{\overline{r} \lor w}{=} 1$

decision level 2

$x \stackrel{\mathsf{d}}{=} 0$

$y \stackrel{u \lor x \lor y}{=} 1$

$z \stackrel{x \lor \overline{y} \lor z}{=} 1$

$\stackrel{\overline{y} \lor \overline{z}}{\bot}$

decision level 3

Conflict Analysis

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$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$p \stackrel{\mathsf{d}}{=} 0$	decision level 1
$u \stackrel{p \lor \overline{u}}{=} 0$	
$q \stackrel{\mathsf{d}}{=} 0$	decision level 2
$r \stackrel{q \lor r}{=} 1$	
$w \stackrel{\overline{r} \lor w}{=} 1$	
$x \stackrel{\mathsf{d}}{=} 0$	decision level 3
$y \stackrel{u \lor x \lor y}{=} 1$	
$z \stackrel{x \lor \overline{y} \lor z}{=} 1$	
$\stackrel{\overline{y} \lor \overline{z}}{\bot}$	

Could backtrack by erasing conflict level & flipping last decision

Conflict Analysis

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$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$p \stackrel{\mathsf{d}}{=} 0$

$u \stackrel{p \lor \overline{u}}{=} 0$

decision level 1

$q \stackrel{\mathsf{d}}{=} 0$

$r \stackrel{q \lor r}{=} 1$

$w \stackrel{\overline{r} \lor w}{=} 1$

decision level 2

$x \stackrel{\mathsf{d}}{=} 0$

$y \stackrel{u \lor x \lor y}{=} 1$

$z \stackrel{x \lor \overline{y} \lor z}{=} 1$

$\stackrel{\overline{y} \lor \overline{z}}{\bot}$

decision level 3

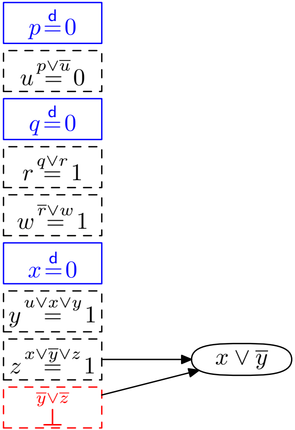
Could backtrack by erasing conflict level & flipping last decision

But want to learn from conflict and cut away as much of search space as possible

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$



Could backtrack by erasing conflict level & flipping last decision

But want to learn from conflict and cut away as much of search space as possible

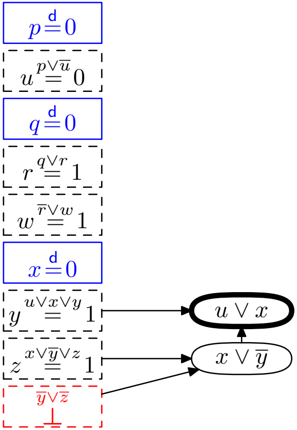
Case analysis over z for last two clauses:

- $x \lor \overline{y} \lor z$ wants $z = 1$
- $\overline{y} \lor \overline{z}$ wants $z = 0$
- Merge clauses & remove z — must satisfy $x \lor \overline{y}$

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$



Could backtrack by erasing conflict level & flipping last decision

But want to learn from conflict and cut away as much of search space as possible

Case analysis over z for last two clauses:

- $x \lor \overline{y} \lor z$ wants $z = 1$
- $\overline{y} \lor \overline{z}$ wants $z = 0$
- Merge clauses & remove z — must satisfy $x \lor \overline{y}$

Repeat until UIP clause with only 1 variable at conflict level after last decision — learn and backjump

Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

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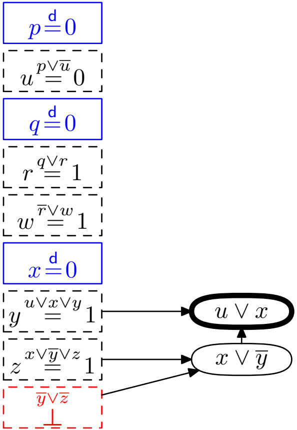

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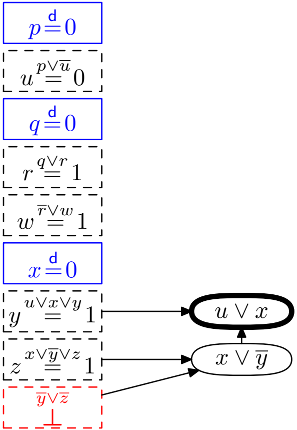

Assertion level 1 (2nd largest level in learned clause) — trim trail to that level

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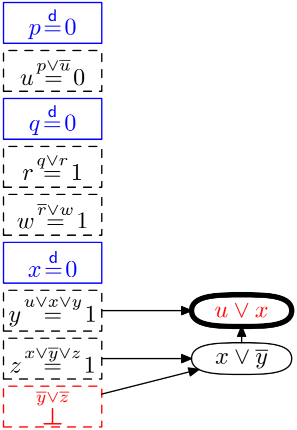

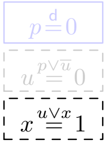

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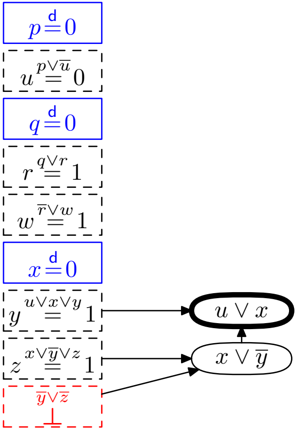

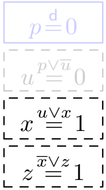

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Then continue as before. . .

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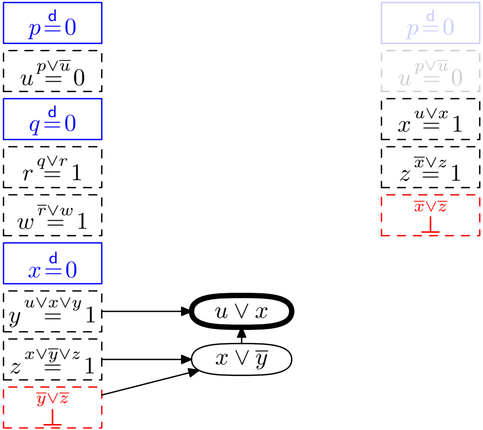

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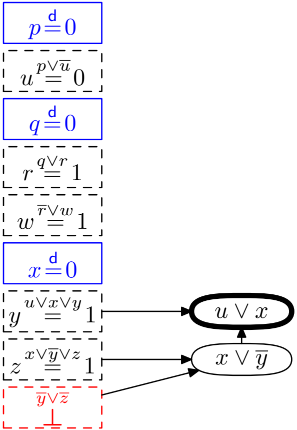

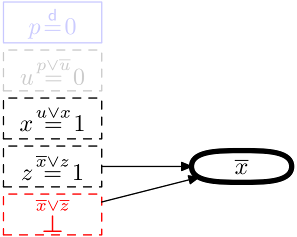

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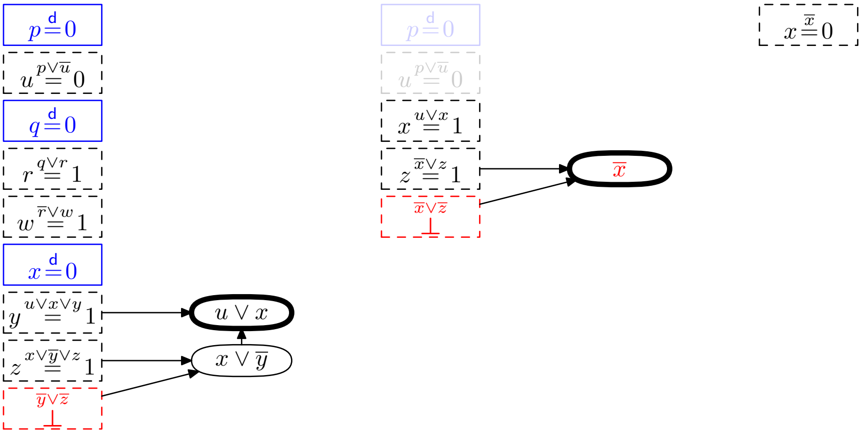

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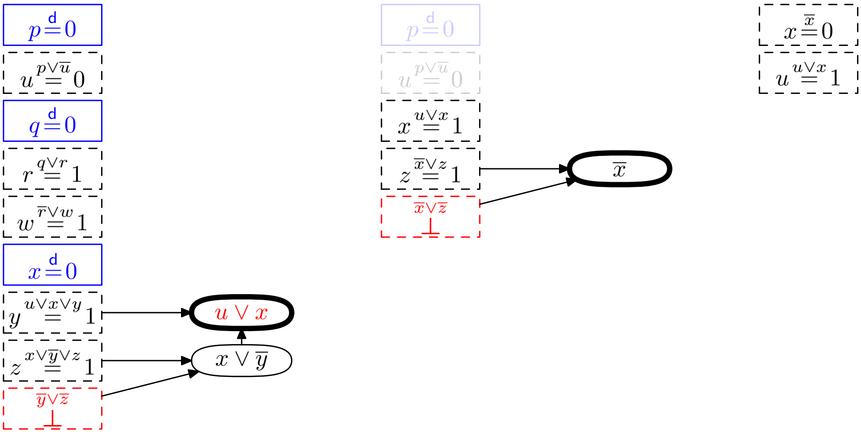

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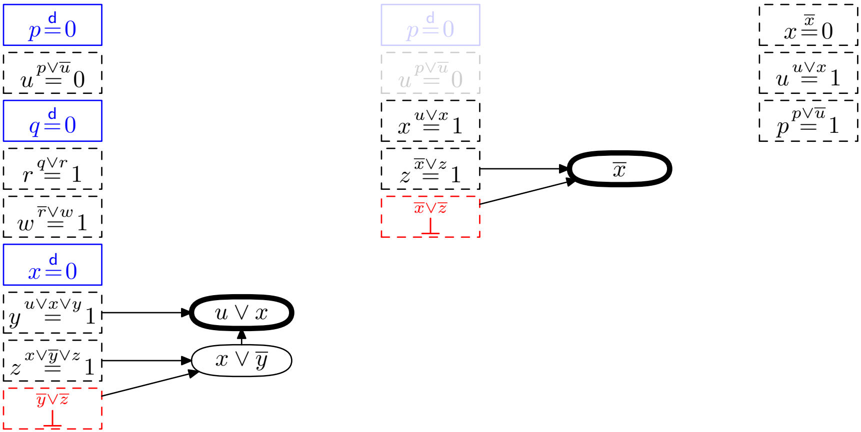

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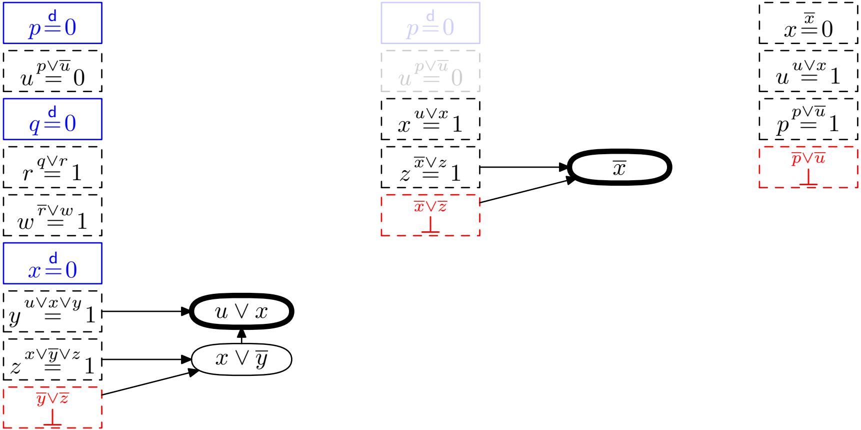

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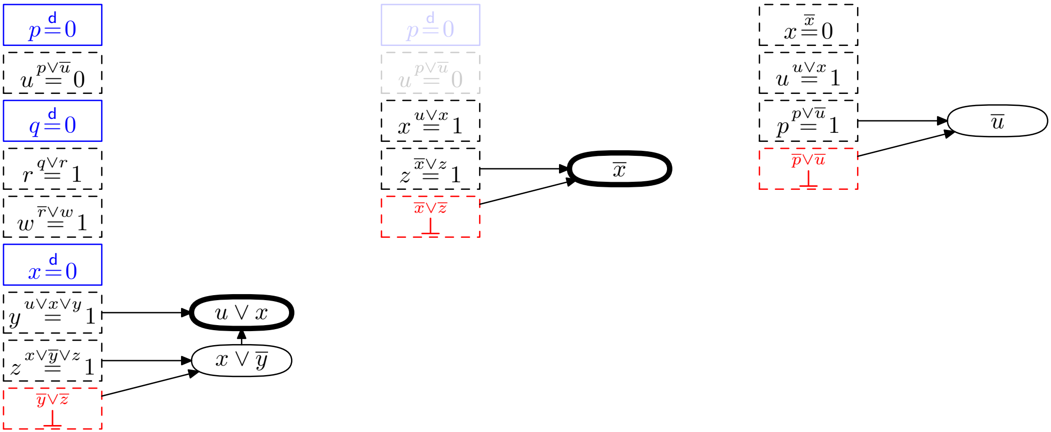

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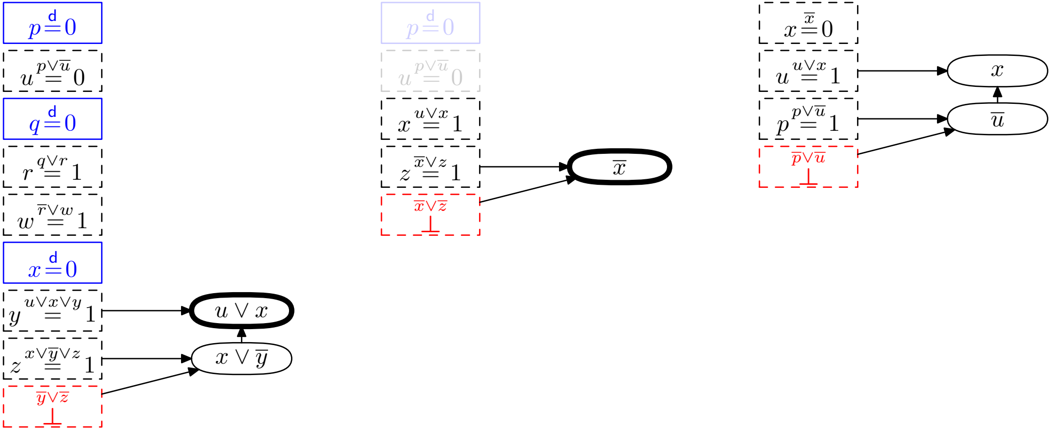

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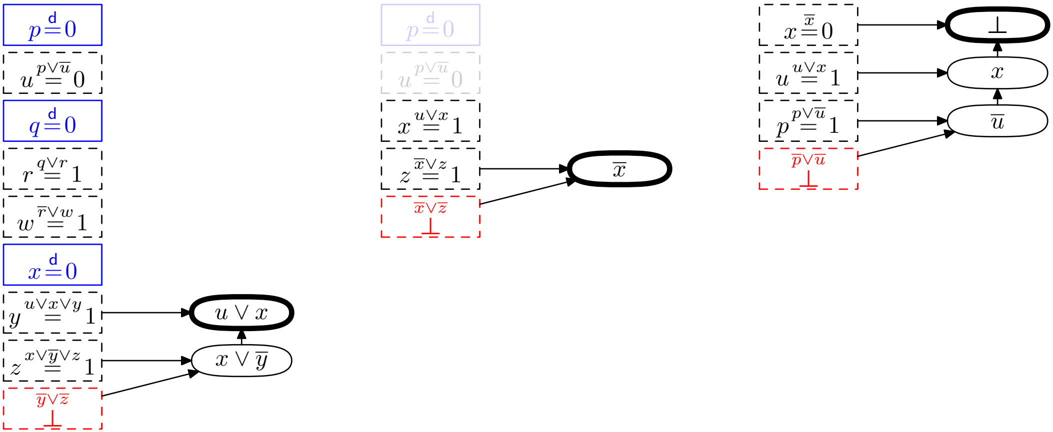

CDCL Reasoning and the Resolution Proof System

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(*) Ignores pre- and inprocessing, but we don't have time to go into this level of detail

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Obtain resolution proof...

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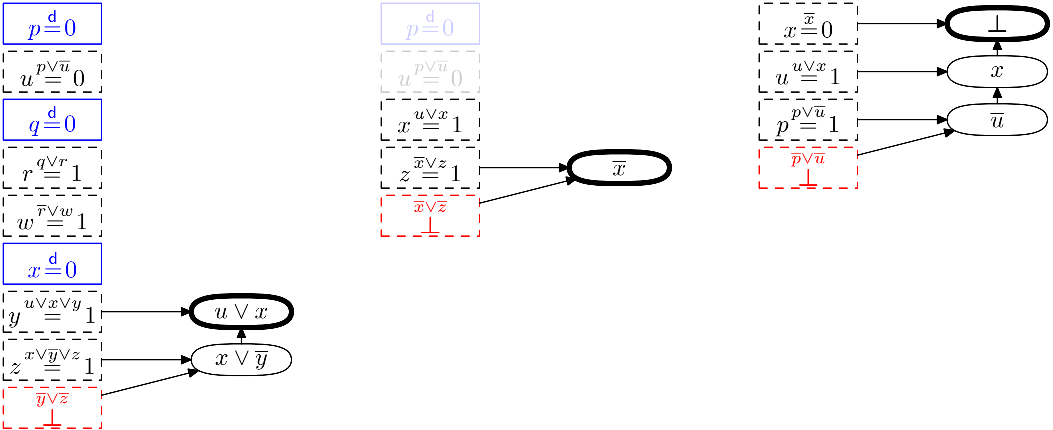

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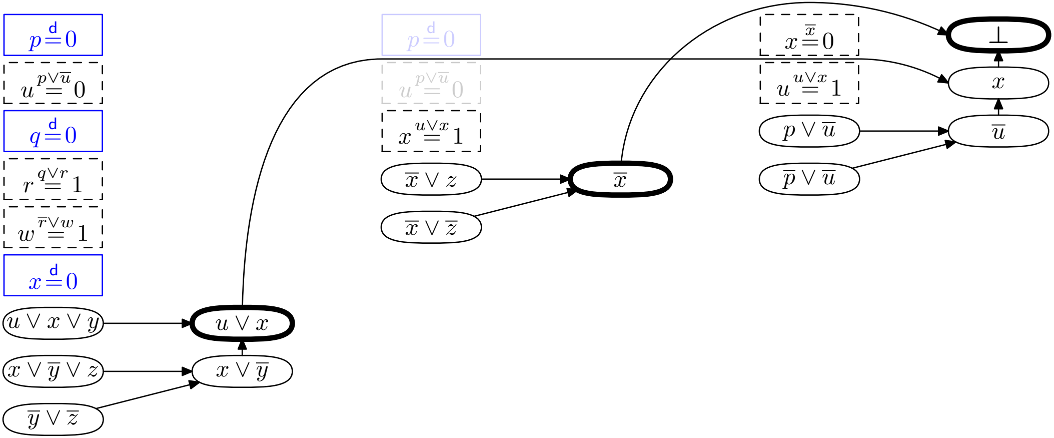

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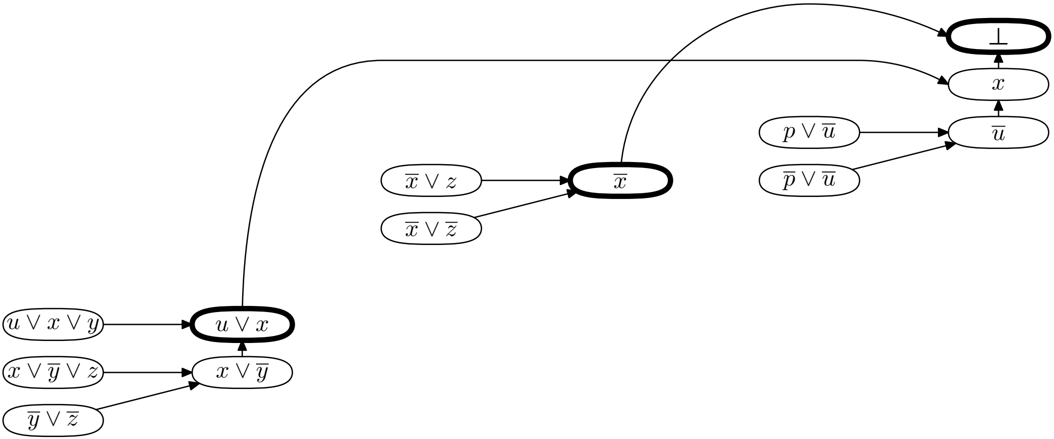

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More Ingredients in Proof Logging for SAT

Fact

RUP proofs can be viewed as shorthand for resolution proofs

See chapter [BN21] in *Handbook of Satisfiability* for more on this

But RUP and resolution are not enough for preprocessing, inprocessing, and some other kinds of advanced SAT solving techniques

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Suppose we want to introduce a new, fresh variable a encoding

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Extended resolution [Tse68]

Resolution rule plus extension rule introducing clauses

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Fact

Extended resolution (RUP + definition of new variables) is essentially equivalent to the DRAT proof logging system [HHW13] most commonly used in proof logging for SAT solving

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Practical limitations of current SAT proof logging technology:

- Difficulties dealing with stronger reasoning efficiently (even for SAT solving)
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Use pseudo-Boolean reasoning with (extension of) the cutting planes proof system [CCT87]

Successful Applications of Pseudo-Boolean Proof Logging

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Surprisingly, pseudo-Boolean reasoning is sufficient to efficiently certify wide range of combinatorial solving techniques:

1. Boolean satisfiability (SAT) solving including advanced techniques such as Gaussian elimination [GN21] and symmetry breaking [BGMN23, ABB+26]
2. SAT-based optimization (MaxSAT) [VDB22, BBN+23, BBN+24, IOT+24]
3. Pseudo-Boolean solving [GMNO22, KLM+25]
4. Subgraph solving (max clique, subgraph isomorphism, max common connected subgraph) [GMN20, GMM+20, GMM+24]
5. Dynamic programming and decision diagrams [DMM+24]
6. Presolving in 0–1 integer linear programming [HOGN24]
7. Constraint programming [EGMN20, GMN22, MM23, MMN24, MM25]
8. Automated planning [DHN+25]

Future Challenges

User-friendliness

- Make proof logging easier for solver authors
- Make proof checking faster
- Improve support beyond $\{0, 1\}$-valued problems for, e.g.,
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Explainability, robustness, fairness

- Certified explanations for decisions or classifications?
- Proofs of neural network robustness to limited input perturbations?
- Rigorous symbolic reasoning about fairness?

Future Challenges

User-friendliness

- Make proof logging easier for solver authors
- Make proof checking faster
- Improve support beyond $\{0,1\}$-valued problems for, e.g.,
  - Mixed integer linear programming *(suggested extension of* VeriPB *in [DEGH23])*
  - Satisfiability modulo theories (SMT) solving *(ongoing work in, e.g., [BBC+23, HS22])*

Explainability, robustness, fairness

- Certified explanations for decisions or classifications?
- Proofs of neural network robustness to limited input perturbations?
- Rigorous symbolic reasoning about fairness?

Lots of other challenging problems and interesting ideas

- This talk will probably sound quite different in two-three years from now

VeriPB Resources

VeriPB tutorials, documentation, and more information can be found at https://VeriPB.org

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Lots of concrete example files at `gitlab.com/MIAOresearch/software/VeriPB`

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We're happy to collaborate — please get in touch if you have computationally challenging problems where correctness is absolutely crucial!

And we're hiring — talk to me if you want to join the proof logging revolution! ☺

Summing up

- Automated reasoning and combinatorial solving are true success stories
- But ensuring correctness is a crucial, and not yet satisfactorily addressed, concern
- Certifying solvers producing machine-verifiable proofs of correctness is the most promising approach
- Cutting planes reasoning with pseudo-Boolean constraints seems to hit a sweet spot between simplicity and expressivity
- **Action point:** What problems can VeriPB solve for you? ☺

Summing up

- Automated reasoning and combinatorial solving are true success stories
- But ensuring correctness is a crucial, and not yet satisfactorily addressed, concern
- Certifying solvers producing machine-verifiable proofs of correctness is the most promising approach
- Cutting planes reasoning with pseudo-Boolean constraints seems to hit a sweet spot between simplicity and expressivity
- **Action point:** What problems can VeriPB solve for you? ☺

Thank you for your attention!

Some Types of Pseudo-Boolean Constraints

1. Clauses

$$x_1 \lor \overline{x}_2 \lor x_3 \quad \Leftrightarrow \quad x_1 + \overline{x}_2 + x_3 \geq 1$$

2. Cardinality constraints

$$x_1 + x_2 + x_3 + x_4 \geq 2$$

3. General pseudo-Boolean constraints

$$x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$$

Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

Input/model axioms

From the input

Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

Input/model axioms From the input

Literal axioms $$\frac{}{\ell_i \geq 0}$$

Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

Input/model axioms From the input

Literal axioms

$$\frac{}{\ell_i \geq 0}$$

Addition

$$\frac{\sum_i a_i \ell_i \geq A \qquad \sum_i b_i \ell_i \geq B}{\sum_i (a_i + b_i)\ell_i \geq A + B}$$

Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

Input/model axioms From the input

Literal axioms
$$\frac{}{\ell_i \geq 0}$$

Addition
$$\frac{\sum_i a_i\ell_i \geq A \qquad \sum_i b_i\ell_i \geq B}{\sum_i (a_i + b_i)\ell_i \geq A + B}$$

Multiplication for any $c \in \mathbb{N}^+$
$$\frac{\sum_i a_i\ell_i \geq A}{\sum_i c a_i\ell_i \geq cA}$$

Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

Input/model axioms — From the input

Literal axioms

$$\frac{}{\ell_i \geq 0}$$

Addition

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Multiplication for any $c \in \mathbb{N}^+$

$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i c a_i \ell_i \geq cA}$$

Division for any $c \in \mathbb{N}^+$
(assumes normalized form)

$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i \left\lceil \frac{a_i}{c} \right\rceil \ell_i \geq \left\lceil \frac{A}{c} \right\rceil}$$

Cutting Planes Toy Example

$$w + 2x + y \geq 2$$

Cutting Planes Toy Example

Multiply by 2 $$\frac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}$$

Cutting Planes Toy Example

Multiply by 2 $\dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}$ $\quad w + 2x + 4y + 2z \geq 5$

Cutting Planes Toy Example

Multiply by 2
$$\frac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}$$

Add
$$\frac{2w + 4x + 2y \geq 4 \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9}$$

Cutting Planes Toy Example

$$\text{Multiply by } 2 \; \frac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} \qquad w + 2x + 4y + 2z \geq 5 \qquad \overline{z} \geq 0$$

$$\text{Add} \; \frac{2w + 4x + 2y \geq 4 \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9}$$

Cutting Planes Toy Example

$$\dfrac{\text{Multiply by 2}\;\dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9}\;\text{Add} \qquad\qquad \dfrac{\overline{z} \geq 0}{2\overline{z} \geq 0}\;\text{Multiply by 2}$$

Cutting Planes Toy Example

$$
\dfrac{\text{Multiply by 2}\ \dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9}\ \text{Add} \qquad \dfrac{\overline{z} \geq 0}{2\overline{z} \geq 0}\ \text{Multiply by 2}
$$

$$
\text{Add}\ \dfrac{3w + 6x + 6y + 2z \geq 9 \qquad 2\overline{z} \geq 0}{3w + 6x + 6y + 2z + 2\overline{z} \geq 9}
$$

Cutting Planes Toy Example

$$\dfrac{\dfrac{\dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}\text{ Multiply by 2} \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9}\text{ Add} \qquad \dfrac{\overline{z} \geq 0}{2\overline{z} \geq 0}\text{ Multiply by 2}}{3w + 6x + 6y + 2 \qquad \geq 9}\text{ Add}$$

Cutting Planes Toy Example

$$\dfrac{\dfrac{\dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}\text{ Multiply by 2} \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9}\text{ Add} \qquad \dfrac{\overline{z} \geq 0}{2\overline{z} \geq 0}\text{ Multiply by 2}}{3w + 6x + 6y \geq 7}\text{ Add}$$

Cutting Planes Toy Example

Multiply by 2: $$\frac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}$$

Add: $$\frac{2w + 4x + 2y \geq 4 \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9}$$

Multiply by 2: $$\frac{\overline{z} \geq 0}{2\overline{z} \geq 0}$$

Add: $$\frac{3w + 6x + 6y + 2z \geq 9 \qquad 2\overline{z} \geq 0}{3w + 6x + 6y \geq 7}$$

Divide by 3: $$\frac{3w + 6x + 6y \geq 7}{w + 2x + 2y \geq 2\tfrac{1}{3}}$$

Cutting Planes Toy Example

$$\dfrac{\dfrac{\dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}\text{Multiply by 2} \quad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9}\text{Add} \quad \dfrac{\overline{z} \geq 0}{2\overline{z} \geq 0}\text{Multiply by 2}}{\dfrac{3w + 6x + 6y \geq 7}{w + 2x + 2y \geq 3}\text{Divide by 3}}\text{Add}$$

Cutting Planes Toy Example

$$
\dfrac{\dfrac{\dfrac{\dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}\,\text{Multiply by } 2 \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9}\,\text{Add} \qquad \dfrac{\overline{z} \geq 0}{2\overline{z} \geq 0}\,\text{Multiply by } 2}{3w + 6x + 6y \geq 7}\,\text{Add}}{w + 2x + 2y \geq 3}\,\text{Divide by } 3
$$

By referring to constraints by labels and to literal axioms by the literal involved as

$$
\begin{aligned}
\texttt{@C1} &\doteq 2x + y + w \geq 2 \\
\texttt{@C2} &\doteq 2x + 4y + 2z + w \geq 5 \\
\texttt{\~{}z} &\doteq \overline{z} \geq 0
\end{aligned}
$$

Cutting Planes Toy Example

$$\dfrac{\dfrac{\dfrac{\dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}\ \text{Multiply by } 2 \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9}\ \text{Add} \qquad \dfrac{\overline{z} \geq 0}{2\overline{z} \geq 0}\ \text{Multiply by } 2}{3w + 6x + 6y \geq 7}\ \text{Add}}{w + 2x + 2y \geq 3}\ \text{Divide by } 3$$

By referring to constraints by labels and to literal axioms by the literal involved as

$$\begin{aligned} \texttt{@C1} &\doteq 2x + y + w \geq 2 \\ \texttt{@C2} &\doteq 2x + 4y + 2z + w \geq 5 \\ \texttt{\~{}z} &\doteq \overline{z} \geq 0 \end{aligned}$$

such a calculation is written in the proof log in reverse Polish notation as

```
pol @C1 2 * @C2 + ~z 2 * + 3 d ;
```

Resolution and Cutting Planes

To simulate resolution step such as

$$\frac{\overline{y} \lor \overline{z} \qquad x \lor \overline{y} \lor z}{x \lor \overline{y}}$$

we can perform the cutting planes steps

$$\text{Add} \; \cfrac{\overline{y} + \overline{z} \geq 1 \qquad x + \overline{y} + z \geq 1}{\text{Divide by } 2 \; \cfrac{x + 2\overline{y} \geq 1}{x + \overline{y} \geq 1}}$$

Resolution and Cutting Planes

To simulate resolution step such as

$$\frac{\overline{y} \vee \overline{z} \qquad x \vee \overline{y} \vee z}{x \vee \overline{y}}$$

we can perform the cutting planes steps

$$\text{Divide by 2}\ \frac{\text{Add}\ \dfrac{\overline{y} + \overline{z} \geq 1 \qquad x + \overline{y} + z \geq 1}{x + 2\overline{y} \geq 1}}{x + \overline{y} \geq 1}$$

Given that the premises are clauses 7 and 5 in our example CNF formula, using references

$$\text{@C7} \doteq \overline{y} + \overline{z} \geq 1$$
$$\text{@C5} \doteq x + \overline{y} + z \geq 1$$

we can write this in the proof log as

```
pol @C7 @C5 + 2 d
```

Pseudo-Boolean Proof Logging — How and Why?

If problem is (special case of) 0-1 integer linear program

- just do proof logging [basically: add print statements to solver code]

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Otherwise

- do trusted or verified translation to 0-1 ILP
- do proof logging for 0-1 ILP formulation [but solver still works with original input]

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If problem is (special case of) 0-1 integer linear program

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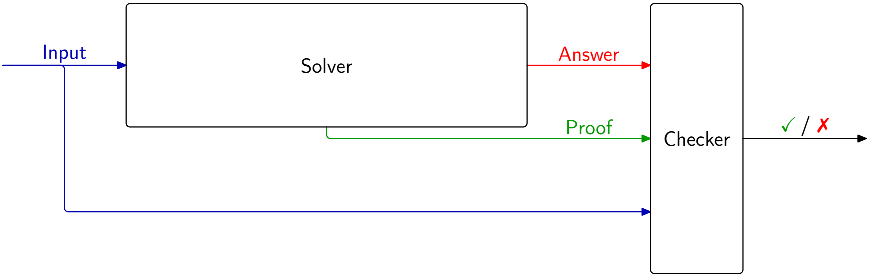
Otherwise

- do trusted or verified translation to 0-1 ILP
- do proof logging for 0-1 ILP formulation [but solver still works with original input]

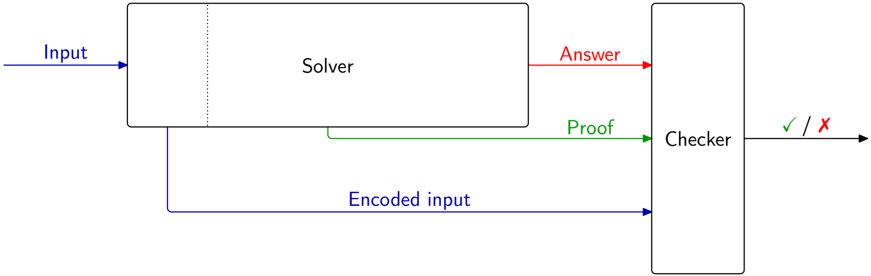
Goldilocks compromise between expressivity and simplicity:

1. 0-1 ILP expressive formalism for combinatorial problems (including objective)
2. Powerful reasoning capturing many combinatorial arguments

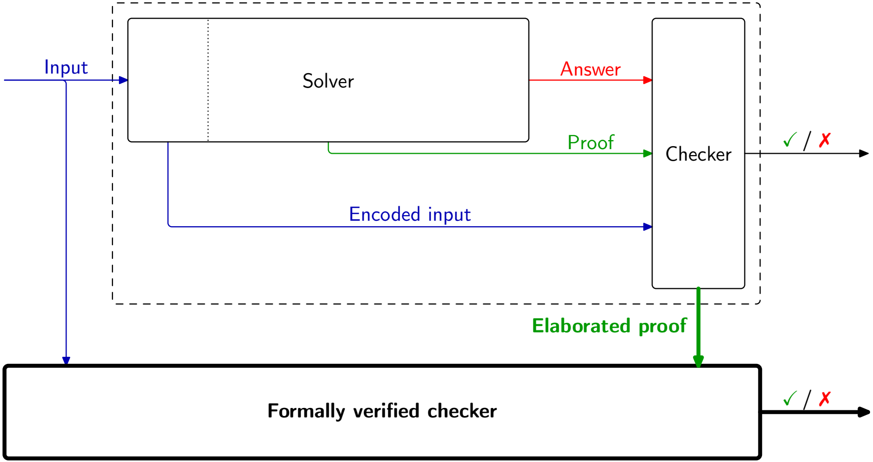
Proof Logging with Formally Verified Checking: Full Workflow



Proof Logging with Formally Verified Checking: Full Workflow



Proof Logging with Formally Verified Checking: Full Workflow



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