Combinatorial Solving with Provably Correct Results

Bart Bogaerts Ciaran McCreesh
Jakob Nordström

Combinatorial Solving and Optimisation

- Revolution last couple of decades in combinatorial solvers for
  - Boolean satisfiability (SAT) solving [BHvMW21][1]
  - Constraint programming (CP) [RvBW06]
  - Mixed integer linear programming (MIP) [AW13, BR07]

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- Solve NP problems (or worse) very successfully in practice!
- Except solvers are sometimes wrong… (Even best commercial ones) [BLB10, CKSW13, AGJ$^+$18, GSD19, GS19]

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The Controversial Slide

In the 2021 constraint programming MiniZinc challenge: for 1.28% of instances, wrong solutions were claimed.

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- Infeasible solutions produced.

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Obviously, *your* solver doesn't have this problem, but how do you convince others of this?

Testing?

Various domain-specific testing methods [BLB10, AGJ+18, GSD19].

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Definitely better than nothing, but is it enough?

- Clearly not: bugs are found in thoroughly tested solvers as well.
- Testing can only reveal the presence of bugs, not their absence.

Formal Methods?

Prove that solver implementation adheres to formal specification.

Current techniques cannot scale to level of complexity in modern solvers.

- In SAT solver competition, formally verified solvers are far behind in terms of performance (and available techniques).
- In constraint programming, even an inefficient implementation of all-different is pushing the limits [Dub20].

A Simple but Crucial Change of Perspective

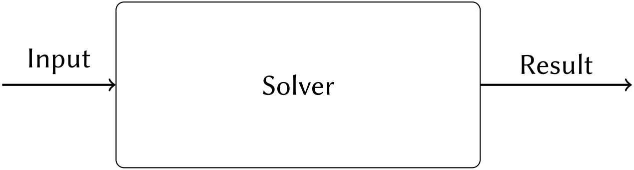
State-of-the-art SAT solvers instead use proof logging.

- Make solvers certifying [ABM⁺11, MMNS11].
- Output proof of correctness in standard format that is independently verified.

A variety of proof logging formats introduced, including

- DRAT [HHW13a, HHW13b, WHH14]
- GRIT [CMS17]
- LRAT [CHH⁺17]
- ...

Proof Logging Workflow



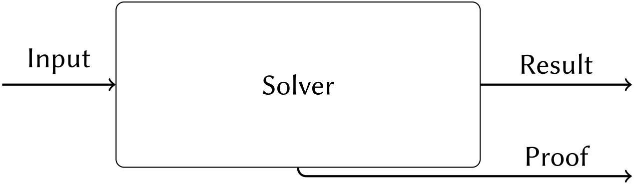
Input

Solver

Result

1. Run solver on problem input.

Proof Logging Workflow



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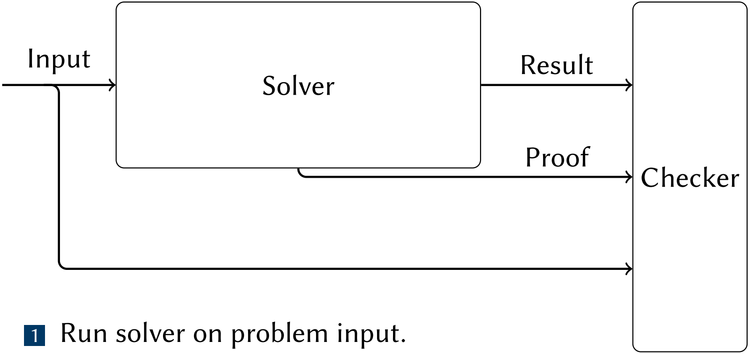

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3. Feed input + result + proof to proof checker.

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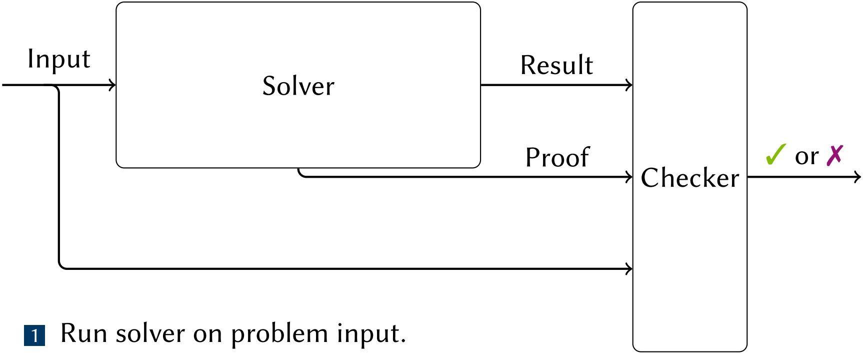

1. Run solver on problem input.
2. Get as output not only result but also proof.
3. Feed input + result + proof to proof checker.
4. Verify that proof checker says result is correct.

Requirements

Proofs produced by certifying solver should:

- Be powerful enough for proof logging to incur minimal overhead.
- Be based on very simple rules.
- Not require knowledge of inner workings of solver.
- Allow verification by stand-alone proof checker.

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- Not require knowledge of inner workings of solver.
- Allow verification by stand-alone proof checker.

Much easier to trust a small, simple checker than a full solver.

- Should even be simple enough to be formally verified.

Does not prove solver correct, but proves solution correct.

The Sales Pitch For Proof Logging

1. Certifies correctness of computed results.
2. Detects errors even if due to compiler bugs, hardware failures, or cosmic rays.
3. Provides debugging support during development [EG21, GMM⁺20, KM21].
4. Facilitates performance analysis.
5. Helps identify potential for further improvements.
6. Enables auditability.
7. Serves as stepping stone towards explainability.

The Rest of This Tutorial

VeriPB (`https://gitlab.com/MIAOresearch/software/VeriPB`)

Versatile proof logging system that can handle

- Subgraph algorithms
- Constraint programming
- Symmetry and dominance reasoning

in a unified way.

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But first we need to tell you about:

- Proof logging for SAT.
- Pseudo-Boolean reasoning and cutting planes.

The SAT Problem

- Variable x: takes value **true** $(=1)$ or **false** $(=0)$
- Literal ℓ: variable x or its negation $\overline{x}$
- Clause $C = \ell_1 \lor \cdots \lor \ell_k$: disjunction of literals
  (Consider as sets, so no repetitions and order irrelevant)
- Conjunctive normal form (CNF) formula $F = C_1 \land \cdots \land C_m$: conjunction of clauses

The SAT Problem

Given a CNF formula F, is it satisfiable?

For instance, what about:

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land$$
$$(x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

Proofs for SAT

For satisfiable instances: just specify a satisfying assignment.

For unsatisfiability: a sequence of clauses (CNF constraints).

- Each clause follows “obviously” from everything we know so far.
- Final clause is empty, meaning contradiction (written $\perp$).
- Means original formula must be inconsistent.

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Unit Propagation

Clause C unit propagates ℓ under partial assignment ρ if ρ falsifies all literals in C except ℓ.

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Proof checker should know how to unit propagate until saturation.

Davis-Putman-Logemann-Loveland (DPLL)

DPLL [DP60, DLL62]: Assign variables and propagate; backtrack when clause violated.

“Proof trace”: when backtracking, write negation of guesses made.

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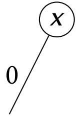

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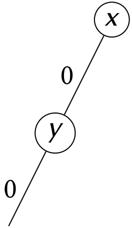

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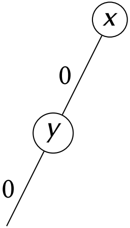

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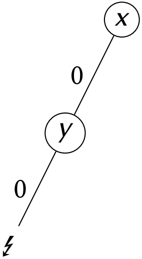

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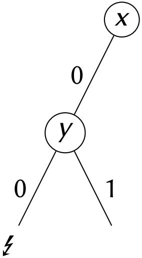

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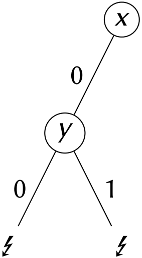

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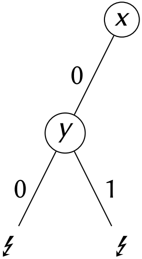
3. x



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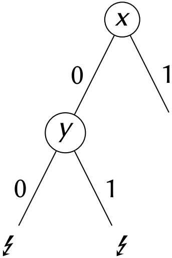

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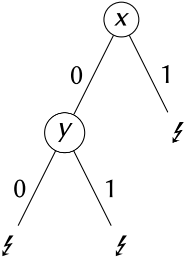

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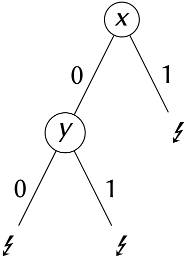
1 $x \vee y$

2 $x \vee \overline{y}$

3 x

4 $\overline{x}$

5 $\perp$



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If so, F clearly implies C, and condition easy to verify efficiently

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Fact

Backtrack clauses from DPLL solver generate a RUP proof.

What About Conflict-Driven Clause Learning (CDCL)?

Run CDCL [BS97, MS99, MMZ+01] on our favourite CNF formula:

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Free choice to assign value to variable

Notation $p \overset{\mathrm{d}}{=} 0$

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Notation $p \overset{\mathrm{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \vee \overline{u}$ forces $u = 0$

Notation $u \overset{p \vee \overline{u}}{=} 0$ ($p \vee \overline{u}$ is reason clause)

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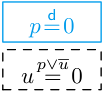
$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$



$\boxed{p \overset{\mathsf{d}}{=} 0}$

$u \overset{p \lor \overline{u}}{=} 0$

Decision

Free choice to assign value to variable

Notation $p \overset{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

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Always propagate if possible, else decide

Add to assignment trail

Until satisfying assignment or conflict

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$\boxed{p \overset{\mathrm{d}}{=} 0}$

$u \overset{p \lor \overline{u}}{=} 0$

$\boxed{q \overset{\mathrm{d}}{=} 0}$

$r \overset{q \lor r}{=} 1$

Decision

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$u \overset{p \lor \overline{u}}{=} 0$

$\boxed{q \overset{\mathsf{d}}{=} 0}$

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$$\boxed{p \overset{\mathsf{d}}{=} 0}$$
$$u \overset{p \lor \overline{u}}{=} 0$$
$$\boxed{q \overset{\mathsf{d}}{=} 0}$$
$$r \overset{q \lor r}{=} 1$$
$$w \overset{\overline{r} \lor w}{=} 1$$
$$\boxed{x \overset{\mathsf{d}}{=} 0}$$

Decision

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$\boxed{x \overset{\mathsf{d}}{=} 0}$

$y \overset{u \vee x \vee y}{=} 1$

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$\boxed{q \overset{\mathsf{d}}{=} 0}$

$r \overset{q\vee r}{=} 1$

$w \overset{\overline{r}\vee w}{=} 1$

$\boxed{x \overset{\mathsf{d}}{=} 0}$

$y \overset{u\vee x\vee y}{=} 1$

$z \overset{x\vee\overline{y}\vee z}{=} 1$

$\overset{\overline{y}\vee\overline{z}}{\perp}$

Decision

Free choice to assign value to variable

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Unit propagation

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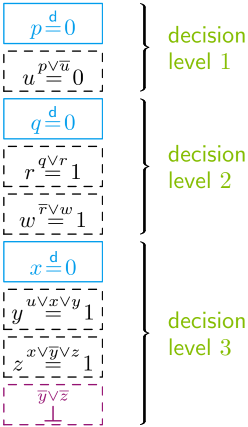


$p \overset{\mathsf{d}}{=} 0$

$u \overset{p \lor \overline{u}}{=} 0$

decision level 1

$q \overset{\mathsf{d}}{=} 0$

$r \overset{q \lor r}{=} 1$

$w \overset{\overline{r} \lor w}{=} 1$

decision level 2

$x \overset{\mathsf{d}}{=} 0$

$y \overset{u \lor x \lor y}{=} 1$

$z \overset{x \lor \overline{y} \lor z}{=} 1$

$\overset{\overline{y} \lor \overline{z}}{\bot}$

decision level 3

Decision

Free choice to assign value to variable

Notation $p \overset{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

Notation $u \overset{p \lor \overline{u}}{=} 0$ ($p \lor \overline{u}$ is reason clause)

Always propagate if possible, else decide

Add to assignment trail

Until satisfying assignment or conflict

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

$$\left.\begin{array}{c} \boxed{p \overset{\mathsf{d}}{=} 0} \\ u \overset{p \vee \overline{u}}{=} 0 \end{array}\right\} \text{decision level } 1$$

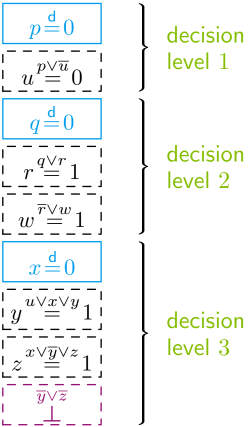
$$\left.\begin{array}{c} \boxed{q \overset{\mathsf{d}}{=} 0} \\ r \overset{q \vee r}{=} 1 \\ w \overset{\overline{r} \vee w}{=} 1 \end{array}\right\} \text{decision level } 2$$

$$\left.\begin{array}{c} \boxed{x \overset{\mathsf{d}}{=} 0} \\ y \overset{u \vee x \vee y}{=} 1 \\ z \overset{x \vee \overline{y} \vee z}{=} 1 \\ \overset{\overline{y} \vee \overline{z}}{\bot} \end{array}\right\} \text{decision level } 3$$

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$



$p \overset{\mathsf{d}}{=} 0$

$u \overset{p \vee \overline{u}}{=} 0$

decision level 1

$q \overset{\mathsf{d}}{=} 0$

$r \overset{q \vee r}{=} 1$

$w \overset{\overline{r} \vee w}{=} 1$

decision level 2

$x \overset{\mathsf{d}}{=} 0$

$y \overset{u \vee x \vee y}{=} 1$

$z \overset{x \vee \overline{y} \vee z}{=} 1$

$\overset{\overline{y} \vee \overline{z}}{\bot}$

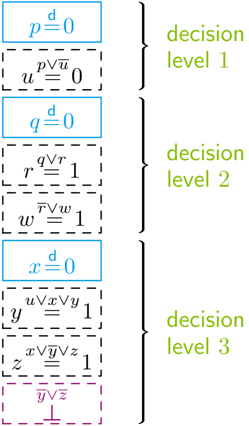
decision level 3

Could backtrack by flipping last decision

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$



$p \overset{\mathsf{d}}{=} 0$

$u \overset{p\vee\overline{u}}{=} 0$

decision level 1

$q \overset{\mathsf{d}}{=} 0$

$r \overset{q\vee r}{=} 1$

$w \overset{\overline{r}\vee w}{=} 1$

decision level 2

$x \overset{\mathsf{d}}{=} 0$

$y \overset{u\vee x\vee y}{=} 1$

$z \overset{x\vee\overline{y}\vee z}{=} 1$

$\overset{\overline{y}\vee\overline{z}}{\bot}$

decision level 3

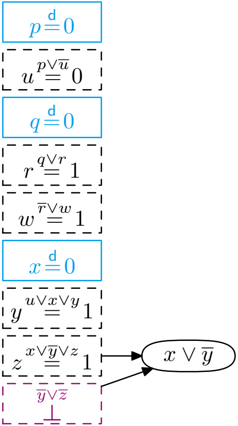
Could backtrack by flipping last decision

But want to learn from conflict and cut away as much of search space as possible

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$



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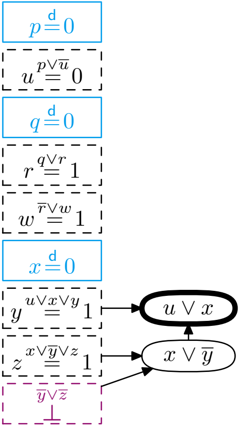
Case analysis over z for last two clauses:

- $x \vee \overline{y} \vee z$ wants $z = 1$
- $\overline{y} \vee \overline{z}$ wants $z = 0$
- Resolve clauses by merging them & removing z — must satisfy $x \vee \overline{y}$

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$



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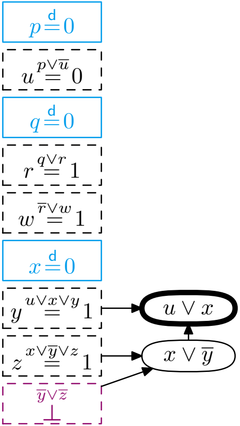
Repeat til UIP clause with only 1 variable at conflict level — learn and backjump

Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$



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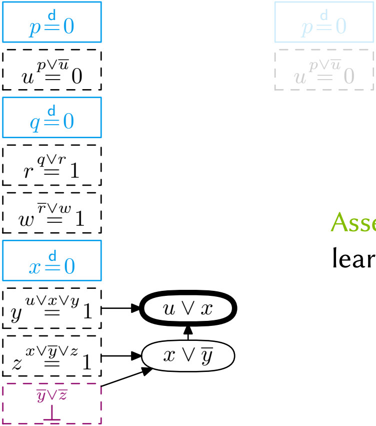


$p \overset{\mathsf{d}}{=} 0$

$u \overset{p \lor \overline{u}}{=} 0$

$q \overset{\mathsf{d}}{=} 0$

$r \overset{q \lor r}{=} 1$

$w \overset{\overline{r} \lor w}{=} 1$

$x \overset{\mathsf{d}}{=} 0$

$y \overset{u \lor x \lor y}{=} 1$

$z \overset{x \lor \overline{y} \lor z}{=} 1$

$\overset{\overline{y} \lor \overline{z}}{\bot}$

$u \lor x$

$x \lor \overline{y}$

$p \overset{\mathsf{d}}{=} 0$

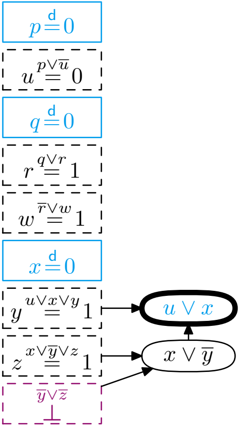
$u \overset{p \lor \overline{u}}{=} 0$

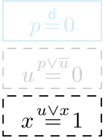
Assertion level 1 (2nd largest level in learned clause) — trim trail to that level

Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$





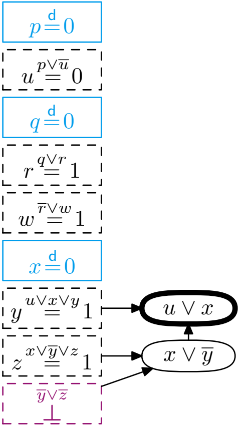
Assertion level 1 (2nd largest level in learned clause) — trim trail to that level

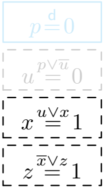
Now UIP literal guaranteed to flip (assert) — but this is a propagation, not a decision

Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$





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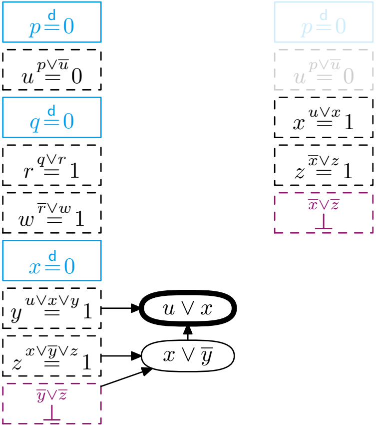
Then continue as before…

Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

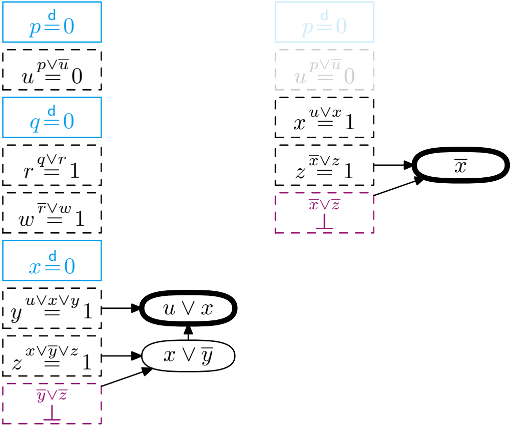
$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$



Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$



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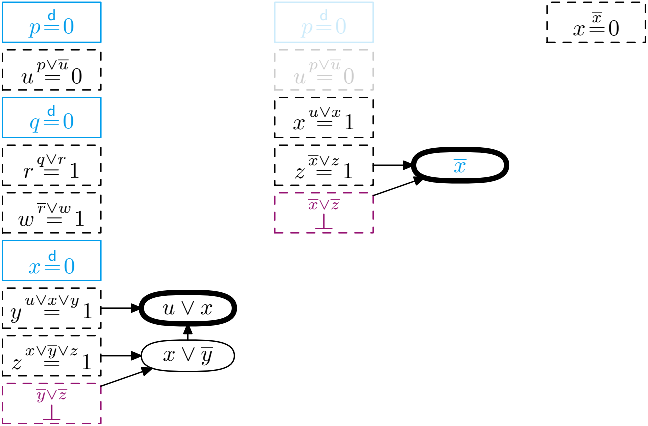

$p \overset{\mathsf{d}}{=} 0$

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$x \overset{\mathsf{d}}{=} 0$

$y \overset{u \vee x \vee y}{=} 1$

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$\overset{\overline{y} \vee \overline{z}}{\perp}$

$u \vee x$

$x \vee \overline{y}$

$p \overset{\mathsf{d}}{=} 0$

$u \overset{p \vee \overline{u}}{=} 0$

$x \overset{u \vee x}{=} 1$

$z \overset{\overline{x} \vee z}{=} 1$

$\overset{\overline{x} \vee \overline{z}}{\perp}$

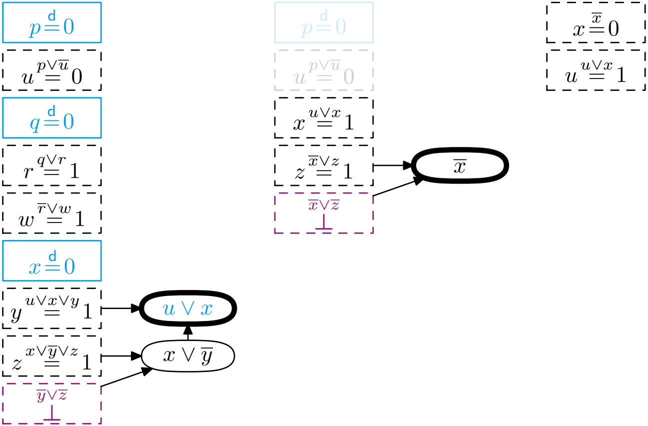
$\overline{x}$

$x \overset{\overline{x}}{=} 0$

Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$



Complete Example of CDCL Execution

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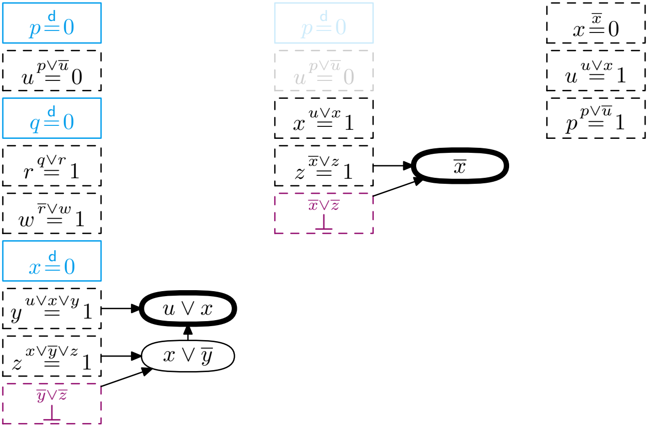
$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$



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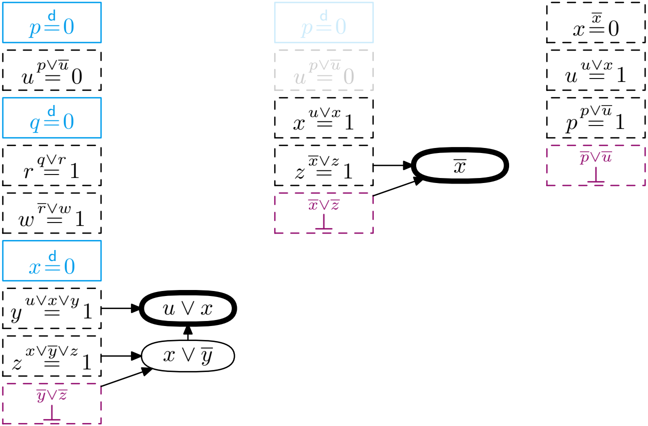

$p \overset{\mathsf{d}}{=} 0$
$u \overset{p \vee \overline{u}}{=} 0$
$q \overset{\mathsf{d}}{=} 0$
$r \overset{q \vee r}{=} 1$
$w \overset{\overline{r} \vee w}{=} 1$
$x \overset{\mathsf{d}}{=} 0$
$y \overset{u \vee x \vee y}{=} 1$
$z \overset{x \vee \overline{y} \vee z}{=} 1$
$\overset{\overline{y} \vee \overline{z}}{\bot}$
$u \vee x$
$x \vee \overline{y}$

$p \overset{\mathsf{d}}{=} 0$
$u \overset{p \vee \overline{u}}{=} 0$
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$\overline{x}$

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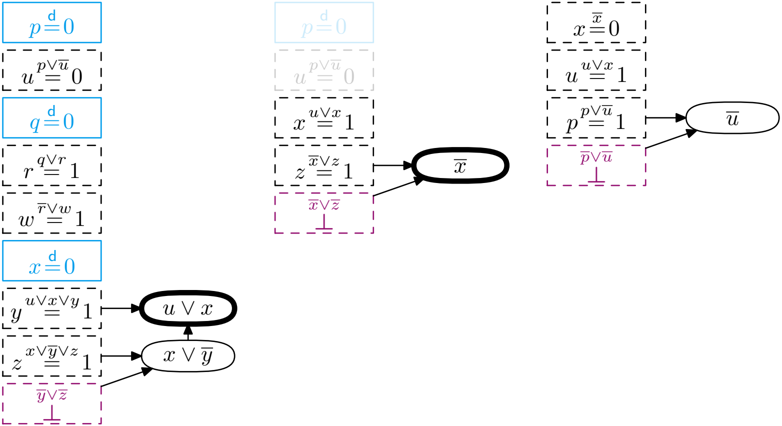
$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$



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$\overline{u}$

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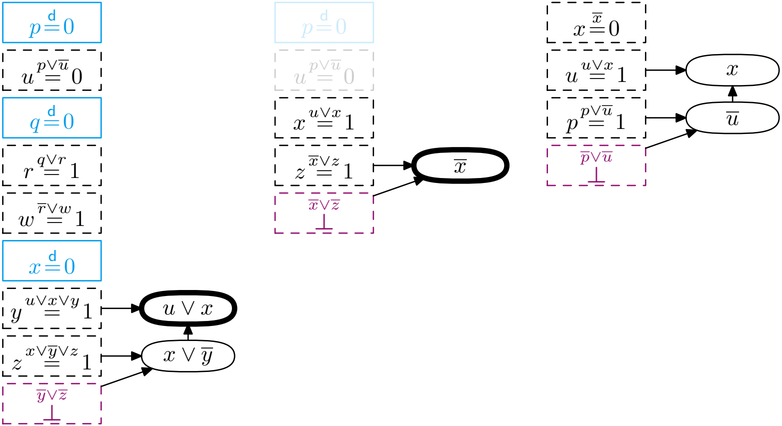

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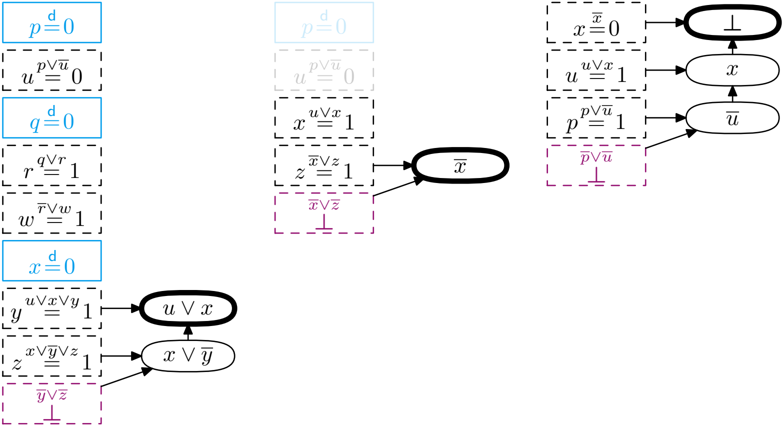

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RUP Proofs and CDCL

Fact

All learned clauses generated by CDCL solver are RUP clauses.

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So short proof of unsatisfiability for

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

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1. $u \vee x$
2. $\overline{x}$
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Writing Proofs in the DRAT Format

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

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In DIMACS

```
p cnf 8 9
1 -4 0
2 3 0
-2 5 0
4 6 7 0
6 -7 8 0
-6 8 0
-7 -8 0
-6 -8 0
-1 -4 0
```

Writing Proofs in the DRAT Format

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

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DPLL Proof in RUP

$x \vee y$

$x \vee \overline{y}$

x

$\overline{x}$

$\perp$

Writing Proofs in the DRAT Format

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

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$x \vee \overline{y}$

x

$\overline{x}$

$\perp$

DPLL Proof in DRAT

```
6 7 0
6 -7 0
6 0
-6 0
0
```

Writing Proofs in the DRAT Format

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

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-6 0
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CDCL Proof in RUP

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$\perp$

Writing Proofs in the DRAT Format

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-6 8 0
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-1 -4 0
```

DPLL Proof in RUP

$x \lor y$

$x \lor \overline{y}$

x

$\overline{x}$

$\bot$

DPLL Proof in DRAT

```
6 7 0
6 -7 0
6 0
-6 0
0
```

CDCL Proof in RUP

$u \lor x$

$\overline{x}$

$\bot$

CDCL Proof in DRAT

```
4 6 0
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```

More Ingredients in Proof Logging for SAT

Fact

RUP proofs are shorthand for so-called Resolution proofs.

See [BN21] for more on this and connections to SAT solving.

But RUP and Resolution aren't enough for preprocessing, inprocessing, and some other kinds of reasoning.

Extension Variables, Part 1

Suppose we want new, fresh variable a encoding

$$a \Leftrightarrow (x \land y)$$

Extended Resolution: allow to introduce clauses

$$a \lor \overline{x} \lor \overline{y} \qquad \overline{a} \lor x \qquad \overline{a} \lor y$$

Should be fine, so long as a doesn't appear anywhere previously.

Extension Variables, Part 1

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Should be fine, so long as a doesn't appear anywhere previously.

Fact

Extended Resolution (RUP + definition of new variables) is essentially equivalent to the DRAT proof logging system.

Deleting Clauses

In practice, important to erase lines to save memory and time during verification.

Very easy to deal with from the point of view of proof logging.

So ignored in this tutorial for simplicity and clarity.

Why Aren’t We Done?

Practical limitations of SAT proof logging technology:

- Difficulties dealing with stronger reasoning efficiently.
- Clausal proofs can’t easily reflect what other algorithms do.

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- Clausal proofs can't easily reflect what other algorithms do.

Surprising claim: a slight change to 0-1 integer linear inequalities does the job!

- Can justify graph reasoning without knowing what a graph is.
- Can justify constraint programming inference without knowing what an integer variable is.
- This even helps justify advanced SAT techniques (cardinality reasoning, Gaussian elimination, symmetry breaking) so far beyond reach for efficient DRAT proof logging.

Pseudo-Boolean Constraints

0-1 integer linear inequalities or pseudo-Boolean constraints:

$$\sum_i a_i \ell_i \geq A$$

- $a_i, A \in \mathbb{Z}$
- literals ℓ_i: x_i or $\overline{x}_i$ (where $x_i + \overline{x}_i = 1$)

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Sometimes convenient to use normalized form [Bar95] with all a_i, A positive (without loss of generality)

Write (partial) assignment ρ as

- set of variable assignments $\rho = \{x \mapsto 1, y \mapsto 0, z \mapsto 1, \ldots\}$, or
- set of true literals $\rho = \{x, \overline{y}, z, \ldots\}$

Some Types of Pseudo-Boolean Constraints

1. Clauses

$$x \vee \overline{y} \vee z \quad \Leftrightarrow \quad x + \overline{y} + z \geq 1$$

2. Cardinality constraints

$$x_1 + x_2 + x_3 + x_4 \geq 2$$

3. General pseudo-Boolean constraints

$$x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$$

RUP Revisited

Can define (reverse) unit propagation in a pseudo-Boolean setting.

Risk for confusion: Constraint programming people might call this (reverse) integer bounds consistency.

- Does the same thing if we're working with clauses.
- More interesting for general pseudo-Boolean constraints.

SAT people beware: constraints can propagate multiple times and multiple literals.

Propagation, Conflict, and Slack

Slack measures how far assignment ρ is from falsifying $\sum_i a_i\ell_i \geq A$
Assuming normalized form:

$$slack\left(\textstyle\sum_i a_i\ell_i \geq A; \rho\right) = \sum_{i\,:\,\rho(\ell_i)\neq 0} a_i \; - A$$

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Consider $C \doteq x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$

ρ	$slack(C;\rho)$	comment

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$\{\}$	8	

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ρ	$slack(C;\rho)$	comment
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$\{\overline{x}_5\}$	3	propagates $\overline{x}_4$ (coefficient > slack)

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Consider $C \doteq x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$

ρ	$slack(C;\rho)$	comment
$\{\}$	8	
$\{\overline{x}_5\}$	3	propagates $\overline{x}_4$ (coefficient > slack)
$\{\overline{x}_5, \overline{x}_4\}$	3	propagation doesn't change slack

Propagation, Conflict, and Slack

Slack measures how far assignment ρ is from falsifying $\sum_i a_i \ell_i \geq A$
Assuming normalized form:

$$slack\left(\textstyle\sum_i a_i \ell_i \geq A; \rho\right) = \sum_{i\,:\,\rho(\ell_i) \neq 0} a_i \; - A$$

Consider $C \doteq x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$

ρ	$slack(C;\rho)$	comment
$\{\}$	8	
$\{\overline{x}_5\}$	3	propagates $\overline{x}_4$ (coefficient > slack)
$\{\overline{x}_5, \overline{x}_4\}$	3	propagation doesn't change slack
$\{\overline{x}_5, \overline{x}_4, \overline{x}_3, x_2\}$	-2	conflict (slack < 0)

Propagation, Conflict, and Slack

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Assuming normalized form:

$$slack\left(\textstyle\sum_i a_i\ell_i \geq A; \rho\right) = \sum_{i\,:\,\rho(\ell_i)\neq 0} a_i \; - A$$

Consider $C \doteq x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$

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$\{\overline{x}_5, \overline{x}_4, \overline{x}_3, x_2\}$	-2	conflict (slack < 0)

Note: constraint can be conflicting though not all variables assigned

Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

Model axioms From the input

Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

Model axioms From the input

Literal axioms $$\frac{}{\ell_i \geq 0}$$

Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

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Addition

$$\frac{\sum_i a_i \ell_i \geq A \qquad \sum_i b_i \ell_i \geq B}{\sum_i (a_i + b_i)\ell_i \geq A + B}$$

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Multiplication
for any $c \in \mathbb{N}^+$

$$\frac{\sum_i a_i\ell_i \geq A}{\sum_i ca_i\ell_i \geq cA}$$

Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

Model axioms From the input

Literal axioms
$$\frac{}{\ell_i \geq 0}$$

Addition
$$\frac{\sum_i a_i \ell_i \geq A \qquad \sum_i b_i \ell_i \geq B}{\sum_i (a_i + b_i)\ell_i \geq A + B}$$

Multiplication
for any $c \in \mathbb{N}^+$
$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i c a_i \ell_i \geq cA}$$

Division
for any $c \in \mathbb{N}^+$
$$\frac{\sum_i c a_i \ell_i \geq A}{\sum_i a_i \ell_i \geq \left\lceil \frac{A}{c} \right\rceil}$$

Cutting Planes Toy Example

$$w + 2x + y \geq 2$$

Cutting Planes Toy Example

$$\text{Mul by 2}\ \frac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}$$

Cutting Planes Toy Example

$$\text{Mul by 2}\ \frac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} \qquad w + 2x + 4y + 2z \geq 5$$

Cutting Planes Toy Example

$$\dfrac{\dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}\text{Mul by 2} \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9}\text{Add}$$

Cutting Planes Toy Example

$$\text{Add}\ \dfrac{\text{Mul by 2}\ \dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9} \qquad \overline{z} \geq 0$$

Cutting Planes Toy Example

$$\text{Add}\ \dfrac{\text{Mul by 2}\ \dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9} \qquad \dfrac{\overline{z} \geq 0}{2\overline{z} \geq 0}\ \text{Mul by 2}$$

Cutting Planes Toy Example

$$\text{Add}\ \cfrac{\text{Add}\ \cfrac{\text{Mul by 2}\ \cfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9} \qquad \cfrac{\overline{z} \geq 0}{2\overline{z} \geq 0}\ \text{Mul by 2}}{3w + 6x + 6y + 2z + 2\overline{z} \geq 9}$$

Cutting Planes Toy Example

$$\text{Add}\ \dfrac{\text{Add}\ \dfrac{\text{Mul by 2}\ \dfrac{w+2x+y\geq 2}{2w+4x+2y\geq 4} \qquad w+2x+4y+2z\geq 5}{3w+6x+6y+2z\geq 9} \qquad \dfrac{\overline{z}\geq 0}{2\overline{z}\geq 0}\ \text{Mul by 2}}{3w+6x+6y+2 \qquad \geq 9}$$

Cutting Planes Toy Example

$$\text{Add}\ \dfrac{\text{Add}\ \dfrac{\text{Mul by 2}\ \dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9} \qquad \dfrac{\overline{z} \geq 0}{2\overline{z} \geq 0}\ \text{Mul by 2}}{3w + 6x + 6y \qquad\qquad \geq 7}$$

Cutting Planes Toy Example

$$\text{Add}\ \dfrac{\text{Add}\ \dfrac{\text{Mul by 2}\ \dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9} \qquad \dfrac{\overline{z} \geq 0}{2\overline{z} \geq 0}\ \text{Mul by 2}}{\text{Div by 3}\ \dfrac{3w + 6x + 6y \qquad \geq 7}{w + 2x + 2y \geq 2\frac{1}{3}}}$$

Cutting Planes Toy Example

$$\text{Div by 3}\ \dfrac{\text{Add}\ \dfrac{\text{Add}\ \dfrac{\text{Mul by 2}\ \dfrac{w+2x+y\geq 2}{2w+4x+2y\geq 4} \qquad w+2x+4y+2z\geq 5}{3w+6x+6y+2z\geq 9} \qquad \dfrac{\overline{z}\geq 0}{2\overline{z}\geq 0}\ \text{Mul by 2}}{3w+6x+6y \qquad \geq 7}}{w+2x+2y\geq 3}$$

Cutting Planes Toy Example

$$\text{Div by 3}\ \dfrac{\text{Add}\ \dfrac{\text{Add}\ \dfrac{\text{Mul by 2}\ \dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9} \qquad \dfrac{\overline{z} \geq 0}{2\overline{z} \geq 0}\ \text{Mul by 2}}{3w + 6x + 6y \qquad \geq 7}}{w + 2x + 2y \geq 3}$$

Such a calculation can be written in a proof line assuming handles

$$\begin{array}{rcl} C_1 & \doteq & 2x + y + w \geq 2 \\ C_2 & \doteq & 2x + 4y + 2z + w \geq 5 \\ Ax(\overline{z}) & \doteq & \overline{z} \geq 0 \end{array}$$

Cutting Planes Toy Example

$$\text{Add}\;\dfrac{\text{Add}\;\dfrac{\text{Mul by 2}\;\dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9} \qquad \dfrac{\overline{z} \geq 0}{2\overline{z} \geq 0}\;\text{Mul by 2}}{\text{Div by 3}\;\dfrac{3w + 6x + 6y \qquad \geq 7}{w + 2x + 2y \geq 3}}$$

Such a calculation can be written in a proof line assuming handles

$$\begin{aligned} C_1 &\doteq 2x + y + w \geq 2 \\ C_2 &\doteq 2x + 4y + 2z + w \geq 5 \\ Ax(\overline{z}) &\doteq \overline{z} \geq 0 \end{aligned}$$

using postfix notation something like

C_1 `2 Mul` C_2 `Add` $Ax(\overline{z})$ `2 Mul Add 3 Div`

Extension Variables, Part 2

Suppose we want new, fresh variable a encoding

$$a \Leftrightarrow (3x + 2y + z + w \geq 3)$$

This time, introduce constraints

$$3\overline{a} + 3x + 2y + z + w \geq 3 \qquad 5a + 3\overline{x} + 2\overline{y} + \overline{z} + \overline{w} \geq 5$$

Again, needs support from the proof system.

Proof Logs for Extended Cutting Planes

For satisfiable instances: just specify a satisfying assignment.

For unsatisfiability: a sequence of pseudo-Boolean constraints in (slight extension of) OPB format [RM16].

- Each constraint follows "obviously" from what is known so far.
- Either implicitly, by RUP…
- Or by an explicit cutting planes derivation…
- Or as an extension variable reifying a new constraint*
- Final constraint is $0 \geq 1$.

(*) Not actually implemented this way — details later…

Enumeration and Optimisation Problems

Enumeration:

- When a solution is found, can log it.
- Introduces a new constraint saying “not this solution”.
- So the proof semantics are “unsatisfiable, except for all the solutions I told you about”.

Enumeration and Optimisation Problems

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- When a solution is found, can log it.
- Introduces a new constraint saying “not this solution”.
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For optimisation:

- Define an objective $f = \sum_i w_i \ell_i$, $w_i \in \mathbb{Z}$, to minimise in the pseudo-Boolean model.
- To maximise, negate objective.
- Log a solution α, get a solution-improving constraint $\sum_i w_i \ell_i \leq -1 + \sum_i w_i \alpha(\ell_i)$.

The VeriPB Format and Tool

`https://gitlab.com/MIAOresearch/software/VeriPB`

Released under MIT Licence.

Various features to help development:

- Extended variable name syntax allowing human-readable names.
- Proof tracing.
- “Trust me” assertions for incremental proof logging.

Full details: Stephan Gocht’s PhD thesis [Goc22].

Progress So Far

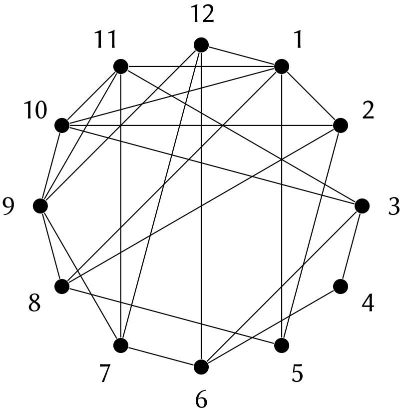
We've seen proof logging, and how it works for SAT.

We've learned about

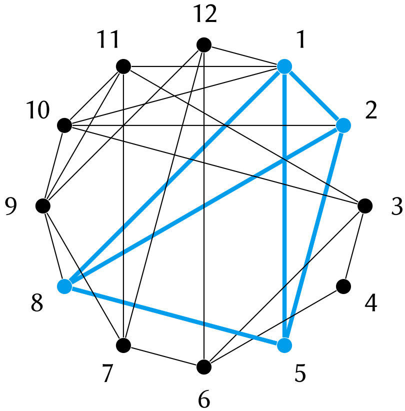
- pseudo-Boolean constraints (0-1 linear inequalities),
- cutting planes reasoning, and
- VeriPB.

Coming next, some worked examples from dedicated graph solvers.

The Maximum Clique Problem



The Maximum Clique Problem



Maximum Clique Solvers

There are a lot of dedicated solvers for clique problems.

But there are issues:

- "State of the art" solvers have been buggy.
- Often undetected: error rate of around 0.1% [MPP19].

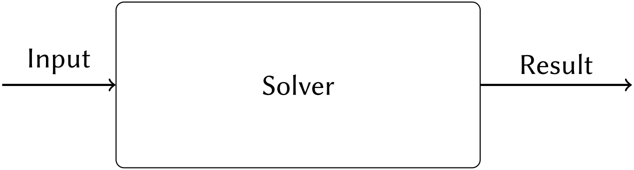
Often used inside other solvers.

- An off-by-one result can cause much larger errors.

Making a Proof-Logging Clique Solver

1. Output a pseudo-Boolean encoding of the problem.
    - Clique problems have several standard file formats.
2. Make the solver log its search tree.
    - Output a small header.
    - Output something on every backtrack.
    - Output something every time a solution is found.
    - Output a small footer.
3. Figure out how to log the bound function.

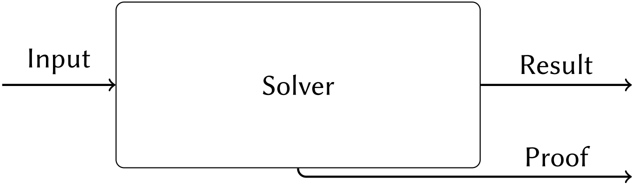
A Slightly Different Workflow


Input

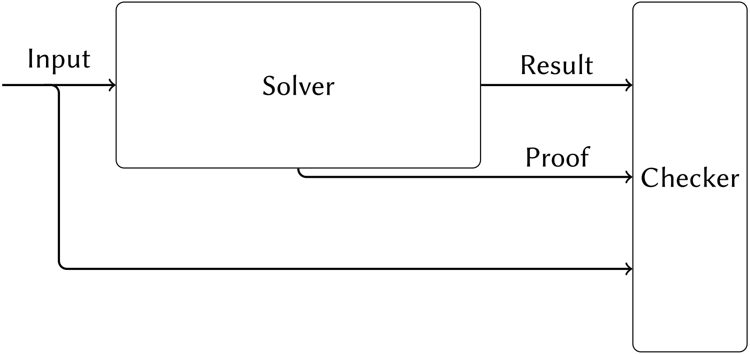
Solver

Result

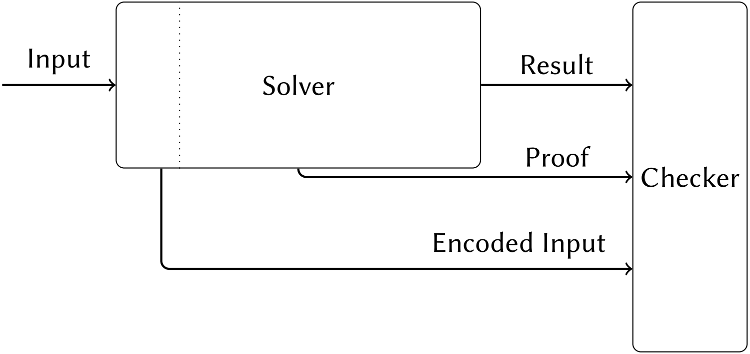
A Slightly Different Workflow



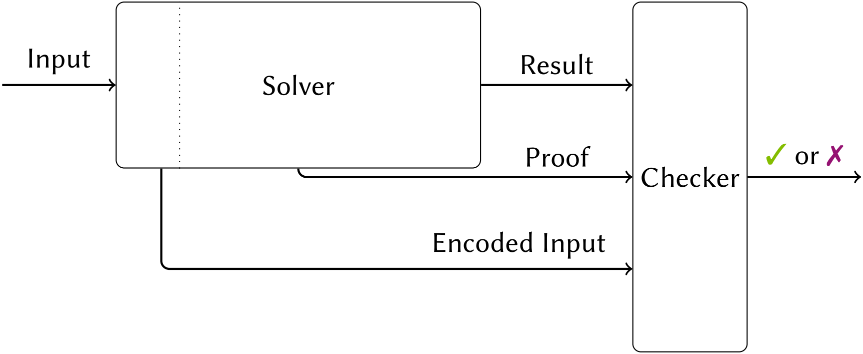
A Slightly Different Workflow



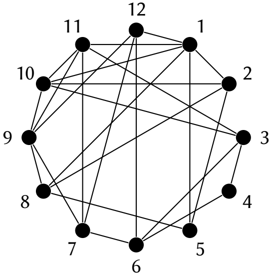
A Slightly Different Workflow



A Slightly Different Workflow



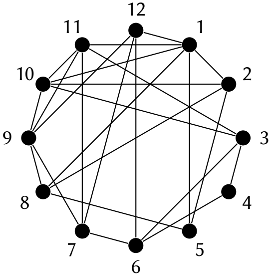
A Pseudo-Boolean Encoding for Clique (in OPB Format)



```
* #variable= 12 #constraint= 41
min: -1 x1 -1 x2 -1 x3 -1 x4 . . . and so on. . .  -1 x11 -1 x12 ;
1 ~x3 1 ~x1 >= 1 ;
1 ~x3 1 ~x2 >= 1 ;
1 ~x4 1 ~x1 >= 1 ;
* . . . and a further 38 similar lines for the remaining non-edges
```

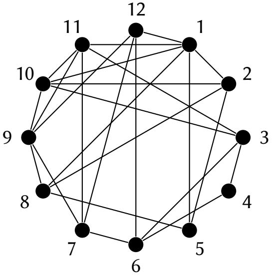
First Attempt at a Proof

```
pseudo-Boolean proof version 1.2
f 41
o x7 x9 x12
u 1 ~x12 1 ~x7 >= 1 ;
u 1 ~x12 >= 1 ;
u 1 ~x11 1 ~x10 >= 1 ;
u 1 ~x11 >= 1 ;
o x1 x2 x5 x8
u 1 ~x8 1 ~x5 >= 1 ;
u 1 ~x8 >= 1 ;
u >= 1 ;
c -1
```



First Attempt at a Proof

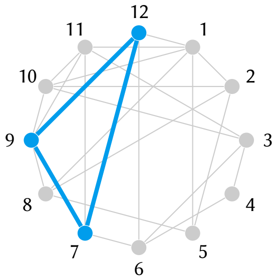
```
pseudo-Boolean proof version 1.2
f 41
o x7 x9 x12
u 1 ~x12 1 ~x7 >= 1 ;
u 1 ~x12 >= 1 ;
u 1 ~x11 1 ~x10 >= 1 ;
u 1 ~x11 >= 1 ;
o x1 x2 x5 x8
u 1 ~x8 1 ~x5 >= 1 ;
u 1 ~x8 >= 1 ;
u >= 1 ;
c -1
```



Start with a header.
Load the 41 problem axioms.

First Attempt at a Proof

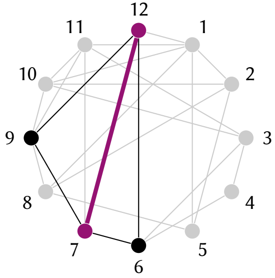
```
pseudo-Boolean proof version 1.2
f 41
o x7 x9 x12
u 1 ~x12 1 ~x7 >= 1 ;
u 1 ~x12 >= 1 ;
u 1 ~x11 1 ~x10 >= 1 ;
u 1 ~x11 >= 1 ;
o x1 x2 x5 x8
u 1 ~x8 1 ~x5 >= 1 ;
u 1 ~x8 >= 1 ;
u >= 1 ;
c -1
```



Branch on 12, 7, 9.
Find a new incumbent.
Now looking for a ≥ 4 vertex clique.

First Attempt at a Proof

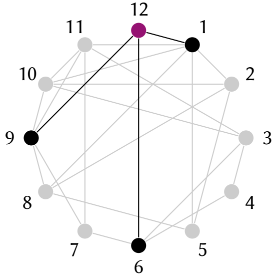
```
pseudo-Boolean proof version 1.2
f 41
o x7 x9 x12
u 1 ~x12 1 ~x7 >= 1 ;
u 1 ~x12 >= 1 ;
u 1 ~x11 1 ~x10 >= 1 ;
u 1 ~x11 >= 1 ;
o x1 x2 x5 x8
u 1 ~x8 1 ~x5 >= 1 ;
u 1 ~x8 >= 1 ;
u >= 1 ;
c -1
```



Backtrack from 12, 7.
Only 6 and 9 feasible.
No ≥ 4 vertex clique possible.
Effectively this deletes the 7–12 edge.

First Attempt at a Proof

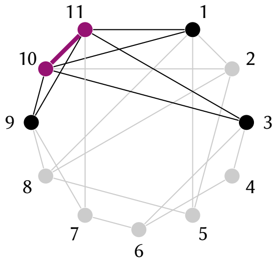
```
pseudo-Boolean proof version 1.2
f 41
o x7 x9 x12
u 1 ~x12 1 ~x7 >= 1 ;
u 1 ~x12 >= 1 ;
u 1 ~x11 1 ~x10 >= 1 ;
u 1 ~x11 >= 1 ;
o x1 x2 x5 x8
u 1 ~x8 1 ~x5 >= 1 ;
u 1 ~x8 >= 1 ;
u >= 1 ;
c -1
```



Backtrack from 12.
Only 1, 6 and 9 feasible.
No ≥ 4 vertex clique possible.
Effectively this deletes vertex 12.

First Attempt at a Proof

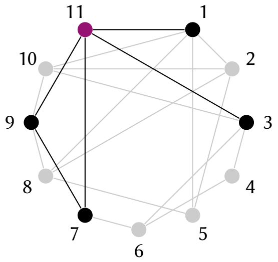
```
pseudo-Boolean proof version 1.2
f 41
o x7 x9 x12
u 1 ~x12 1 ~x7 >= 1 ;
u 1 ~x12 >= 1 ;
u 1 ~x11 1 ~x10 >= 1 ;
u 1 ~x11 >= 1 ;
o x1 x2 x5 x8
u 1 ~x8 1 ~x5 >= 1 ;
u 1 ~x8 >= 1 ;
u >= 1 ;
c -1
```



Branch on 11 then 10.
Only 1, 3 and 9 feasible.
No ≥ 4 vertex clique possible.
Backtrack, deleting the edge.

First Attempt at a Proof

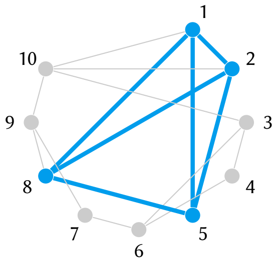
```
pseudo-Boolean proof version 1.2
f 41
o x7 x9 x12
u 1 ~x12 1 ~x7 >= 1 ;
u 1 ~x12 >= 1 ;
u 1 ~x11 1 ~x10 >= 1 ;
u 1 ~x11 >= 1 ;
o x1 x2 x5 x8
u 1 ~x8 1 ~x5 >= 1 ;
u 1 ~x8 >= 1 ;
u >= 1 ;
c -1
```



Backtrack from 11.
Clearly no ≥ 4 clique.
Delete the vertex.

First Attempt at a Proof

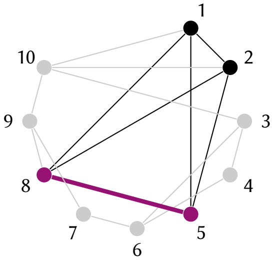
```
pseudo-Boolean proof version 1.2
f 41
o x7 x9 x12
u 1 ~x12 1 ~x7 >= 1 ;
u 1 ~x12 >= 1 ;
u 1 ~x11 1 ~x10 >= 1 ;
u 1 ~x11 >= 1 ;
o x1 x2 x5 x8
u 1 ~x8 1 ~x5 >= 1 ;
u 1 ~x8 >= 1 ;
u >= 1 ;
c -1
```



Branch on 8, 5, 1, 2.
Find a new incumbent.
Now looking for a ≥ 5 vertex clique.

First Attempt at a Proof

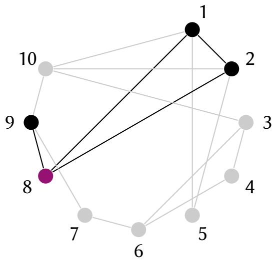
```
pseudo-Boolean proof version 1.2
f 41
o x7 x9 x12
u 1 ~x12 1 ~x7 >= 1 ;
u 1 ~x12 >= 1 ;
u 1 ~x11 1 ~x10 >= 1 ;
u 1 ~x11 >= 1 ;
o x1 x2 x5 x8
u 1 ~x8 1 ~x5 >= 1 ;
u 1 ~x8 >= 1 ;
u >= 1 ;
c -1
```



Backtrack from 8, 5.
Only 4 vertices, can't have a ≥ 5 clique.
Delete the edge.

First Attempt at a Proof

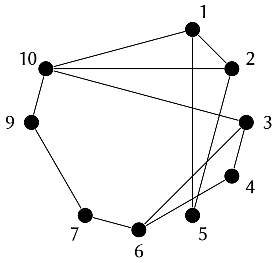
```
pseudo-Boolean proof version 1.2
f 41
o x7 x9 x12
u 1 ~x12 1 ~x7 >= 1 ;
u 1 ~x12 >= 1 ;
u 1 ~x11 1 ~x10 >= 1 ;
u 1 ~x11 >= 1 ;
o x1 x2 x5 x8
u 1 ~x8 1 ~x5 >= 1 ;
u 1 ~x8 >= 1 ;
u >= 1 ;
c -1
```



Backtrack from 8.
Still not enough vertices.
Delete the vertex.

First Attempt at a Proof

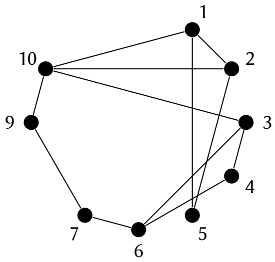
```
pseudo-Boolean proof version 1.2
f 41
o x7 x9 x12
u 1 ~x12 1 ~x7 >= 1 ;
u 1 ~x12 >= 1 ;
u 1 ~x11 1 ~x10 >= 1 ;
u 1 ~x11 >= 1 ;
o x1 x2 x5 x8
u 1 ~x8 1 ~x5 >= 1 ;
u 1 ~x8 >= 1 ;
u >= 1 ;
c -1
```



Now obvious to solver that claim of ≥ 5 clique is contradictory (we'll see why).

First Attempt at a Proof

```
pseudo-Boolean proof version 1.2
f 41
o x7 x9 x12
u 1 ~x12 1 ~x7 >= 1 ;
u 1 ~x12 >= 1 ;
u 1 ~x11 1 ~x10 >= 1 ;
u 1 ~x11 >= 1 ;
o x1 x2 x5 x8
u 1 ~x8 1 ~x5 >= 1 ;
u 1 ~x8 >= 1 ;
u >= 1 ;
c -1
```



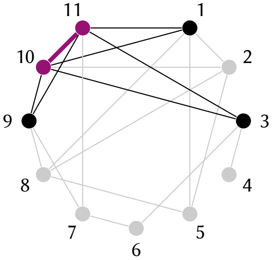
Assert previous line has derived contradiction, ending proof.

Verifying This Proof (Or Not…)

```
$ veripb clique.opb clique-attempt-one.veripb
Verification failed.
Failed in proof file line 6.
Hint: Failed to show '1 ~x10 1 ~x11 >= 1' by reverse unit propagation.
```

Verifying This Proof (Or Not…)

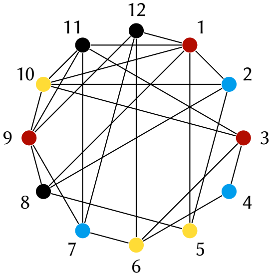
```
$ veripb clique.opb clique-attempt-one.veripb
Verification failed.
Failed in proof file line 6.
Hint: Failed to show '1 ~x10 1 ~x11 >= 1' by reverse unit propagation.
```



Verifying This Proof (Or Not. . .)

```
$ veripb --trace clique.opb clique-attempt-one.veripb
line 002: f 41
  ConstraintId 001: 1 ~x1 1 ~x3 >= 1
  ConstraintId 002: 1 ~x2 1 ~x3 >= 1
...
  ConstraintId 041: 1 ~x11 1 ~x12 >= 1
line 003: o x7 x9 x12 ~x1 ~x2 ~x3 ~x4 ~x5 ~x6 ~x8 ~x10 ~x11
  ConstraintId 042: 1 x1 1 x2 1 x3 1 x4 1 x5 1 x6 1 x7 1 x8 1 x9 1 x10 1 x1
line 004: u 1 ~x12 1 ~x7 >= 1 ;
  ConstraintId 043: 1 ~x7 1 ~x12 >= 1
line 005: u 1 ~x12 >= 1 ;
  ConstraintId 044: 1 ~x12 >= 1
line 006: u 1 ~x11 1 ~x10 >= 1 ;
Verification failed.
Failed in proof file line 6.
Hint: Failed to show '1 ~x10 1 ~x11 >= 1' by reverse unit propagation.
```

Bound Functions



Given a *k*-colouring of a subgraph, that subgraph cannot have a clique of more than *k* vertices.

- Each colour class describes an at-most-one constraint.

This does *not* follow by reverse unit propagation.

Recovering At-Most-One Constraints

Practically infeasible to list every colour class we *might* use in the pseudo-Boolean input.

But we can use cutting planes to recover colour classes lazily!

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$$\begin{aligned}
&(\overline{x}_1 + \overline{x}_6 \geq 1) && \\
+\,&(\overline{x}_1 + \overline{x}_9 \geq 1) && = 2\overline{x}_1 + \overline{x}_6 + \overline{x}_9 \geq 2 \\
+\,&(\overline{x}_6 + \overline{x}_9 \geq 1) && = 2\overline{x}_1 + 2\overline{x}_6 + 2\overline{x}_9 \geq 3 \\
&/\,2 && = \overline{x}_1 + \overline{x}_6 + \overline{x}_9 \geq 2 \\
& && \text{i.e. } x_1 + x_6 + x_9 \leq 1
\end{aligned}$$

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& && \text{i.e. } x_1 + x_6 + x_9 \leq 1
\end{aligned}$$

This generalises for arbitrarily large colour classes.

- Each non-edge is used exactly once, $v(v-1)$ additions.
- $v-3$ multiplications and $v-2$ divisions.

Solvers don't need to "understand" cutting planes to write this out.

What This Looks Like

```
pseudo-Boolean proof version 1.2
f 41
o x12 x7 x9
u 1 ~x12 1 ~x7 >= 1 ;
* bound, colour classes [ x1 x6 x9 ]
p $7_{1\nsim 6}$ $19_{1\nsim 9}$ + $24_{6\nsim 9}$ + 2 d
p $42_{\mathrm{obj}}$ -1 +
u 1 ~x12 >= 1 ;
* bound, colour classes [ x1 x3 x9 ]
p $1_{1\nsim 3}$ $19_{1\nsim 9}$ + $21_{3\nsim 9}$ + 2 d
p $42_{\mathrm{obj}}$ -1 +
u 1 ~x11 1 ~x10 >= 1 ;
* bound, colour classes [ x1 x3 x7 ] [ x9 ]
p $1_{1\nsim 3}$ $10_{1\nsim 7}$ + $12_{3\nsim 7}$ + 2 d
p $42_{\mathrm{obj}}$ -1 +
u 1 ~x11 >= 1 ;
o x8 x5 x2 x1
u 1 ~x8 1 ~x5 >= 1 ;
* bound, colour classes [ x1 x9 ] [ x2 ]
p $53_{\mathrm{obj}}$ $19_{1\nsim 9}$ +
u 1 ~x8 >= 1 ;
* bound, colour classes [ x1 x3 x7 ] [ x2 x4 x9 ] [ x5 x6 x10 ]
p $1_{1\nsim 3}$ $10_{1\nsim 7}$ + $12_{3\nsim 7}$ + 2 d
p $53_{\mathrm{obj}}$ -1 +
p $4_{2\nsim 4}$ $20_{2\nsim 9}$ + $22_{4\nsim 9}$ + 2 d
p $53_{\mathrm{obj}}$ -3 + -1 +
p $9_{5\nsim 6}$ $26_{5\nsim 10}$ + $27_{6\nsim 10}$ + 2 d
p $53_{\mathrm{obj}}$ -5 + -3 + -1 +
u >= 1 ;
c -1
```

Verifying This Proof (For Real, This Time)

```
$ veripb --trace clique.opb clique-attempt-two.veripb
=== begin trace ===
line 002: f 41
  ConstraintId 001: 1 ~x1 1 ~x3 >= 1
  ConstraintId 002: 1 ~x2 1 ~x3 >= 1
...
  ConstraintId 041: 1 ~x11 1 ~x12 >= 1
line 003: o x7 x9 x12 ~x1 ~x2 ~x3 ~x4 ~x5 ~x6 ~x8 ~x10 ~x11
  ConstraintId 042: 1 x1 1 x2 1 x3 1 x4 1 x5 1 x6 1 x7 1 x8 1 x9 1 x10 1 x11 1 x12 >= 4
line 004: u 1 ~x12 1 ~x7 >= 1 ;
  ConstraintId 043: 1 ~x7 1 ~x12 >= 1
line 005: * bound, colour classes [ x1 x6 x9 ]
line 006: p 7 19 + 24 + 2 d
  ConstraintId 044: 1 ~x1 1 ~x6 1 ~x9 >= 2
line 007: p 42 43 +
  ConstraintId 045: 1 x1 1 x2 1 x3 1 x4 1 x5 1 x6 1 x8 1 x9 1 x10 1 x11 >= 3
...
  ConstraintId 061: 1 ~x5 1 ~x6 1 ~x10 >= 2
line 028: p 53 57 + 59 + 61 +
  ConstraintId 062: 1 x8 1 x11 1 x12 >= 2
line 029: u >= 1 ;
  ConstraintId 063: >= 1
line 030: c -1
=== end trace ===

Verification succeeded.
```

Different Clique Algorithms

Different search orders?

- ✓ Irrelevant for proof logging.

Using local search to initialise?

- ✓ Just log the incumbent.

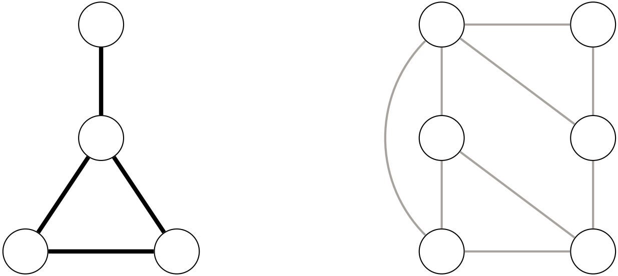
Different bound functions?

- ■ Is cutting planes strong enough to justify every useful bound function ever invented?
- ■ So far, seems like it…

Weighted cliques?

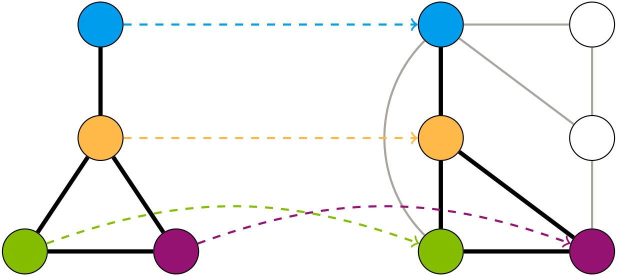
- ✓ Multiply a colour class by its largest weight.
- ✓ Also works for vertices "split between colour classes".

Subgraph Isomorphism



- Find the *pattern* inside the *target*.
- Applications in compilers, biochemistry, model checking, pattern recognition, …
- Often want to find *all* matches.

Subgraph Isomorphism



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Subgraph Isomorphism in Pseudo-Boolean Form

Each pattern vertex gets a target vertex:

$$\sum_{t \in \mathrm{V}(T)} x_{p,t} = 1 \qquad\qquad p \in \mathrm{V}(P)$$

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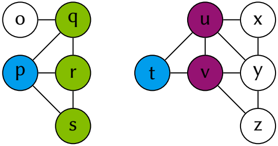
Adjacency constraints, if p is mapped to t, then p's neighbours must be mapped to t's neighbours:

$$\overline{x}_{p,t} + \sum_{u \in \mathrm{N}(t)} x_{q,u} \geq 1 \qquad p \in \mathrm{V}(P),\ q \in \mathrm{N}(p),\ t \in \mathrm{V}(T)$$

Degree Reasoning in Cutting Planes

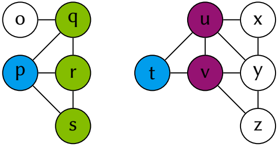


A pattern vertex p of degree $\deg(p)$ can never be mapped to a target vertex t of degree $\deg(p) - 1$ or lower in any subgraph isomorphism.

Observe $\mathrm{N}(p) = \{q, r, s\}$ and $\mathrm{N}(t) = \{u, v\}$.

We wish to derive $\overline{x}_{p,t} \geq 1$.

Degree Reasoning in Cutting Planes



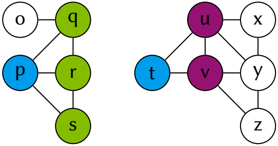
We have the three adjacency constraints,

$$\overline{x}_{p,t} + x_{q,u} + x_{q,v} \geq 1$$
$$\overline{x}_{p,t} + x_{r,u} + x_{r,v} \geq 1$$
$$\overline{x}_{p,t} + x_{s,u} + x_{s,v} \geq 1$$

Their sum is

$$3\overline{x}_{p,t} + x_{q,u} + x_{q,v} + x_{r,u} + x_{r,v} + x_{s,u} + x_{s,v} \geq 3$$

Degree Reasoning in Cutting Planes



Continuing with the sum

$$3\overline{x}_{p,t} + x_{q,u} + x_{q,v} + x_{r,u} + x_{r,v} + x_{s,u} + x_{s,v} \geq 3$$

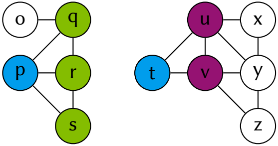
Due to injectivity,

$$-x_{o,u} + -x_{p,u} + -x_{q,u} + -x_{r,u} + -x_{s,u} \geq -1$$
$$-x_{o,v} + -x_{p,v} + -x_{q,v} + -x_{r,v} + -x_{s,v} \geq -1$$

Add all these together, getting

$$3\overline{x}_{p,t} + -x_{o,u} + -x_{o,v} + -x_{p,u} + -x_{p,v} \geq 1$$

Degree Reasoning in Cutting Planes



We're more or less there. We have:

$$3\overline{x}_{p,t} + -x_{o,u} + -x_{o,v} + -x_{p,u} + -x_{p,v} \geq 1$$

Add the literal axioms $x_{o,u} \geq 0$, $x_{o,v} \geq 0$, $x_{p,u} \geq 0$ and $x_{p,v} \geq 0$ to get

$$3\overline{x}_{p,t} \geq 1$$

Divide by 3 to get the desired

$$\overline{x}_{p,t} \geq 1$$

Degree Reasoning in VeriPB

p $18_{p\sim t:q}$ $19_{p\sim t:r}$ + $20_{p\sim t:s}$ + * sum adjacency constraints
$12_{inj(u)}$ + $13_{inj(v)}$ + * sum injectivity constraints
`xo_u + xo_v +` * cancel stray xo_*
`xp_u + xp_v +` * cancel stray xp_*
`3 d` * divide, and we're done

Or we can ask VeriPB to do the last bit of simplification automatically:

p $18_{p\sim t:q}$ $19_{p\sim t:r}$ + $20_{p\sim t:s}$ + * sum adjacency constraints
$12_{inj(u)}$ + $13_{inj(v)}$ + * sum injectivity constraints
`j -1 1 ~xp_t >= 1 ;` * desired conclusion is implied

Other Forms of Reasoning

We can also log all of the other things state of the art subgraph solvers do:

- Injectivity reasoning and filtering.
- Distance filtering.
- Neighbourhood degree sequences.
- Path filtering.
- Supplemental graphs.

Other Forms of Reasoning

We can also log all of the other things state of the art subgraph solvers do:

- Injectivity reasoning and filtering.
- Distance filtering.
- Neighbourhood degree sequences.
- Path filtering.
- Supplemental graphs.

Proof steps are “efficient” using cutting planes.

- The length of the proof steps are no worse than the time complexity of the reasoning algorithms.
- Most proof steps require only trivial additional computations.

Limitations

Why trust the encoding?

- Here we can formally verify the correctness of the encoding!
  Work in progress…

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Works up to moderately-sized hard instances

- Even an $O(n^3)$ encoding is painful.
- Particularly bad when the pseudo-Boolean encoding talks about "non-edges" but large sparse graphs are "easy".

Code

`https://github.com/ciaranm/glasgow-subgraph-solver`

Released under MIT Licence.

What About Constraint Programming?

Non-Boolean variables?

Constraints?

- Encoding constraints as Pseudo-Boolean constraints?
- Justifying inference?

Reformulation?

Compiling CP Variables

Given $A \in \{-3 \dots 9\}$, the direct encoding is:

$$\begin{aligned} a_{=-3} + a_{=-2} + a_{=-1} + a_{=0} + a_{=1} + a_{=2} + a_{=3} & \\ + a_{=4} + a_{=5} + a_{=6} + a_{=7} + a_{=8} + a_{=9} &= 1 \end{aligned}$$

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This doesn't work for large domains.

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This doesn't work for large domains.

We could use a binary encoding:

$$\begin{aligned} -16a_{\text{neg}} + 1a_{\text{b0}} + 2a_{\text{b1}} + 4a_{\text{b2}} + 8a_{\text{b3}} &\geq -3 \text{ and} \\ 16a_{\text{neg}} + -1a_{\text{b0}} + -2a_{\text{b1}} + -4a_{\text{b2}} + -8a_{\text{b3}} &\geq -9 \end{aligned}$$

This doesn't propagate much, but that isn't a problem for proof logging.

Compiling CP Variables

We can mix binary and an order encoding. Where needed, define:

$$
\begin{aligned}
a_{\geq 4} &\Leftrightarrow -16a_{\text{neg}} + 1a_{\text{b0}} + 2a_{\text{b1}} + 4a_{\text{b2}} + 8a_{\text{b3}} \geq 4 \\
a_{\geq 5} &\Leftrightarrow -16a_{\text{neg}} + 1a_{\text{b0}} + 2a_{\text{b1}} + 4a_{\text{b2}} + 8a_{\text{b3}} \geq 5 \\
a_{=4} &\Leftrightarrow a_{\geq 4} \wedge \overline{a}_{\geq 5}
\end{aligned}
$$

When creating $a_{=i}$, also introduce pseudo-Boolean constraints encoding

$$a_{\geq i} \Rightarrow a_{\geq j} \quad \text{and} \quad a_{\geq h} \Rightarrow a_{\geq i}$$

for the closest values $j < i < h$ that already exist.

We can do this:

- Inside the pseudo-Boolean model, where needed.
- Otherwise lazily during proof logging.

Compiling Constraints

- Also need to compile every constraint to pseudo-Boolean form.
- Doesn't need to be a propagating encoding.
- Can use additional variables.

Compiling Constraints

Given $2A + 3B + 4C \geq 42$, where $A, B, C \in \{-3 \ldots 9\}$,

$$\begin{aligned} & -32a_{\mathrm{neg}} + 2a_{\mathrm{b0}} + 4a_{\mathrm{b1}} + 8a_{\mathrm{b2}} + 16a_{\mathrm{b3}} \\ + & -48b_{\mathrm{neg}} + 3b_{\mathrm{b0}} + 6b_{\mathrm{b1}} + 12b_{\mathrm{b2}} + 24b_{\mathrm{b3}} \\ + & -64c_{\mathrm{neg}} + 4c_{\mathrm{b0}} + 8c_{\mathrm{b1}} + 16c_{\mathrm{b2}} + 32c_{\mathrm{b3}} \geq 42 \end{aligned}$$

Compiling Constraints

Constraints can be specified *extensionally* as list of feasible tuples, called a *table*. We have to pick one of the tuples from the table, and give it to the associated variables.

Given a table constraint $(A, B, C) \in [(1, 2, 3), (1, 3, 4), (2, 2, 5)]$, define

$$\begin{aligned} 3\bar{t}_0 + a_{=1} + b_{=2} + c_{=3} &\geq 3 \quad \text{i.e.} \quad && t_0 \Rightarrow (a_{=1} \wedge b_{=2} \wedge c_{=3}) \\ 3\bar{t}_1 + a_{=1} + b_{=4} + c_{=4} &\geq 3 \quad \text{i.e.} \quad && t_1 \Rightarrow (a_{=1} \wedge b_{=4} \wedge c_{=4}) \\ 3\bar{t}_2 + a_{=2} + b_{=2} + c_{=5} &\geq 3 \quad \text{i.e.} \quad && t_2 \Rightarrow (a_{=2} \wedge b_{=2} \wedge c_{=5}) \end{aligned}$$

using a tuple selector variable

$$t_0 + t_1 + t_2 = 1$$

Encoding Constraint Definitions

We already know how to do it for any constraint that has a sane encoding using some combination of

- CNF,
- Integer linear inequalities,
- Table constraints,
- Auxiliary variables.

Simplicity is important, propagation strength isn't.

Justifying Search

Mostly this works as in earlier examples.

Restarts are easy.

No need to justify guesses or decisions. We only justify backtracking.

Justifying Inference

If it follows from unit propagation, nothing needed.

Some propagators and encodings need RUP steps for inferences.

- A lot of propagators are effectively “doing a little bit of lookahead” but in an efficient way.

A few need explicit cutting planes justifications.

- Linear inequalities just need to multiply and add.
- All-different needs a bit more.

Justifying All-Different Failures

$$\begin{array}{rl} V \in \{ & 1 \quad \quad \quad 4 \quad 5 \; \} \\ W \in \{ & 1 \quad 2 \quad 3 \quad \quad \; \} \\ X \in \{ & \quad 2 \quad 3 \quad \quad \; \} \\ Y \in \{ & 1 \quad \quad 3 \quad \quad \; \} \\ Z \in \{ & 1 \quad \quad 3 \quad \quad \; \} \end{array}$$

Justifying All-Different Failures

$$\begin{array}{rcccccccc}
V & \in & \{ & 1 & & & 4 & 5 & \} \\
W & \in & \{ & 1 & 2 & 3 & & & \} \\
X & \in & \{ & & 2 & 3 & & & \} \\
Y & \in & \{ & 1 & & 3 & & & \} \\
Z & \in & \{ & 1 & & 3 & & & \}
\end{array}$$

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$$\begin{array}{rl} V \in & \{\, 1 \quad\quad\quad 4 \quad 5 \,\} \\ W \in & \{\, 1 \quad 2 \quad 3 \quad\quad\quad \} \\ X \in & \{\, \quad\; 2 \quad 3 \quad\quad\quad \} \\ Y \in & \{\, 1 \quad\quad 3 \quad\quad\quad \} \\ Z \in & \{\, 1 \quad\quad 3 \quad\quad\quad \} \end{array}$$

$$w_{=1} + w_{=2} + w_{=3} \geq 1$$

Justifying All-Different Failures

$$\begin{array}{ll}
V \in \{1 \quad\quad 4 \quad 5\} & \\
W \in \{1 \quad 2 \quad 3 \quad\quad\} & w_{=1} + w_{=2} + w_{=3} \geq 1 \\
X \in \{\quad 2 \quad 3 \quad\quad\} & x_{=2} + x_{=3} \geq 1 \\
Y \in \{1 \quad\quad 3 \quad\quad\} & y_{=1} + y_{=3} \geq 1 \\
Z \in \{1 \quad\quad 3 \quad\quad\} & z_{=1} + z_{=3} \geq 1
\end{array}$$

Justifying All-Different Failures

$$\begin{array}{rlllllllll}
V \in \{ & 1 & & & 4 & 5 \} & & & & \\
W \in \{ & 1 & 2 & 3 & & \} & w_{=1} + & w_{=2} + & w_{=3} & & & \geq 1 \\
X \in \{ & & 2 & 3 & & \} & & x_{=2} + & x_{=3} & & & \geq 1 \\
Y \in \{ & 1 & & 3 & & \} & y_{=1} & + & y_{=3} & & & \geq 1 \\
Z \in \{ & 1 & & 3 & & \} & z_{=1} & + & z_{=3} & & & \geq 1
\end{array}$$

$$\begin{array}{l}
\rightarrow \quad -v_{=1} + -w_{=1} + \qquad -y_{=1} + -z_{=1} \geq -1 \\
\quad \rightarrow \quad -w_{=2} + -x_{=2} \geq -1 \\
\qquad \rightarrow \quad -w_{=3} + -x_{=3} + -y_{=3} + -z_{=3} \geq -1
\end{array}$$

Justifying All-Different Failures

$$\begin{array}{llcr}
V \in \{\, 1 \qquad\quad 4 \quad 5 \,\} & & & \\
W \in \{\, 1 \quad 2 \quad 3 \qquad\quad \} & w_{=1} + w_{=2} + w_{=3} & & \geq 1 \\
X \in \{\, \quad\; 2 \quad 3 \qquad\quad \} & x_{=2} + x_{=3} & & \geq 1 \\
Y \in \{\, 1 \qquad\; 3 \qquad\quad \} & y_{=1} + y_{=3} & & \geq 1 \\
Z \in \{\, 1 \qquad\; 3 \qquad\quad \} & z_{=1} + z_{=3} & & \geq 1 \\
 & & & \\
\rightarrow & -v_{=1} + -w_{=1} + -y_{=1} + -z_{=1} & & \geq -1 \\
\quad \rightarrow & -w_{=2} + -x_{=2} & & \geq -1 \\
\qquad \rightarrow & -w_{=3} + -x_{=3} + -y_{=3} + -z_{=3} & & \geq -1 \\
 & & & \\
 & -v_{=1} & & \geq 1
\end{array}$$

Justifying All-Different Failures

$$\begin{array}{rl}
V \in \{\, 1 \qquad\quad 4 \quad 5 \,\} & \\
W \in \{\, 1 \quad 2 \quad 3 \qquad\quad \} & w_{=1} + w_{=2} + w_{=3} \geq 1 \\
X \in \{\, \quad\; 2 \quad 3 \qquad\quad \} & x_{=2} + x_{=3} \geq 1 \\
Y \in \{\, 1 \qquad\; 3 \qquad\quad \} & y_{=1} + y_{=3} \geq 1 \\
Z \in \{\, 1 \qquad\; 3 \qquad\quad \} & z_{=1} + z_{=3} \geq 1
\end{array}$$

$$\begin{array}{rl}
\longrightarrow & -v_{=1} + -w_{=1} + -y_{=1} + -z_{=1} \geq -1 \\
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\longrightarrow & -w_{=3} + -x_{=3} + -y_{=3} + -z_{=3} \geq -1
\end{array}$$

$$\begin{array}{l}
-v_{=1} \geq 1 \\
v_{=1} \geq 0
\end{array}$$

Justifying All-Different Failures

$$\begin{array}{ll}
V \in \{\, 1 \qquad\quad 4 \quad 5 \,\} & \\
W \in \{\, 1 \quad 2 \quad 3 \qquad\quad \,\} & w_{=1} + w_{=2} + w_{=3} \geq 1 \\
X \in \{\, \quad\; 2 \quad 3 \qquad\quad \,\} & x_{=2} + x_{=3} \geq 1 \\
Y \in \{\, 1 \qquad\; 3 \qquad\quad \,\} & y_{=1} + y_{=3} \geq 1 \\
Z \in \{\, 1 \qquad\; 3 \qquad\quad \,\} & z_{=1} + z_{=3} \geq 1
\end{array}$$

$$\begin{array}{ll}
\rightarrow & -v_{=1} + -w_{=1} + -y_{=1} + -z_{=1} \geq -1 \\
\quad\rightarrow & -w_{=2} + -x_{=2} \geq -1 \\
\qquad\rightarrow & -w_{=3} + -x_{=3} + -y_{=3} + -z_{=3} \geq -1
\end{array}$$

$$\begin{array}{l}
-v_{=1} \geq 1 \\
v_{=1} \geq 0
\end{array}$$

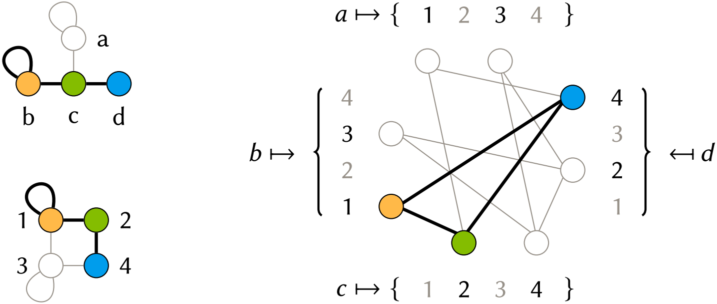
$$0 \geq 1$$

Reformulation

Auto-tabulation is possible.

- Heavy use of extension variables.

Can re-encode maximum common subgraph as a clique problem, without changing the pseudo-Boolean model.



High Level Modelling Languages?

High level modelling languages like MiniZinc and Essence have complicated compilers.

How do we know we’re giving a proof for the problem the user actually specified?

Future research…

Code

`https://github.com/ciaranm/glasgow-constraint-solver`

Released under MIT Licence.

Supports proof logging for global constraints including:

- All-different.
- Integer linear inequality (including for very large domains).
- Table.
- Minimum / maximum of an array.
- Element.
- Absolute value.

Details in [GMN22].

What’s Left?

- The truth about extension variables (redundance rule)
- Some applications of this rule (parity reasoning & PB-to-CNF translations)
- Extensions of the redundance rule to optimization
- Symmetry Breaking

The Truth About Extension Variables

Recall: we want new, fresh variable *a* encoding

$$a \Leftrightarrow (x \wedge y)$$

Introduce clauses

$$a \vee \overline{x} \vee \overline{y} \qquad \overline{a} \vee x \qquad \overline{a} \vee y$$

Or, in pseudo-Boolean language, constraints

$$a + \overline{x} + \overline{y} \geq 1 \qquad 2\overline{a} + x + y \geq 2$$

Resolution and cutting planes proof system inherently cannot certify such derivations: they are not implied!

Redundance-Based Strengthening

C is redundant with respect to F if F and $F \wedge C$ are equisatisfiable

Adding redundant constraints should be OK

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Redundance-based strengthening [BT19, GN21] (extending RAT)

C is redundant with respect to F iff there is a substitution ω (mapping variables to truth values or literals), called a witness, for which

$$F \wedge \neg C \models (F \wedge C)\restriction_{\omega}$$

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Fact

α satisfies $\phi\restriction_\omega$ iff $\alpha \circ \omega$ satisfies ϕ

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Proof sketch for interesting direction: If α satisfies F but falsifies C, then $\alpha \circ \omega$ satisfies $F \wedge C$

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Proof sketch for interesting direction: If α satisfies F but falsifies C, then $\alpha \circ \omega$ satisfies $F \wedge C$

Witness ω should be specified, and implication be efficiently verifiable (which is the case, e.g., if all constraints in $(F \wedge C)\restriction_\omega$ are RUP)

Deriving $a \Leftrightarrow (x \wedge y)$ Using the Redundance Rule

Want to derive

$$a + \overline{x} + \overline{y} \geq 1 \qquad 2\overline{a} + x + y \geq 2$$

using condition $F \wedge \neg C \models (F \wedge C)\restriction_{\omega}$

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1. $F \wedge \neg(2\overline{a} + x + y \geq 2) \models (F \wedge (2\overline{a} + x + y \geq 2))\restriction_{\omega}$
   Choose $\omega = \{a \mapsto 0\}$ — F untouched; new constraint satisfied

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2. $F \wedge (2\overline{a} + x + y \geq 2) \wedge \neg(a + \overline{x} + \overline{y} \geq 1) \models$
   $(F \wedge (2\overline{a} + x + y \geq 2) \wedge (a + \overline{x} + \overline{y} \geq 1))\restriction_{\omega}$
   Choose $\omega = \{a \mapsto 1\}$ — F untouched; new constraint satisfied
   $\neg(a + \overline{x} + \overline{y} \geq 1)$ forces $x \mapsto 1$ and $y \mapsto 1$, hence $2\overline{a} + x + y \geq 2$ remains satisfied after forcing a to be true

CDCL Solvers on Pseudo-Boolean Inputs

Can re-encode to CNF and run CDCL:

- *MiniSat+* [ES06]
- *Open-WBO* [MML14]
- *NaPS* [SN15]

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E.g., encode pseudo-Boolean constraint

$$x_1 + x_2 + x_3 + x_4 \geq 2$$

to clauses with extension variables

$$s_{i,k} \Leftrightarrow \sum_{j=1}^{i} x_j \geq k$$

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$$k \cdot \overline{s}_{i,k} + \sum_{j=1}^{i} x_j \geq k$$

$$(i - k + 1) \cdot s_{i,k} + \sum_{j=1}^{i} \overline{x}_j \geq i - k + 1$$

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How to know translation is correct?

VeriPB can certify pseudo-Boolean-to-CNF rewriting

Parity (XOR) Reasoning

Given clauses

$$x \lor y \lor z$$

$$x \lor \overline{y} \lor \overline{z}$$

$$\overline{x} \lor y \lor \overline{z}$$

$$\overline{x} \lor \overline{y} \lor z$$

and

$$y \lor z \lor w$$

$$y \lor \overline{z} \lor \overline{w}$$

$$\overline{y} \lor z \lor \overline{w}$$

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want to derive

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$$x + y + z = 1 \pmod 2$$
$$y + z + w = 1 \pmod 2$$

imply

$$x + w = 0 \pmod 2$$

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But used in *CryptoMiniSat* [Cry]

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DRAT proof logging like [PR16] too inefficient in practice!

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Could add XORs to language, but prefer to keep things super-simple

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$$x + y + z + 2a = 3$$
$$y + z + w + 2b = 3$$

(“=” syntactic sugar for “≥” plus “≤”)

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VeriPB can certify XOR reasoning [GN21]

Redundance and Dominance Rules for Optimisation

Redundance-based strengthening, optimisation version

Add constraint C to formula F if exists witness substitution ω s.t.

$$F \wedge \neg C \models (F \wedge C)\restriction_{\omega} \wedge f\restriction_{\omega} \leq f$$

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Can be more aggressive if witness ω strictly improves solution.

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Add constraint C to formula F if exists witness substitution ω s.t.

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Redundance and Dominance Rules for Optimisation

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- Applying ω should strictly decrease f.
- If so, don't need to show that $C\upharpoonright_{\omega}$ holds!

Soundness of Dominance Rule

Dominance-based strengthening (simplified)

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Why is this sound?

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7. ...

Soundness of Dominance Rule

Dominance-based strengthening (simplified)

Add constraint C to formula F if exists witness substitution ω s.t.

$$F \land \neg C \models F\restriction_\omega \ \land\ f\restriction_\omega < f$$

Why is this sound?

1. Suppose α satisfies F but falsifies C (i.e. satisfies $\neg C$).
2. Then $\alpha \circ \omega$ satisfies F and $f(\alpha \circ \omega) < f(\alpha)$.
3. If $\alpha \circ \omega$ satisfies C, we're done.
4. Otherwise $(\alpha \circ \omega) \circ \omega$ satisfies F and $f\big((\alpha \circ \omega) \circ \omega\big) < f\big(\alpha \circ \omega\big)$.
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7. …
8. Can't go on forever, so finally reach α' satisfying $F \land C$.

Strength of Dominance Rule

Dominance-based strengthening (stronger, still simplified)

If $C_1, C_2, \ldots, C_{m-1}$ have been derived from F (maybe using dominance), then can derive C_m if exists witness substitution ω s.t.

$$F \land \bigwedge_{i=1}^{m-1} C_i \land \neg C_m \models F\!\upharpoonright_\omega \land f\!\upharpoonright_\omega < f$$

Only consider F — no need to show that any $C_i\!\upharpoonright_\omega$ implied!

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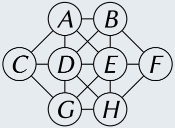
Further extensions:

- Define dominance rule w.r.t. order independent of objective.
- Switch between different orders in same proof.
- See [BGMN22a] for details.

Symmetry Elimination (CP)

The Crystal Maze Puzzle



Place numbers 1 to 8 without repetition, adjacent circles cannot have consecutive numbers.

Symmetry Elimination (CP)

Human modellers might add:

- $A < G$ (mirror vertically)
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- $A \leq 4$ (value symmetry)

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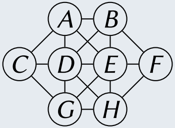

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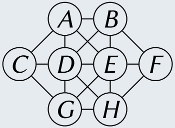

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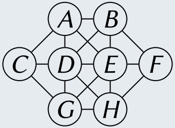

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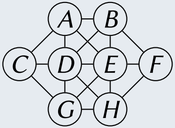

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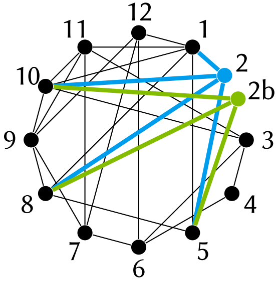
Research challenge: a CP toolchain supporting this.

Lazy Global Domination For Maximum Clique [MP16]



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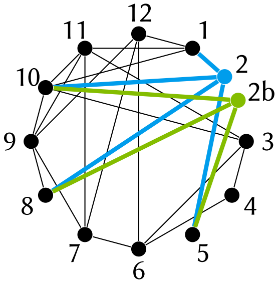

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Dominance rule can justify this.

- Even when detected dynamically during search.

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3. Derive CNF encoding of lex-leader constraints from PB constraint (in same spirit as [GMNO22])

$$\begin{array}{ll} y_0 & \overline{y}_j \vee \overline{\sigma(x_j)} \vee x_j \\ \overline{y}_{j-1} \vee \overline{x}_j \vee \sigma(x_j) & y_j \vee \overline{y}_{j-1} \vee \overline{x}_j \\ \overline{y}_j \vee y_{j-1} & y_j \vee \overline{y}_{j-1} \vee \sigma(x_j) \end{array}$$

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Example: Pigeonhole principle formula

- Variables p_{ij} $(1 \leq i \leq 4, 1 \leq j \leq 3)$ true iff pigeon i in hole j
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Order: “Pigeon 1 preferred in the smallest hole; next pigeon 2, ...”

$$f \doteq 2^{11} \cdot p_{13} + 2^{10} \cdot p_{12} + 2^{9} \cdot p_{11} + 2^{8} \cdot p_{23} + \cdots + 1 \cdot p_{41}$$

Breaking a Single Simple Symmetry (Example)

- F is a formula expressing PHP constraints with $F\restriction_{\sigma_{(12)}} = F$
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Similar to DRAT symmetry breaking [HHW15]

Breaking More/Other symmetries

Problem

This idea does not generalize.

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Can "restore" its truth by applying $\sigma_{(1234)}$ once, twice, or thrice.

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Definition

Given a symmetry σ, the (pseudo-Boolean) breaking constraint of σ is

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Theorem

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Why does it work?

- Witness need not satisfy all derived constraints
- Sufficient to just produce “better” assignment

Making Your Solver Output Proofs

The VeriPB proof verifier lives at

`https://gitlab.com/MIAOresearch/software/VeriPB`

And it's documented!

See [GMM⁺20, EGMN20, BGMN22b, GN22, GMN22] for worked examples, and even more in Stephan Gocht's PhD thesis [Goc22].

We're happy to collaborate with you! And we're hiring!

Challenges and Work In Progress

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The end. Or rather, the beginning!

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https://gitlab.com/MIAOresearch/software/VeriPB

https://github.com/ciaranm/glasgow-constraint-solver

https://github.com/ciaranm/glasgow-subgraph-solver

https://bitbucket.org/krr/breakid

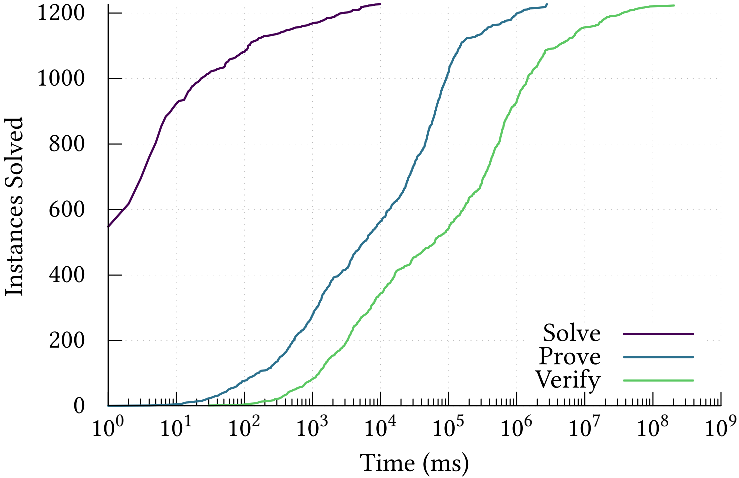
Clique Results

- Implemented in the Glasgow Subgraph Solver.
    - Bit-parallel, can perform a colouring and recursive call in under a microsecond.
- 59 of the 80 DIMACS instances take under 1,000 seconds to solve without logging.
- Produced and verified proofs for 57 of these 59 instances (the other two reached 1TByte disk space).
- Mean slowdown from proof logging is 80.1 (due to disk I/O).
- Mean verification slowdown a further 10.1.
- Approximate implementation effort: one Masters student.

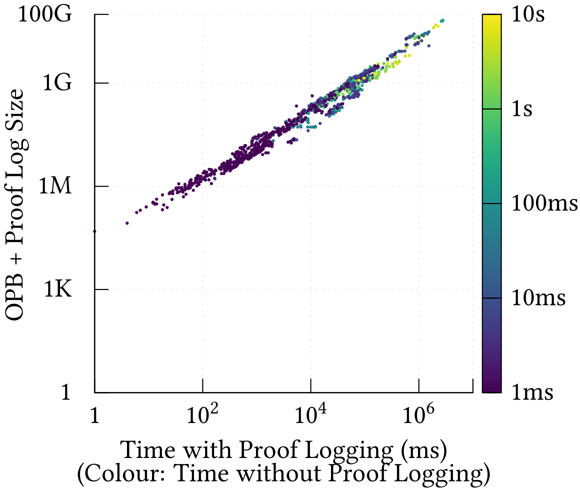
Subgrap Isomorphism Results

- The Pseudo-Boolean models can be large: had to restrict to instances with no more than 260 vertices in the target graph.
- Took enumeration instances which could be solved without proof logging in under ten seconds.
- 1,227 instances from Solnon's benchmark collection:
  - 789 unsatisfiable, up to 50,635,140 solutions in the rest.
  - 498 instances solved without guessing.
  - Hardest solved satisfiable and unsatisfiable instances required 53,605,482 and 2,074,386 recursive calls.

Subgrap Isomorphism Results



Subgrap Isomorphism Results



How Expensive is Proof Logging?

- Laurent D. Michel, Pierre Schaus, Pascal Van Hentenryck: MiniCP: a lightweight solver for constraint programming. Math. Program. Comput. 13(1) (2021).
- Five benchmark problems allowing comparison of solvers “doing the same thing”:
    - Simple models.
    - Fixed search order and well-defined propagation consistency levels.
    - Few global constraints (although we don’t have circuit yet).
- Probably close to the worst case for proof logging performance.
- Also: Crystal Maze and World’s Hardest Sudoku.

How Expensive is Proof Logging?

- Our solver: faster than the fastest of MiniCP, OscaR, and Choco.
- Proof logging slowdown: between 8.4 to 61.1.
  - 800,000 to 3,000,000 inferences per second.
  - Proof logs can be hundreds of GBytes.
  - No effort put into making the proof-writing code run fast.
- Verification slowdown: a further 10 to 100.
  - Probably possible to reduce this substantially if we are prepared to put more care into writing proofs.

PB-to-CNF Translation: Experiments

- Certified translations for the following CNF encodings:[2]
  - Sequential counter [Sin05]
  - Totalizer [BB03]
  - Generalized totalizer [JMM15]
  - Adder network [ES06]
- Proof verified by proof checker VeriPB
- Benchmarks from PB 2016 Evaluation:[3]
  - `SMALLINT` decision benchmarks without purely clausal formulas
  - 3 subclasses of benchmarks:
    - Only cardinality constraints (sequential counter, totalizer)
    - Only general 0-1 ILP constraints (generalized totalizer, adder network)
    - Mixed cardinality & general 0-1 ILP constraints (sequential counter + adder network)

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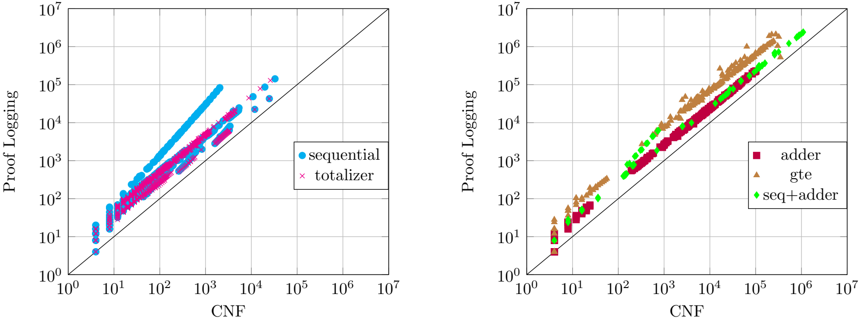
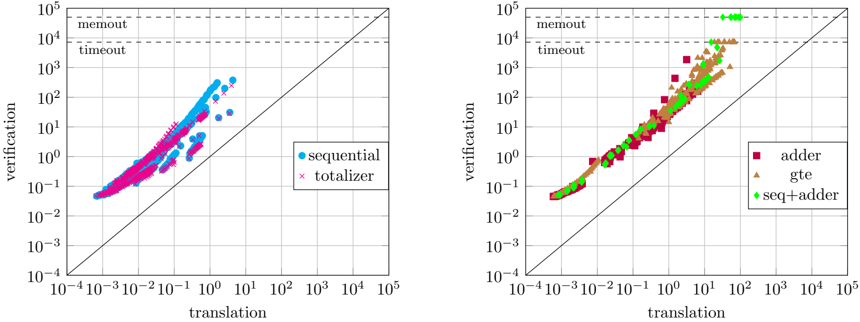
[2] `https://github.com/forge-lab/VeritasPBLib`

[3] `http://www.cril.univ-artois.fr/PB16/`

PB-to-CNF: CNF Size vs Proof Size in KiB

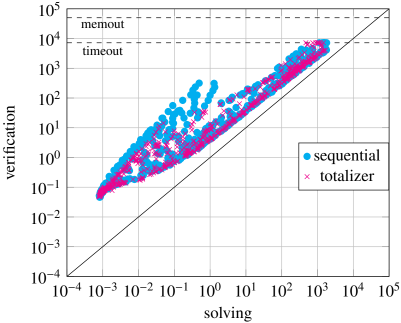


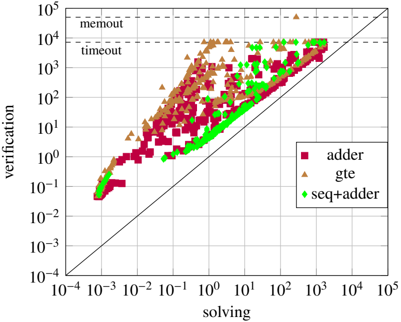
PB-to-CNF: Translation vs Verification Time in Seconds



- Translation just generates clauses and proof
- Verification slower, as reasoning has to be performed

PB-to-CNF: Solving Time vs Verification Time in Seconds





- Solved with fork of Kissat[4] syntactically modified to output pseudo-Boolean proofs
- Room for improvement, but clearly shows approach is viable

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[4] `https://gitlab.com/MIAOresearch/tools-and-utilities/kissat_fork`

PB-to-CNF: Future Work

Improving performance:

- Cutting Planes derivations instead of reverse unit propagations [VDB22]
- Backwards checking/trimming for verification (as in DRAT-trim [HHW13a])

PB-to-CNF: Future Work

Improving performance:

- Cutting Planes derivations instead of reverse unit propagations [VDB22]
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Extend proof logging further:

- Sorting networks like odd-even mergesort, bitonic sorter [Bat68]
- MaxSAT solving

Parity Reasoning: Experiments

Implemented parity reasoning and PB proof logging engine[5]

Also DRAT proof logging as described in [PR16]

Experiments with MiniSat[6]

Set-up:[7]

- Intel Core i5-1145G7 @2.60GHz × 4
- Memory limit 8GiB
- Disk write speed roughly 200 MiB/s
- Read speed of 2 GiB/s

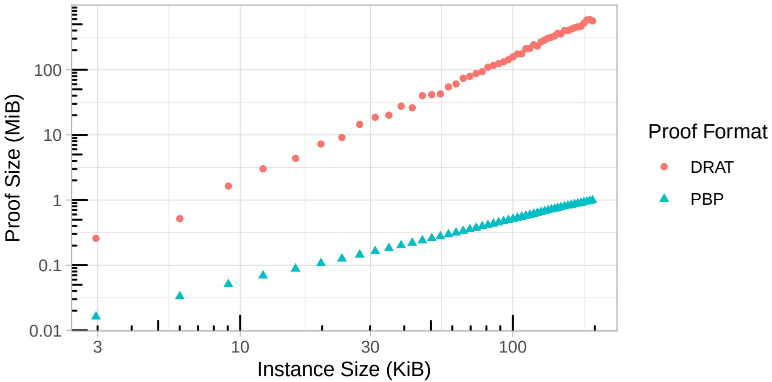
---

[5] `https://gitlab.com/MIAOresearch/tools-and-utlities/xorengine`

[6] `http://minisat.se/`

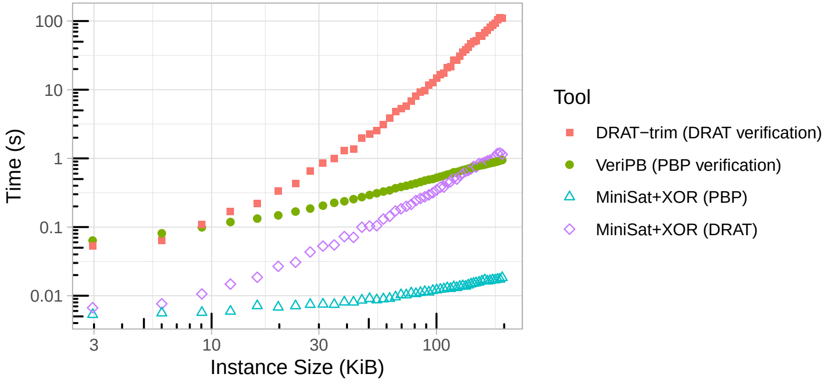
[7] Tools, benchmarks, data and evaluation scripts available at `https://doi.org/10.5281/zenodo.7083485`

Parity Reasoning: Proof Size



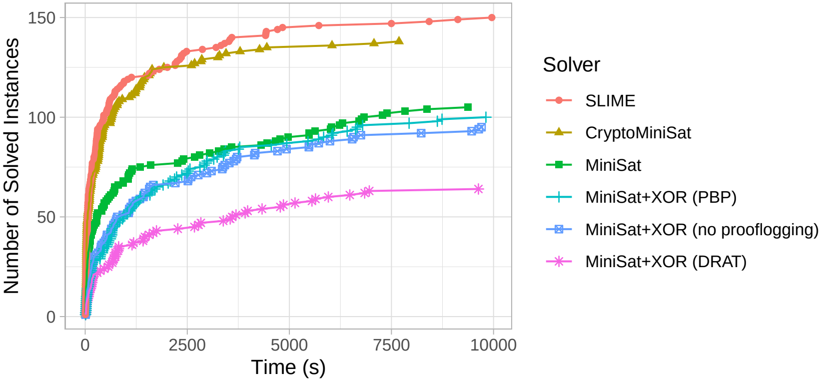
Proof sizes for Tseitin formulas using DRAT and PB proof logging

Parity Reasoning: Solving and Verification Time



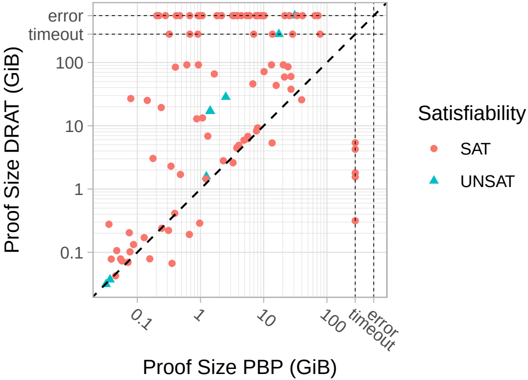
Solving and verification time for Tseitin formulas

Parity Reasoning: Crypto Track of SAT 2021 Competition



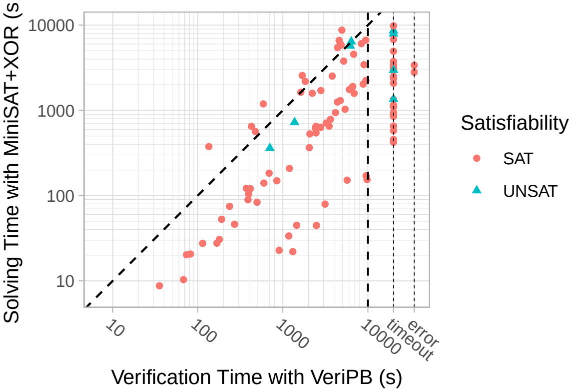
Cumulative plot for the crypto track of the SAT Competition 2021

Parity Reasoning: Crypto Track Proof Size



DRAT and PB proof sizes for crypto track of SAT Competition 2021

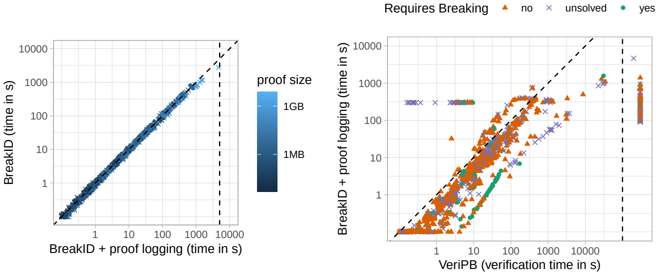
Parity Reasoning: Crypto Track Verification Time



Time required for solving and verifying crypto instances

Experimental Evaluation of SAT Symmetry Breaking

- Evaluated on SAT competition benchmarks
- BreakID [DBBD16, Bre] used to find and break symmetries



- proof logging overhead negligible
- verification at most 20 times slower than solving for 95% of instances

Strategy for SAT Symmetry Breaking

1. Pretend to solve optimisation problem minimizing $f \doteq \sum_{i=1}^{n} 2^{n-i} \cdot x_i$
(search lexicographically smallest assignment satisfying formula)

2. Derive pseudo-Boolean lex-leader constraint

$$C_\sigma \quad \doteq \quad f \leq f\restriction_\sigma \quad \doteq \quad \sum_{i=1}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0$$

3. Derive CNF encoding of lex-leader constraints from PB constraint (in same spirit as [GMNO22])

$$\begin{array}{ll} y_0 & \overline{y}_j \vee \overline{\sigma(x_j)} \vee x_j \\ \overline{y}_{j-1} \vee \overline{x}_j \vee \sigma(x_j) & y_j \vee \overline{y}_{j-1} \vee \overline{x}_j \\ \overline{y}_j \vee y_{j-1} & y_j \vee \overline{y}_{j-1} \vee \sigma(x_j) \end{array}$$

Symmetry Breaking in CNF

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- We use the encoding of BreakID [DBBD16]:

$$y_0$$

$$\overline{y}_{j-1} \vee \overline{x}_j \vee \sigma(x_j)$$

$$\overline{y}_j \vee y_{j-1}$$

$$\overline{y}_j \vee \overline{\sigma(x_j)} \vee x_j$$

$$y_j \vee \overline{y}_{j-1} \vee \overline{x}_j$$

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$y_j \vee \overline{y}_{j-1} \vee \overline{x}_j$

$y_j \vee \overline{y}_{j-1} \vee \sigma(x_j)$

Define y_j to be true if x_k equals $\sigma(x_k)$ for all $k \leq j$

$$y_k \Leftrightarrow y_{k-1} \wedge (x_k \Leftrightarrow \sigma(x_k))$$

(derivable with redundance rule)

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$$y_j \vee \overline{y}_{j-1} \vee \overline{x}_j$$

$$y_j \vee \overline{y}_{j-1} \vee \sigma(x_j)$$

(derivable with redundance rule) If y_k is true, x_k is at most $\sigma(x_k)$ (derivable from the PB breaking constraint)

Detailed Derivation of CNF Breaking Constraints

Derived constraints (D):

Pseudo-Boolean breaking constraint

$$\sum_{i=1}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0$$

Detailed Derivation of CNF Breaking Constraints

Derived constraints (D):

$$\sum_{i=1}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0$$

y_0

Derivable by redundance with witness $\omega : y_0 \mapsto 1$

$$F \wedge D \wedge \{\overline{y}_0\} \models (F \wedge D)\restriction_{\omega} \wedge \{y_0\}\restriction_{\omega}$$

Detailed Derivation of CNF Breaking Constraints

Derived constraints (D):

$$\sum_{i=1}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0$$

y_0

Derivable by redundance with witness $\omega : y_0 \mapsto 1$

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$\overline{y}_0 \vee \overline{x}_1 \vee \sigma(x_1)$

Derivable by RUP

$$F \wedge D \wedge \neg(\overline{y}_0 \vee \overline{x}_1 \vee \sigma(x_1))$$

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$$\sum_{i=1}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0$$

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Detailed Derivation of CNF Breaking Constraints

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Derivable by RUP

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$$\sum_{i=1}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0$$

$$2^{n-1} \cdot (-1) + \sum_{i=2}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0$$

with

$$\sum_{i=2}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \leq 2^{n-1} - 1$$

Detailed Derivation of CNF Breaking Constraints

Derived constraints (D):

$$\sum_{i=1}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0$$

y_0

$\overline{y}_0 \vee \overline{x}_1 \vee \sigma(x_1)$

$\overline{y}_1 \vee y_0$

Derivable by redundance with witness $\omega : y_1 \mapsto 0$

$$F \wedge D \wedge \neg(\overline{y}_1 \vee y_0)$$
$$\models (F \wedge D)\restriction_{\omega} \wedge \{\overline{y}_1 \vee y_0\}\restriction_{\omega}$$

Detailed Derivation of CNF Breaking Constraints

Derived constraints (D):

$$\sum_{i=1}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0$$

y_0

$\overline{y}_0 \vee \overline{x}_1 \vee \sigma(x_1)$

$\overline{y}_1 \vee y_0$

Derivable by redundance with witness $\omega : y_1 \mapsto 0$

$$F \wedge D \wedge \neg(\overline{y}_1 \vee y_0) \models (F \wedge D)\restriction_\omega \wedge \{\overline{y}_1 \vee y_0\}\restriction_\omega$$

$$F \wedge D \wedge \neg(\overline{y}_1 \vee y_0) \models (F \wedge D)\restriction_\omega \wedge \{1 \vee y_0\}$$

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$\omega : y_1 \mapsto 0$
(same argument)

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$y_1 \vee \overline{y}_0 \vee \overline{x}_1$

Derivable by redundance with witness $\omega : y_1 \mapsto 1$

$$F \wedge D \wedge \neg(y_1 \vee \overline{y}_0 \vee \overline{x}_1)$$
$$\models (F \wedge D)\restriction_\omega \wedge \{y_1 \vee \overline{y}_0 \vee \overline{x}_1\}\restriction_\omega$$

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$$\models \cdots \wedge D\restriction_\omega \wedge \ldots$$

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$$\sum_{i=1}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0$$

$$+ 2^{n-1} \cdot \left(\overline{y}_1 + \overline{\sigma(x_1)} + x_1 \geq 1\right)$$

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$$2^{n-1} \cdot \overline{y}_1 + \sum_{i=2}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0$$

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The clause to derive is RUP with respect to this constraint