

# An Introduction to Pseudo-Boolean Proof Logging

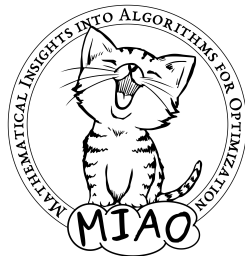
Jakob Nordström

University of Copenhagen and Lund University

*2nd International Workshop on Highlights in  
Organizing and Optimizing Proof-logging Systems*

Paris, France

September 13–14, 2025



## Based on Joint Work With...

- Markus Anders
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# Combinatorial Solving and Optimisation

- Revolution last couple of decades in **combinatorial solvers** for
  - Boolean satisfiability (SAT) solving [BHvMW21]<sup>1</sup>
  - Constraint programming (CP) [RvBW06]
  - Mixed integer linear programming (MIP) [AW13, BR07]
- Solve NP-complete problems (or worse) very successfully in practice!
- **Except solvers are sometimes wrong...** (Even best commercial ones) [BLB10, CKSW13, AGJ<sup>+</sup>18, GSD19, BMN22, GCS23]
- Solvers can propose infeasible “solutions” (but erroneous claims can in principle be checked)
- More challenging: How to achieve reliable claims of infeasibility?
- Or that a solution is optimal? (Even off-by-one mistakes can snowball into large errors if solver used as subroutine)

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<sup>1</sup>See end of slides for all references with bibliographic details

# What Can Be Done About Solver Bugs?

## ■ Software testing

Very useful, but bugs slip through even with careful domain-specific testing

Progress using fuzzing and delta debugging [BB09, BLB10, KB22, NPB22, PB23]

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Prove that solver implementation adheres to formal specification

Current techniques cannot scale to level of complexity in modern solvers

(Despite valiant efforts in, e.g., [Fle20])



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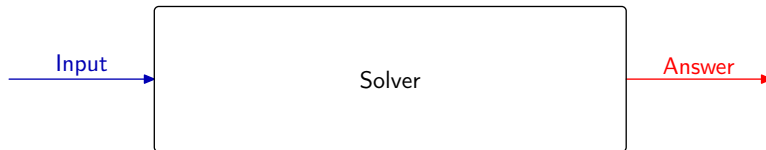
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## ■ Proof logging

Make solver **certifying** [ABM<sup>+</sup>11, MMNS11] by adding code so that it outputs

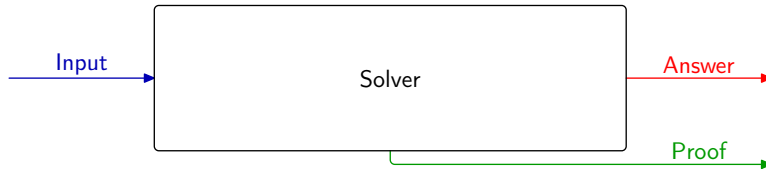
- 1 not only **answer** but also
- 2 simple, machine-verifiable **proof** that answer is correct

# Proof Logging with Certifying Solvers: Workflow



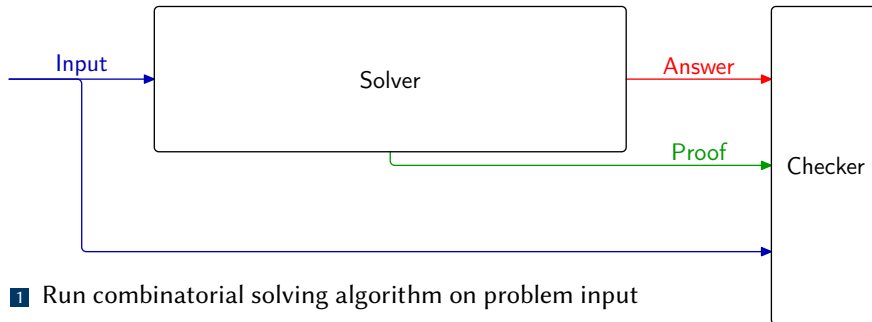
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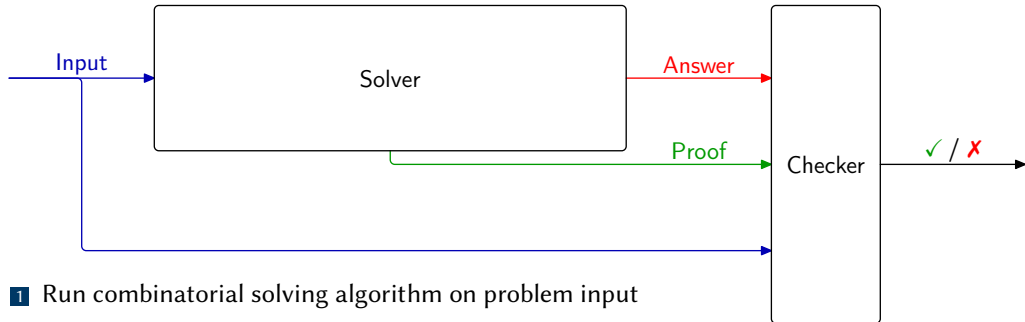
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# Proof Logging with Certifying Solvers: Workflow



- 1 Run combinatorial solving algorithm on problem input
- 2 Get as output not only answer but also proof
- 3 Feed input + answer + proof to proof checker

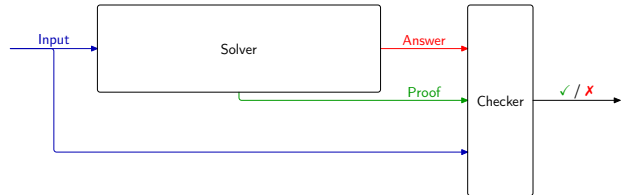
# Proof Logging with Certifying Solvers: Workflow



- 1 Run combinatorial solving algorithm on problem input
- 2 Get as output not only answer but also proof
- 3 Feed input + answer + proof to proof checker
- 4 Verify that proof checker says answer is correct

# Proof Logging Desiderata

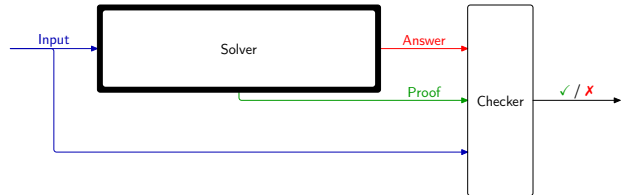
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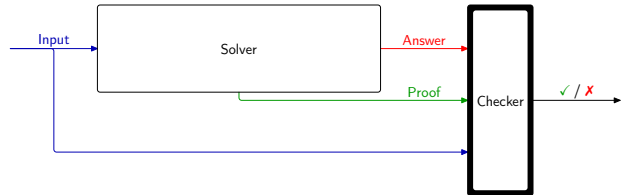
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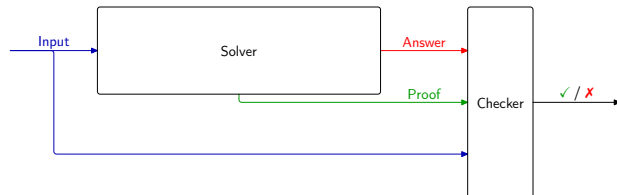
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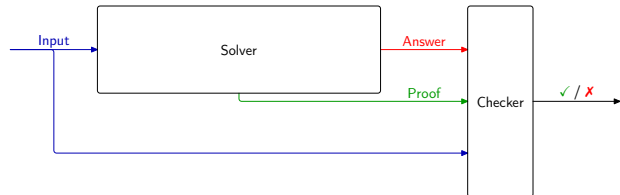


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Clear conflict expressivity vs. simplicity!

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Clear conflict expressivity vs. simplicity!

Asking for both perhaps a little bit too good to be true?

# Take-Away Message from This Tutorial Day

Proof logging for combinatorial optimisation is possible with **single, unified method!**

# Take-Away Message from This Tutorial Day

Proof logging for combinatorial optimisation is possible with **single, unified method!**

- Build on successes in proof logging for SAT solvers with proof formats such as DRAT [HHW13a, HHW13b, WHH14], GRIT [CMS17], LRAT [CHH<sup>+</sup>17], ...
- But represent constraints as **0–1 linear inequalities**
- Formalize reasoning using **cutting planes** [CCT87] proof system
- Add well-chosen **strengthening rules** [Goc22, GN21, BGMN23]
- Implemented in **VERIPB** (<https://gitlab.com/MIAOresearch/software/VeriPB>)

# The Sales Pitch For Proof Logging

- 1 Certifies correctness of computed results
- 2 Detects errors even if due to compiler bugs, hardware failures, or cosmic rays
- 3 Provides debugging support during software development  
[GMM<sup>+</sup>20, KM21, BBN<sup>+</sup>23, EG23, KLM<sup>+</sup>25]
- 4 Facilitates performance analysis
- 5 Helps identify potential for further improvements
- 6 Enables auditability
- 7 Serves as stepping stone towards explainability

# Design Principles for Proof Logging

## Proof logging implementation

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- Proof logging overhead small constant fraction of running time ( $\lesssim 10\%$ )
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## Proof system

- Keep language simple — no XOR constraints, CP propagators, symmetries, ...
- But reason efficiently about such notions using power of proof system
- Combine proof logging with formally verified proof checker



# Program for This Tutorial Day

Explain how to use **VERIPB** as a unified proof logging method for

**09:00** SAT solving (Jakob Nordström)

**10:00** Subgraph solving (Ciaran McCreesh)

**11:30** Constraint programming (Matthew McIlree)

**14:00** Pseudo-Boolean optimisation (Wietze Koops)

**15:30** Preprocessing/presolving in MaxSAT and 0–1 linear programming (Andy Oertel)

**16:30** Symmetry breaking (Markus Anders)

# But Let Us Start from the Beginning...

Review of some basic concepts:

- SATISFIABILITY (SAT) problem
- Unit propagation
- DPLL and CDCL algorithms
- Resolution proof system

# The SATISFIABILITY (SAT) Problem

- **Variable**  $x$ : takes value **true** (=1) or **false** (=0)
- **Literal**  $\ell$ : variable  $x$  or its negation  $\bar{x}$
- **Clause**  $C = \ell_1 \vee \dots \vee \ell_k$ : disjunction of literals  
(Consider as sets, so no repetitions and order irrelevant)
- **Conjunctive normal form (CNF) formula**  $F = C_1 \wedge \dots \wedge C_m$ : conjunction of clauses

## The SAT Problem

Given a CNF formula  $F$ , is it satisfiable?

For instance, what about:

$$(p \vee \bar{u}) \wedge (q \vee r) \wedge (\bar{r} \vee w) \wedge (u \vee x \vee y) \wedge \\ (x \vee \bar{y} \vee z) \wedge (\bar{x} \vee z) \wedge (\bar{y} \vee \bar{z}) \wedge (\bar{x} \vee \bar{z}) \wedge (\bar{p} \vee \bar{u})$$

# Proofs for SAT

For satisfiable instances: just specify satisfying assignment

For unsatisfiability: a sequence of clauses (CNF constraints)

- Each clause follows “obviously” from everything we know so far
- Final clause is empty, meaning contradiction (written  $\perp$ )
- Means original formula must be inconsistent

# What Is Obvious? Unit Propagation

## Unit Propagation

Clause  $C$  **unit propagates**  $\ell$  under partial assignment  $\rho$  if  $\rho$  falsifies all literals in  $C$  except  $\ell$

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**Example:** Unit propagate for  $\rho = \{p \mapsto 0, q \mapsto 0\}$  on our formula

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- $p \vee \bar{u}$  propagates  $u \mapsto 0$



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Proof checker should know how to unit propagate until saturation

# Davis-Putman-Logemann-Loveland (DPLL)

DPLL [DP60, DLL62]: Assign variables and propagate; backtrack when clause violated

“Proof trace”: when backtracking, write negation of guesses made

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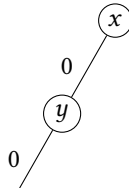


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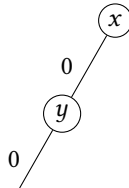


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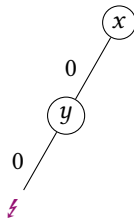
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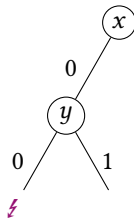
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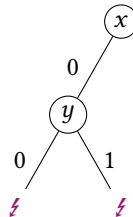
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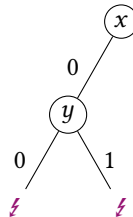
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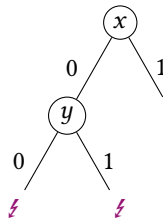
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# Davis-Putman-Logemann-Loveland (DPLL)

DPLL [DP60, DLL62]: Assign variables and propagate; backtrack when clause violated

“Proof trace”: when backtracking, write negation of guesses made

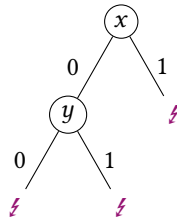
$$(p \vee \bar{u}) \wedge (q \vee r) \wedge (\bar{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \bar{y} \vee z) \wedge (\bar{x} \vee z) \wedge (\bar{y} \vee \bar{z}) \wedge (\bar{x} \vee \bar{z}) \wedge (\bar{p} \vee \bar{u})$$

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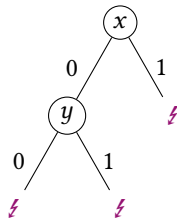
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5  $\perp$



# Reverse Unit Propagation (RUP)

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## Fact

Backtrack clauses from DPLL solver generate a RUP proof

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Run CDCL [BS97, MS99, MMZ<sup>+</sup>01] on our favourite CNF formula:

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Free choice to assign value to variable

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decision  
level 1

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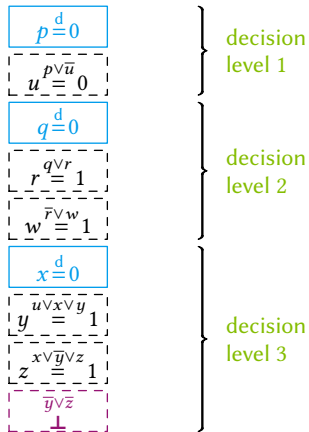
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# Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \vee \bar{u}) \wedge (q \vee r) \wedge (\bar{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \bar{y} \vee z) \wedge (\bar{x} \vee z) \wedge (\bar{y} \vee \bar{z}) \wedge (\bar{x} \vee \bar{z}) \wedge (\bar{p} \vee \bar{u})$$

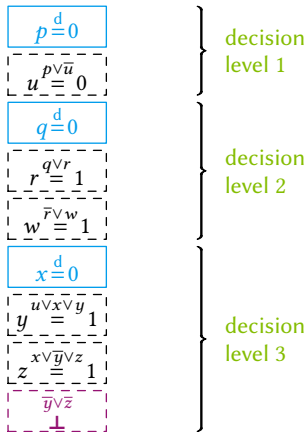




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decision  
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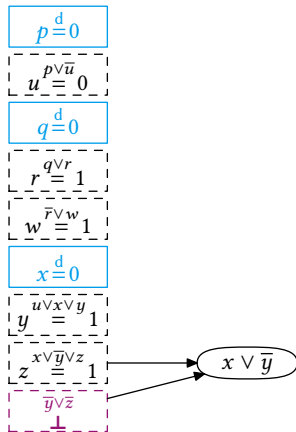
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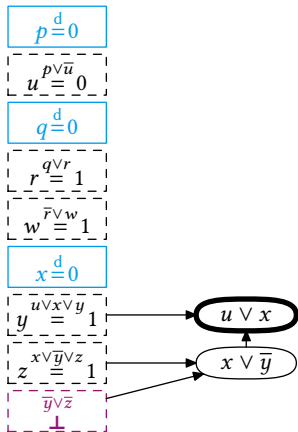
Case analysis over  $z$  for last two clauses:

- $x \vee \bar{y} \vee z$  wants  $z = 1$
- $\bar{y} \vee \bar{z}$  wants  $z = 0$
- **Resolve** clauses by merging them & removing  $z$  — must satisfy  $x \vee \bar{y}$

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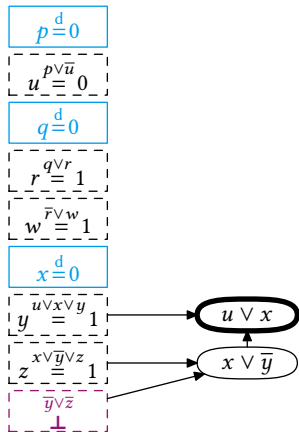
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Repeat until **UIP clause** with only 1 variable at conflict level after last decision — **learn** and **backjump**

# Complete Example of CDCL Execution

**Backjump:** undo max #decisions while learned clause propagates

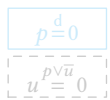
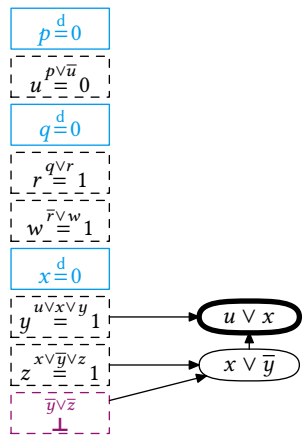
$$(p \vee \bar{u}) \wedge (q \vee r) \wedge (\bar{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \bar{y} \vee z) \wedge (\bar{x} \vee z) \wedge (\bar{y} \vee \bar{z}) \wedge (\bar{x} \vee \bar{z}) \wedge (\bar{p} \vee \bar{u})$$



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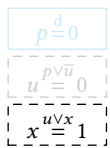
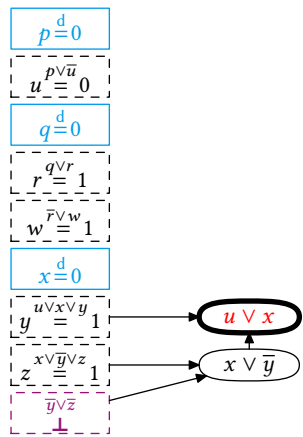


**Assertion level 1** (2nd largest level in learned clause) — trim trail to that level

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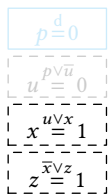
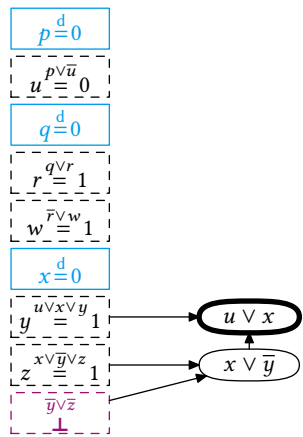
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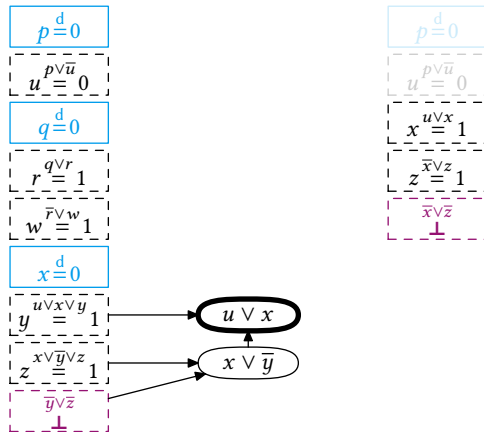
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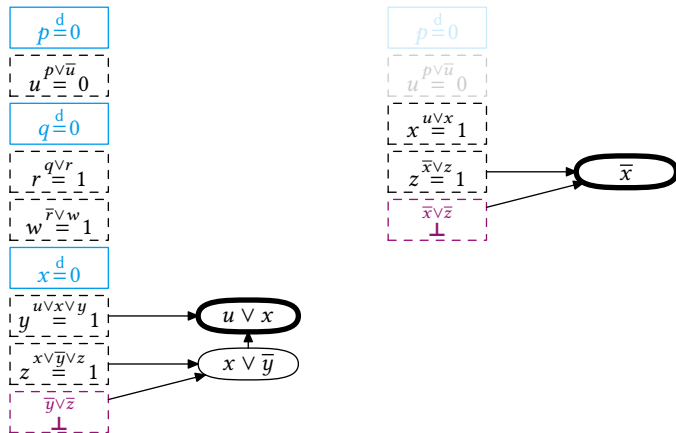
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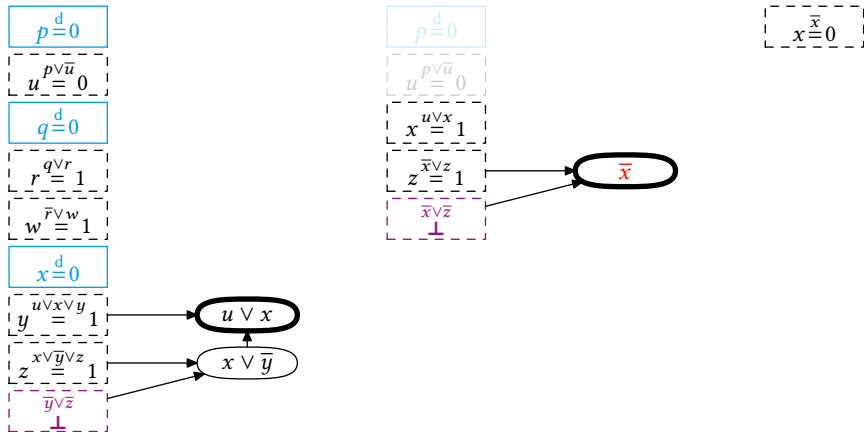
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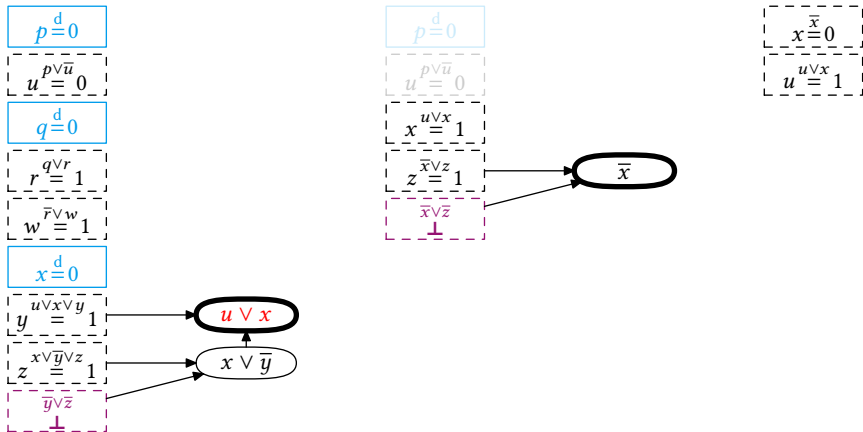
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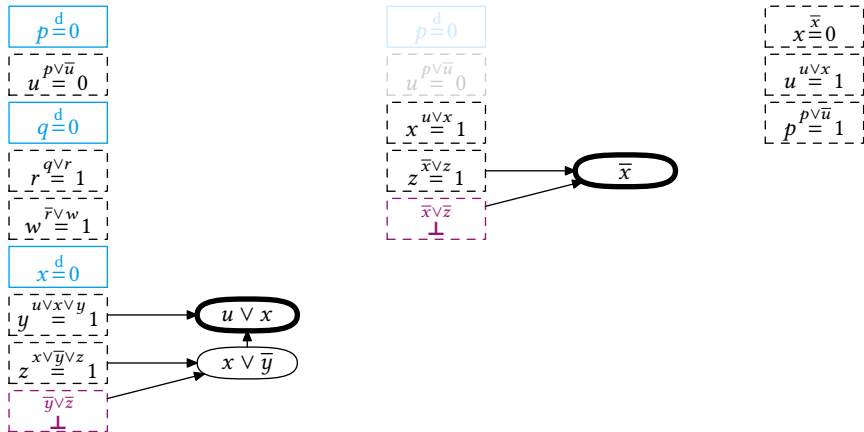
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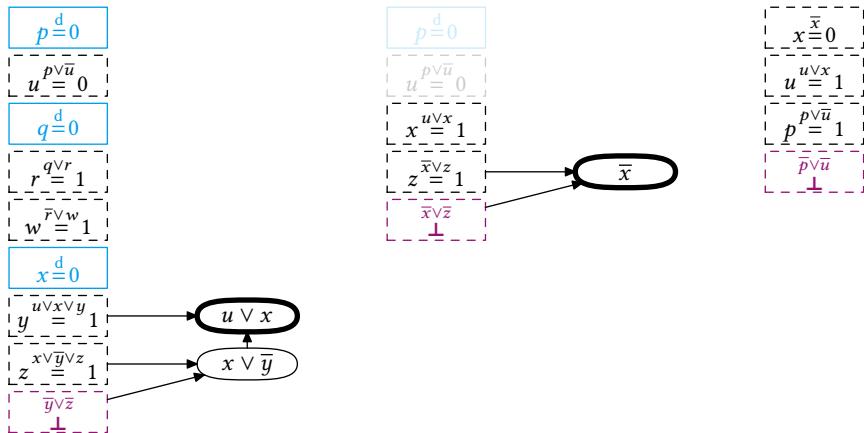
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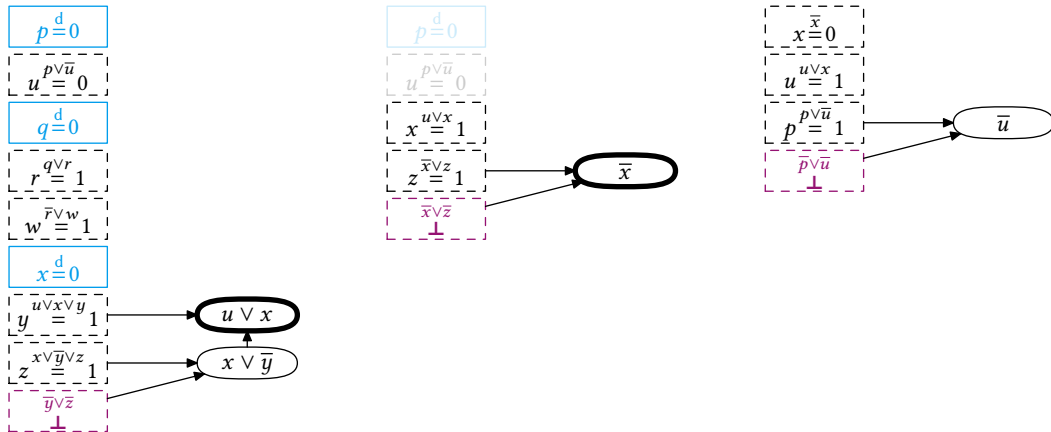
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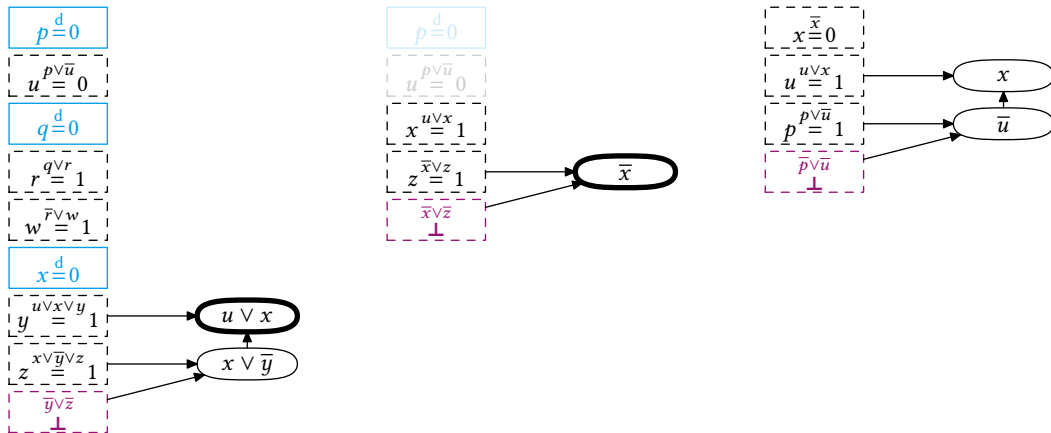
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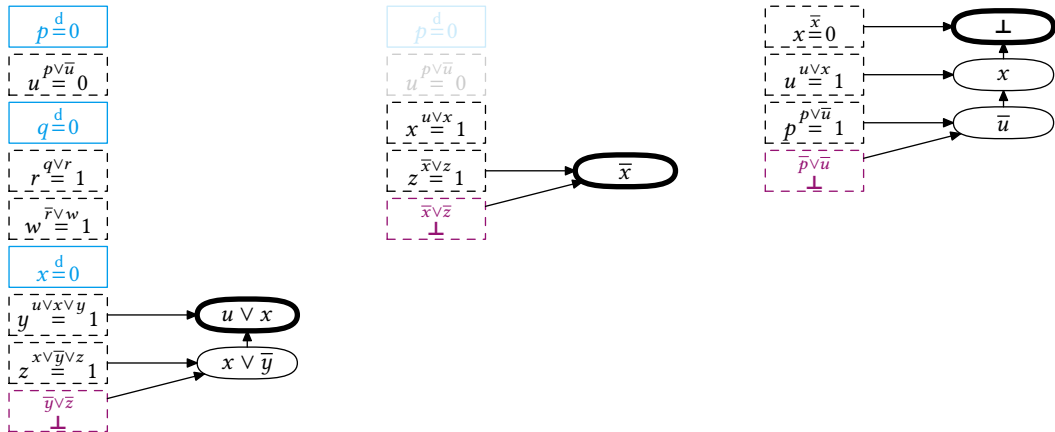




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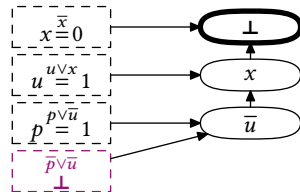
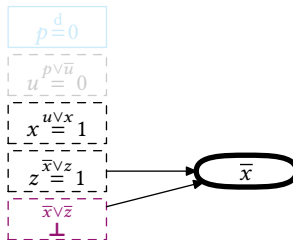
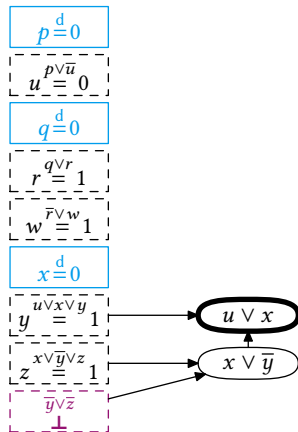
(\*) Ignores pre- and inprocessing, but we will get there...

# Resolution Proofs from CDCL Executions

Obtain resolution proof...

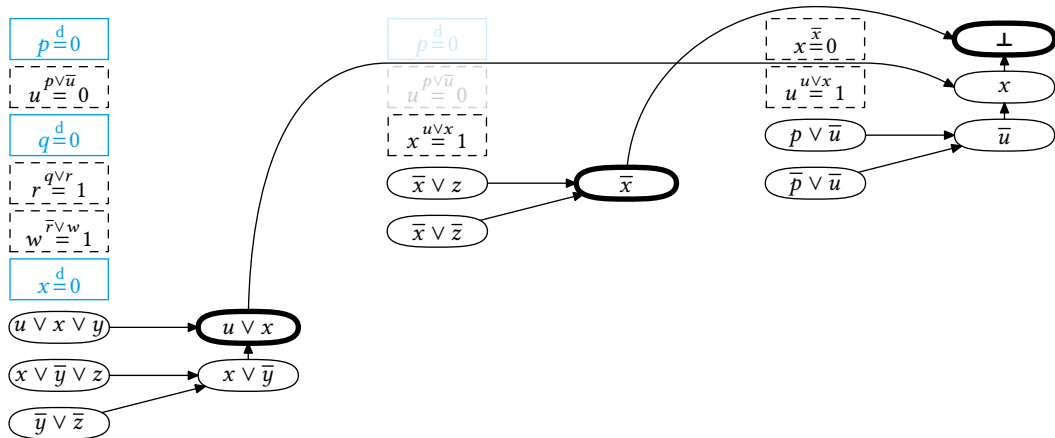
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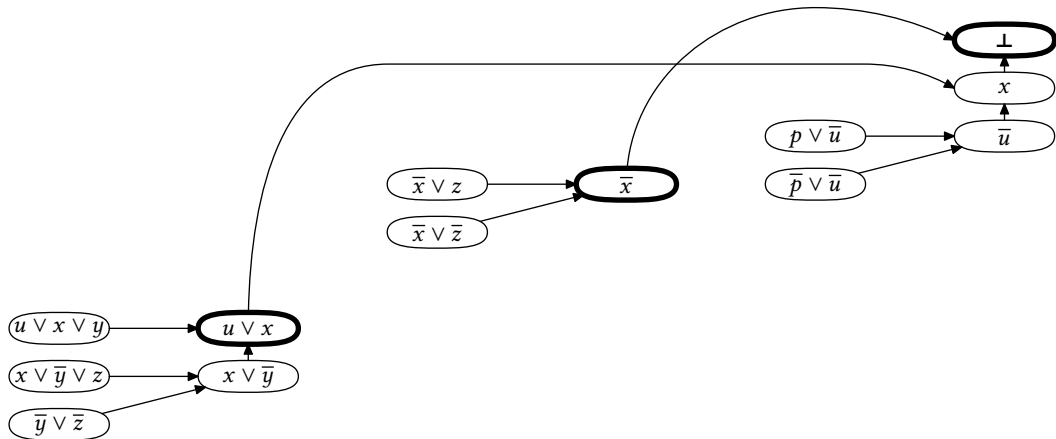
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# More Ingredients in Proof Logging for SAT

## Fact

RUP proofs can be viewed as shorthand for resolution proofs

See proof complexity and SAT solving survey [BN21] for more on this

But RUP and resolution are not enough for preprocessing, inprocessing, and some other kinds of reasoning

# Extension Variables, Part 1

Suppose we want a variable  $a$  encoding

$$a \Leftrightarrow (x \wedge y)$$

## Extended resolution [Tse68]

Resolution rule plus **extension rule** introducing clauses

$$a \vee \bar{x} \vee \bar{y} \quad \bar{a} \vee x \quad \bar{a} \vee y$$

for fresh variable  $a$  (this is fine since  $a$  doesn't appear anywhere previously)

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## Fact

Extended resolution (RUP + definition of new variables) is essentially equivalent to the DRAT proof logging system most commonly used for SAT solving

# Why Aren't We Done?

Practical limitations of current SAT proof logging technology:

- Difficulties dealing with stronger reasoning efficiently (even for SAT solving)
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- Can justify **graph reasoning** without knowing what a graph is
- Can justify **constraint programming** inference without knowing what an integer variable is

# Pseudo-Boolean Constraints

0–1 integer linear inequalities or (linear) pseudo-Boolean constraints:

$$\sum_i a_i \ell_i \geq A$$

- $a_i, A \in \mathbb{Z}$
- **literals**  $\ell_i$ :  $x_i$  or  $\bar{x}_i$  (where  $x_i + \bar{x}_i = 1$ )

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Sometimes convenient to use **normalised form** [Bar95] with **all  $a_i, A$  positive** (without loss of generality)

# Some Types of Pseudo-Boolean Constraints

## 1 Clauses

$$x_1 \vee \bar{x}_2 \vee x_3 \quad \Leftrightarrow \quad x_1 + \bar{x}_2 + x_3 \geq 1$$

## 2 Cardinality constraints

$$x_1 + x_2 + x_3 + x_4 \geq 2$$

## 3 General pseudo-Boolean constraints

$$x_1 + 2\bar{x}_2 + 3x_3 + 4\bar{x}_4 + 5x_5 \geq 7$$



# Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

Input/model axioms

From the input

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**Multiplication** for any  $c \in \mathbb{N}^+$

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**Saturation** for any  $c \in \mathbb{N}^+$   
(assumes normalised form)

$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i \min(a_i, A) \cdot \ell_i \geq A}$$

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 & \frac{\bar{z} \geq 0}{2\bar{z} \geq 0} & \text{Multiply by 2} \\
 \text{Add} & \frac{3w + 6x + 6y + 2z \geq 9}{3w + 6x + 6y} & \geq 7
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Multiply by 2

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Naming constraints by integers and literal axioms by the literal involved (with  $\sim$  for negation) as

$$\text{Constraint 1} \doteq 2x + y + w \geq 2$$

$$\text{Constraint 2} \doteq 2x + 4y + 2z + w \geq 5$$

$$\sim z \doteq \bar{z} \geq 0$$

# Cutting Planes Toy Example

Multiply by 2

$w + 2x + y \geq 2$

---

$2w + 4x + 2y \geq 4$

Add

$w + 2x + 4y + 2z \geq 5$

---

$3w + 6x + 6y + 2z \geq 9$

Add

$\bar{z} \geq 0$

---

$2\bar{z} \geq 0$

Divide by 3

$3w + 6x + 6y \geq 9$

---

$w + 2x + 2y \geq 3$

Naming constraints by integers and literal axioms by the literal involved (with ~ for negation) as

Constraint 1  $\doteq 2x + y + w \geq 2$

Constraint 2  $\doteq 2x + 4y + 2z + w \geq 5$

$\sim z \doteq \bar{z} \geq 0$

such a calculation is written in the proof log in reverse Polish notation as

```
pol 1 2 * 2 + ~z 2 * + 3 d ;
```

# Resolution and Cutting Planes

To simulate resolution step such as

$$\frac{\bar{y} \vee \bar{z} \quad x \vee \bar{y} \vee z}{x \vee \bar{y}}$$

we can perform the cutting planes steps

$$\begin{array}{l} \text{Add} \quad \frac{\bar{y} + \bar{z} \geq 1 \quad x + \bar{y} + z \geq 1}{x + 2\bar{y} \geq 1} \\ \text{Divide by 2} \quad \frac{x + 2\bar{y} \geq 1}{x + \bar{y} \geq 1} \end{array}$$

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Given that the premises are clauses 7 and 5 in our example CNF formula, using references

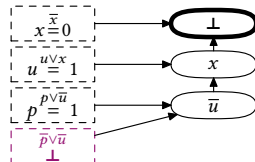
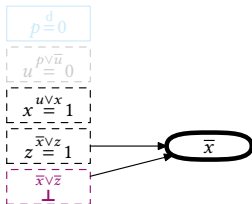
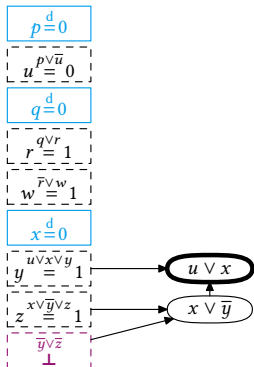
$$\text{Constraint 7} \doteq \bar{y} + \bar{z} \geq 1$$

$$\text{Constraint 5} \doteq x + \bar{y} + z \geq 1$$

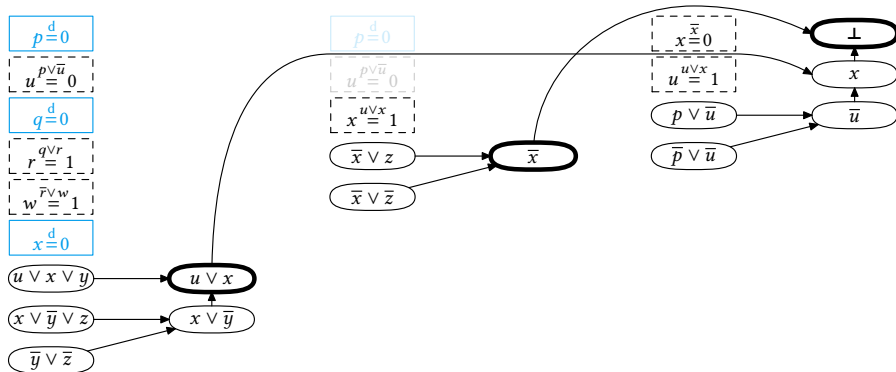
we can write this in the proof log as

pol 7 5 + 2 d ;

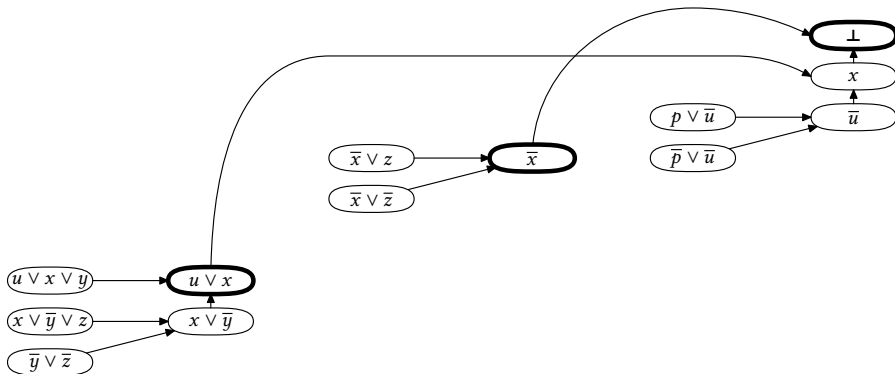
# Pseudo-Boolean Proof Logging for Example CDCL Conflict Analyses



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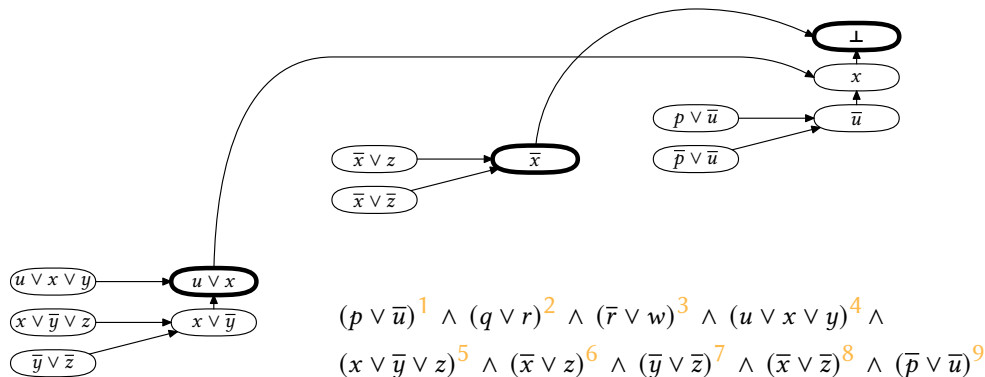


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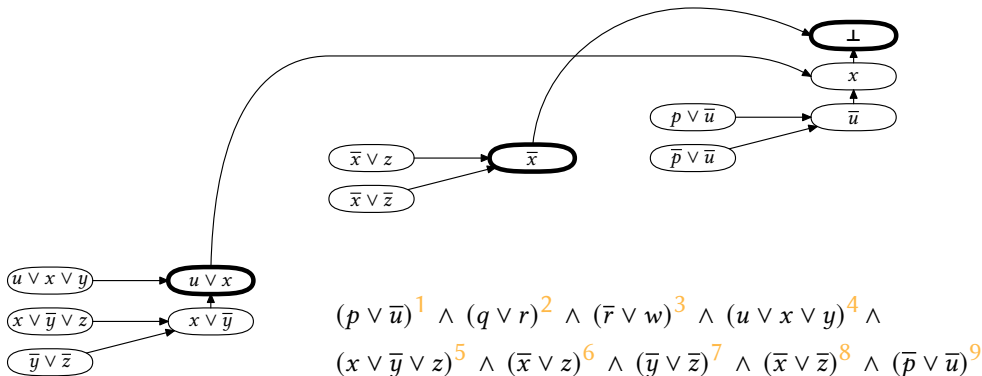




# Pseudo-Boolean Proof Logging for Example CDCL Conflict Analyses



# Pseudo-Boolean Proof Logging for Example CDCL Conflict Analyses



pol 7 5 + 2 d 4 + 2 d ;

pol 8 6 + 2 d ;

pol 9 1 + 2 d 10 + 2 d 11 + 2 d ;

$\rightsquigarrow$  Constraint 10  $\doteq u + x \geq 1$

$\rightsquigarrow$  Constraint 11  $\doteq \bar{x} \geq 1$

$\rightsquigarrow$  Constraint 12  $\doteq 0 \geq 1$  ⚡

# RUP Revisited

Can define (reverse) unit propagation in a pseudo-Boolean setting [EGMN20]

Constraint  $C$  propagates variable  $x$  if setting  $x$  to “wrong value” would make  $C$  unsatisfiable

E.g., if  $x_5$  is false,

$$x_1 + 2\bar{x}_2 + 3x_3 + 4\bar{x}_4 + 5x_5 \geq 7$$

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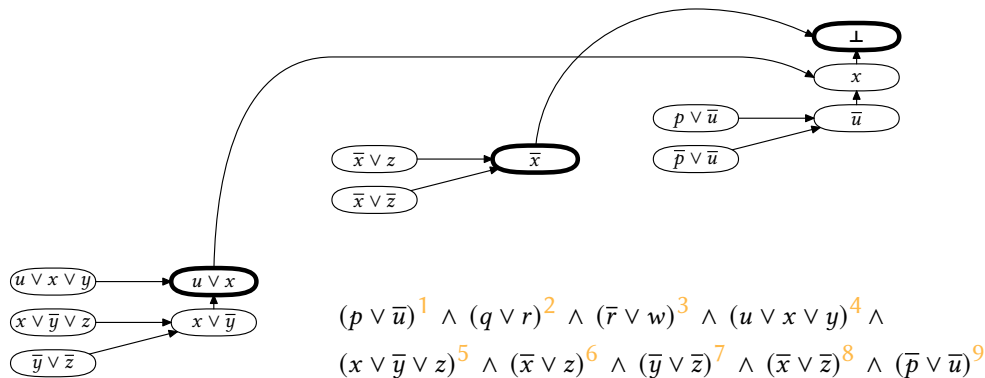
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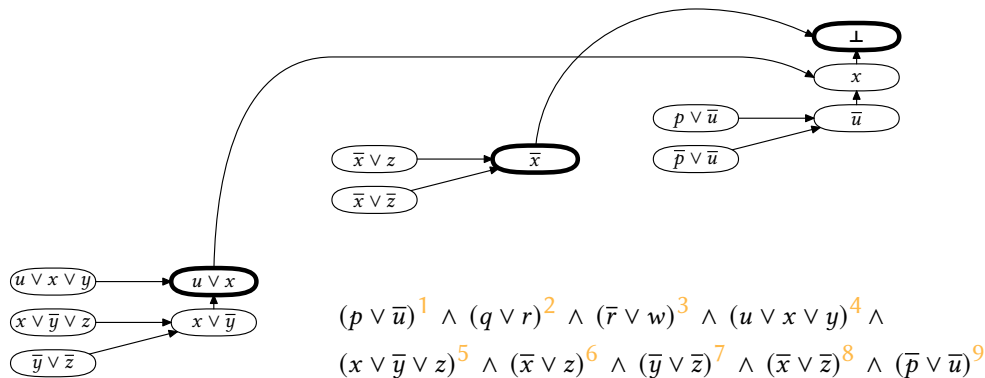
## Risk for confusion!

- Constraint programming people might call this (reverse) integer bounds consistency
  - Does the same thing if we're working with clauses
  - More interesting for general pseudo-Boolean constraints
- SAT people beware: constraints can propagate multiple times and multiple variables

# Pseudo-Boolean Proof Logging for Example CDCL Execution with RUP



# Pseudo-Boolean Proof Logging for Example CDCL Execution with RUP



rup 1 u 1 x >= 1 ;

rup 1 ~x >= 1 ;

rup >= 1 ;

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## Extension Variables, Part 2

Suppose we want new, fresh variable  $a$  encoding

$$a \Leftrightarrow (3x + 2y + z + w \geq 3)$$

This time, introduce constraints

$$3\bar{a} + 3x + 2y + z + w \geq 3 \qquad 5a + 3\bar{x} + 2\bar{y} + \bar{z} + \bar{w} \geq 5$$

Again, needs support from the proof system in the form of **strengthening rules**

# Proof Logs for “Cutting Planes with Strengthening”

For satisfiable instances: just specify a satisfying assignment.

For unsatisfiability: a sequence of **pseudo-Boolean constraints** in (slight extension of) **OPB format** [RM16]

- Each constraint follows “obviously” from what is known so far
- Either implicitly, by RUP...
- Or by an explicit cutting planes derivation...
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(\*) Not actually implemented this way — more details in a few slides ...

# Deleting Constraints

In practice, important to erase constraints to save memory and time during verification

Unsatisfiability proofs: fairly straightforward to deal with from point of view of proof logging

Optimisation proofs: significantly more delicate

We will mostly ignore deletions during this tutorial day for simplicity and clarity

# Enumeration and Optimisation Problems

## Enumeration:

- When a solution is found, can log it with solx rule
- Introduces a new constraint saying “not this solution”
- So the proof semantics is “infeasible, except for all the solutions I told you about”

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## Optimisation:

- Define an objective  $f = \sum_i w_i \ell_i$ ,  $w_i \in \mathbb{Z}$ , to minimise subject to the constraints in the formula
- To maximise, negate objective
- Log solution  $\alpha$  with soli rule  $\Rightarrow$  objective-improving constraint  $\sum_i w_i \ell_i \leq -1 + \sum_i w_i \alpha(\ell_i)$
- Semantics for proof of optimality: “infeasible to find better solution than best so far”
- Can also derive (potentially non-tight) lower bound  $\sum_i w_i \ell_i \geq LB$

# Pseudo-Boolean Proof Logging — How and Why?

If problem is (special case of) 0-1 integer linear program

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- 1 0-1 ILP **expressive formalism** for combinatorial problems (including objective)
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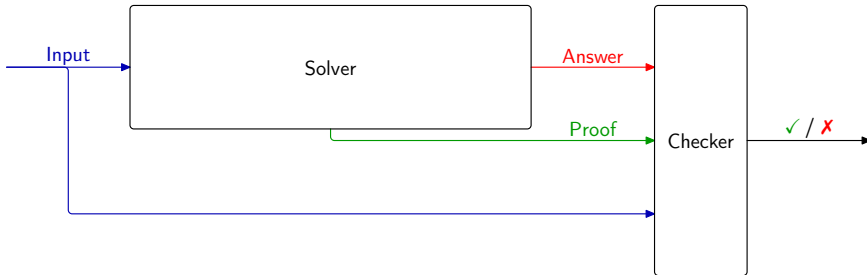
$$r \Rightarrow x_1 + 2\bar{x}_2 + 3x_3 + 4\bar{x}_4 + 5x_5 \geq 7$$

$$7\bar{r} + x_1 + 2\bar{x}_2 + 3x_3 + 4\bar{x}_4 + 5x_5 \geq 7$$

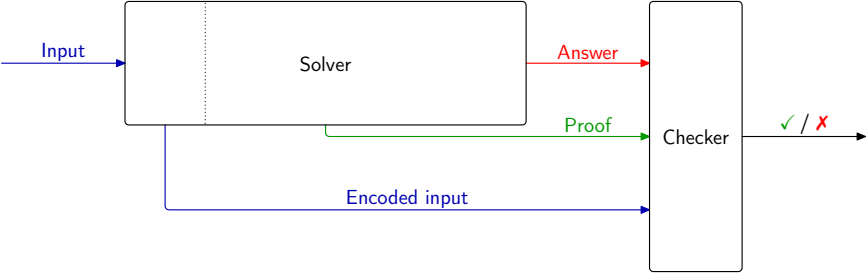
$$r \Leftarrow x_1 + 2\bar{x}_2 + 3x_3 + 4\bar{x}_4 + 5x_5 \geq 7$$

$$9r + \bar{x}_1 + 2x_2 + 3\bar{x}_3 + 4x_4 + 5\bar{x}_5 \geq 9$$

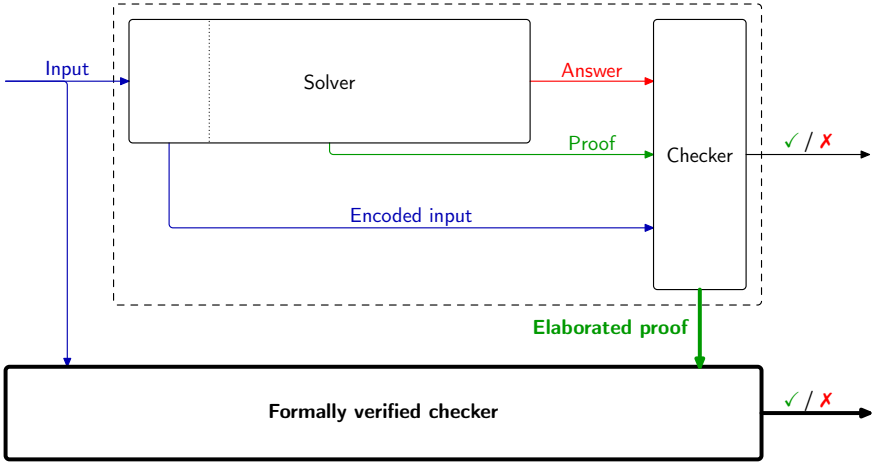
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we introduced pseudo-Boolean constraints

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Wish to allow **without-loss-of-generality** arguments that can derive non-implied constraints

# Redundance-Based Strengthening

$C$  is “**redundant**” with respect to  $F$  if  $F$  and  $F \cup \{C\}$  are **equisatisfiable**

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$C$  is redundant with respect to  $F$  if and only if there is a **substitution**  $\omega$  (mapping variables to truth values or literals), called a **witness**, for which

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Witness  $\omega$  should be specified, and implication needs to be **efficiently verifiable** — every  $D \in (F \cup \{C\}) \upharpoonright_{\omega}$  should follow from  $F \cup \{\neg C\}$  either

- “obviously” (e.g., by reverse unit propagation) or
- by explicitly presented derivation

# Deriving $a \Leftrightarrow (3x + 2y + z + w \geq 3)$ Using the Redundance Rule

Want to derive

$$3\bar{a} + 3x + 2y + z + w \geq 3 \quad 5a + 3\bar{x} + 2\bar{y} + \bar{z} + \bar{w} \geq 5$$

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$$1 \quad F \cup \{\neg(3\bar{a} + 3x + 2y + z + w \geq 3)\} \models (F \cup \{3\bar{a} + 3x + 2y + z + w \geq 3\}) \upharpoonright_{\omega}$$

Choose  $\omega = \{a \mapsto 0\}$  —  $F$  untouched; new constraint satisfied

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Choose  $\omega = \{a \mapsto 1\}$  —  $F$  untouched; new constraint satisfied

$\neg(5a + 3\bar{x} + 2\bar{y} + \bar{z} + \bar{w} \geq 5)$  forces  $3\bar{x} + 2\bar{y} + \bar{z} + \bar{w} \leq 4$

This is the same constraint as  $3x + 2y + z + w \geq 3$

And VERIPB can automatically detect this

# Redundance and Dominance Rules for Optimisation

## Redundance-based strengthening, optimisation version

Add constraint  $C$  to formula  $F$  if exists witness substitution  $\omega$  s.t.

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- Applying  $\omega$  should **strictly decrease**  $f$
- If so, don't need to show that  $C \upharpoonright_{\omega}$  holds!

# Soundness of Dominance Rule

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- 7 ...
- 8 Can't go on forever, so finally reach  $\alpha'$  satisfying  $F \wedge C$

# Strength of Dominance Rule

## Dominance-based strengthening (stronger, still simplified)

If  $C_1, C_2, \dots, C_{m-1}$  have been derived from  $F$  (maybe using dominance), then can derive  $C_m$  if exists witness substitution  $\omega$  s.t.

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Further extensions:

- Define dominance rule w.r.t. order independent of objective
- Switch between different orders in same proof
- See [BGMN23] for details

# Strengthening Rules: Proof Format

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red ⟨Constraint C⟩ : ⟨var1⟩ -> ⟨val1⟩ . . . ⟨varN⟩ -> ⟨valN⟩ : subproof
  subproofs for proof goals
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- Witness  $\omega$  should be explicitly specified in proof log
- Subproofs of proof goals should also be explicit
- Except can be skipped for proof goals that “obviously” follow, e.g.,
  - by reverse unit propagation (RUP)
  - by simple syntactic implication from other constraint
  - since the proof goal is not affected by the witness substitution  $\omega$

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- 1 **Boolean satisfiability (SAT) solving** including advanced techniques such as
  - Gaussian elimination [GN21]
  - symmetry breaking [BGMN23]
- 2 **SAT-based optimization (MaxSAT)** [VDB22, BBN<sup>+</sup>23, BBN<sup>+</sup>24, IOT<sup>+</sup>24]
- 3 (Linear) **Pseudo-Boolean solving** [GMNO22, KLM<sup>+</sup>25]
- 4 **Subgraph solving** (max clique, subgraph isomorphism, max common connected subgraph) [GMN20, GMM<sup>+</sup>20, GMM<sup>+</sup>24]
- 5 **Dynamic programming** and **decision diagrams** [DMM<sup>+</sup>24]
- 6 **Presolving** in 0–1 integer linear programming [HOGN24]
- 7 **Constraint programming** [EGMN20, GMN22, MM23, MMN24, MM25]
- 8 **Automated planning** [DHN<sup>+</sup>25]

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Specific details on different proof logging techniques covered in research papers

[EGMN20, GMN20, GMM<sup>+</sup>20, GN21, GMN22, GMNO22, VDB22, BBN<sup>+</sup>23, BGMN23, MM23, BBN<sup>+</sup>24, DMM<sup>+</sup>24, GMM<sup>+</sup>24, HOGN24, IOT<sup>+</sup>24, MMN24, DHN<sup>+</sup>25, JBBJ25, KLM<sup>+</sup>25, MM25]

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Lots of concrete example files at [gitlab.com/MIA0research/software/VeriPB](https://gitlab.com/MIA0research/software/VeriPB)

# Future Research Directions

## Performance and reliability of pseudo-Boolean proof logging and checking

- Trim proof while checking (as in *DRAT-Trim* [HHW13a])
- Compress proof file using binary format
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- Model counting
- Mixed integer linear programming (*suggested extension of VERIPB in [DEGH23]*)
- Satisfiability modulo theories (SMT) solving (*work on solvers cvc5, SMTInterpol, Z3, ... [BBC<sup>+</sup>23, HS22]*)

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- Lots of other challenging problems and interesting ideas
- Talk to us if you want to join the pseudo-Boolean proof logging revolution! ☺  
We're happy to **collaborate**, and **we're hiring**

# Summing up

- Combinatorial solving and optimisation is a true success story
- But ensuring correctness is a crucial, and not yet satisfactorily addressed, concern
- Certifying solvers producing machine-verifiable proofs of correctness seems like most promising approach
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