

Faster Certified Symmetry Breaking Using Orders With Auxiliary Variables (Extended Version Including Appendix)

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Abstract

Symmetry breaking is a crucial technique in modern combinatorial solving, but it is difficult to be sure it is implemented correctly. The most successful approach to deal with bugs is to make solvers *certifying*, so that they output not just a solution, but also a mathematical proof of correctness in a standard format, which can then be checked by a formally verified checker. This requires justifying symmetry reasoning within the proof, but developing efficient methods for this has remained a long-standing open challenge. A fully general approach was recently proposed by Bogaerts et al. (2023), but it relies on encoding lexicographic orders with big integers, which quickly becomes infeasible for large symmetries. In this work, we develop a method for instead encoding orders with auxiliary variables. We show that this leads to orders-of-magnitude speed-ups in both theory and practice by running experiments on proof logging and checking for SAT symmetry breaking using the state-of-the-art SATSUMA symmetry breaker and the VERIPB proof checking toolchain.

1 Introduction

An important challenge in combinatorial solving is to avoid repeatedly exploring different parts of the search space that are equivalent under symmetries. In a wide range of combinatorial solving paradigms, *symmetry breaking* is deployed as a default technique, including mixed integer programming (Achterberg and Wunderling 2013; Bolusani et al. 2024) and constraint programming (Walsh 2006). The importance of symmetry breaking is supported by both theoretical considerations (Urquhart 1999) and experimental results (Pfetsch and Rehn 2019). A detailed discussion of symmetry breaking is given, e.g., by Sakallah (2021).

Symmetry breaking has not been adopted as mainstream in Boolean satisfiability (SAT) solving, however, despite a body of work (Aloul, Markov, and Sakallah 2003; Devriendt et al. 2016; Anders, Brenner, and Rattan 2024) showing the potential for speed-ups also in this setting. One reason for this could perhaps be the higher cost, relatively speaking, of symmetry breaking compared to low-level SAT reasoning, but state-of-the-art symmetry detection is efficient enough to use by default without degrading performance (Anders, Brenner, and Rattan 2024). A more important concern is that the SAT community places a strong emphasis on provable correctness. For over a decade, SAT solvers taking part in the annual SAT competitions have had to generate machine-verifiable proofs for their results. Such proofs are especially important for sophisticated techniques such as symmetry breaking, which is notoriously difficult to implement correctly. However, except for some special cases (Heule, Hunt Jr., and Wetzler 2015), it has not been known how to generate proofs for symmetry breaking in the DRAT proof format (Wetzler, Heule, and Hunt Jr. 2014) used in the competitions, or whether this is even possible.

The way symmetry breaking is typically done in SAT solving is by introducing *lex-leader* constraints, which are encoded as the clauses

$$\begin{array}{lll} s_1 \vee \bar{x}_1 & s_1 \vee y_1 & y_1 \vee \bar{x}_1 \\ s_{i+1} \vee \bar{s}_i \vee \bar{x}_{i+1} & s_{i+1} \vee \bar{s}_i \vee y_{i+1} & \bar{s}_i \vee y_{i+1} \vee \bar{x}_{i+1} \end{array} \quad (1)$$

that can be thought of as encoding a circuit enforcing $(x_1, \dots, x_n) \preceq_{\text{lex}} (y_1, \dots, y_n)$ —here, s_i are fresh auxiliary variables encoding that the x - and y -variables are equal up to position i ; using these, we enforce that x_i is false and y_i true the first time this does not hold. Such clauses are

clearly not implied by the original formula, and the problem is how to prove that they can be added without changing the satisfiability of the input. Although the RAT rule (Järvisalo, Heule, and Biere 2012) in DRAT can handle a single symmetry (Kołodziejczyk and Thapen 2024), once the first symmetry is broken it is not known how or even if the other symmetries found by the symmetry breaker could be proven correct using DRAT.

Bogaerts et al. (2023) finally resolved this long-standing open problem by introducing a stronger proof format, which operates with *pseudo-Boolean* (i.e., 0–1 linear) inequalities rather than clauses, and reasons in terms of *dominance* (Chu and Stuckey 2015) to support fully general symmetry breaking without any limitations on the number of symmetries that can be handled. One benefit of this richer format is that a single inequality

$$2^{n-1}x_1 + \dots + 2x_{n-1} + x_n \leq 2^{n-1}y_1 + \dots + 2y_{n-1} + y_n, \quad (2)$$

can be used in the proof to encode lexicographic order, and from this constraint it is straightforward to derive the clauses (1) used by the solver. However, at least n^2 bits are needed to represent the coefficients in (2), while the representation of (1) scales linearly with n . This means that proof generation incurs a linear overhead compared to solving. Also, the algorithm by Anders, Brenner, and Rattan (2024) can break a symmetry in quasi-linear time measured in the number of variables k remapped by the symmetry, which introduces yet another asymptotic slowdown in proof generation if $k \ll n$. Furthermore, the exponentially growing integer coefficients in (2) require expensive arbitrary-precision arithmetic, which slows down proof checking. All of these problems combine to make the proof logging approach proposed by Bogaerts et al. (2023) infeasible for large-scale problems requiring non-trivial symmetry breaking.

In this work, we present an asymptotically faster method for generating and checking proofs of correctness for symmetry breaking. The main new technical idea is to use *auxiliary variables* to encode the lexicographic order used for the dominance reasoning, similar to the clausal encoding in (1). Unfortunately, this breaks the fundamental invariant of Bogaerts et al. (2023) that all low-level proofs should be implicational. When one needs to prove that a symmetry-breaking constraint respects lexicographical order, the encoding of this order will contain auxiliary variables that are not mentioned in the premises, and so this property cannot possibly be implied. We therefore need to make a substantial redesign of the proof system of Bogaerts et al. (2023) to work with auxiliary variables. Very briefly, our key technical twist is to split the encoding of the order into two parts, putting one part into the premises, so that the property of implicational low-level proofs can be maintained. Our redesigned proof system supports fully general symmetry breaking in a similar fashion to Bogaerts et al. (2023), but is significantly more efficient. Specifically, we prove that our approach leads to asymptotic gains for proof logging and checking for symmetry breaking by at least a linear factor in the size n of the lexicographic order used.

We have implemented support for our new proof system in the proof checker VERIPB (Bogaerts et al. 2023; Gocht

and Nordström 2021; Gocht 2022) with its formally verified backend CAKEPB (Gocht et al. 2024). Together, these yield an efficient, end-to-end verified proof checking toolchain for symmetry breaking proofs. We have also enhanced the state-of-the-art SAT symmetry breaker SATSUMA (Anders, Brenner, and Rattan 2024) to generate proofs of correctness in our new format as well as that of Bogaerts et al. (2023) for a comparative evaluation of performance. Our experimental findings match our theoretical results and show that only a constant overhead in running time is required for proof logging with our new method. Proof checking performance is also vastly better compared to Bogaerts et al. (2023), although here there might be room for further improvements.

Our paper is organized as follows. After reviewing preliminaries in Section 2, we present our new proof logging system in Section 3. Sections 4 and 5 discuss how proof logging and checking can be improved asymptotically using our new method, which is confirmed by our experiments in Section 6. We conclude with a brief discussion of future work in Section 7. In this full-length version, we also provide five appendices. Appendix A gives an overview of the cutting planes proof system and the syntax used by VERIPB for implicational reasoning. Appendix B provides further details on the extended proof system introduced in Sections 3.1 up to 3.5, including all proofs. Appendix C provides all details on how the proof logging is implemented in SATSUMA. Appendix D presents a concrete toy example of a VERIPB proof generated by SATSUMA. Appendix E contains an overview of the crafted benchmarks that we used in our experimental evaluation.

2 Preliminaries

We start with a brief review of pseudo-Boolean reasoning. For more details, we refer the reader to, e.g., Buss and Nordström (2021) or Bogaerts et al. (2023). A *Boolean variable* takes values 0 or 1. A *literal* over a Boolean variable x is x itself or its negation $\bar{x} = 1 - x$. A *pseudo-Boolean (PB) constraint* C is an integer linear inequality over literals

$$C \doteq \sum_i a_i \ell_i \geq A, \quad (3)$$

where we use \doteq to denote syntactic equivalence. Without loss of generality the coefficients a_i and the right-hand side A are non-negative and the literals ℓ_i are over distinct variables. The *trivially false constraint* is $\perp \doteq 0 \geq 1$. The *negation* $\neg C$ of the pseudo-Boolean constraint C in (3) is the pseudo-Boolean constraint $\neg C \doteq \sum_i a_i \bar{\ell}_i \geq \sum_i a_i - A + 1$. A *pseudo-Boolean formula* F is a conjunction $F \doteq \bigwedge_i C_i$ or equivalently a set $F \doteq \bigcup_i \{C_i\}$ of pseudo-Boolean constraints C_i , whichever view is more convenient. A *(disjunctive) clause* $\bigvee_i \ell_i$ is equivalent to the pseudo-Boolean constraint $\sum_i \ell_i \geq 1$. Hence, formulas in *conjunctive normal form (CNF)* are special cases of pseudo-Boolean formulas.

An *assignment* is a function mapping from Boolean variables to $\{0, 1\}$. *Substitutions* (or *witnesses*) generalize assignments by allowing variables to be mapped to literals, too. A substitution ω is extended to literals by $\omega(\bar{x}) = \overline{\omega(x)}$, and to preserve truth values, i.e., $\omega(0) = 0$ and $\omega(1) = 1$. For a substitution ω , the support $\text{supp}(\omega)$ is the set of vari-

ables x where $\omega(x) \neq x$. A substitution α can be composed with another substitution ω by applying ω first and then α , i.e., $(\alpha \circ \omega)(x) = \alpha(\omega(x))$. Applying a substitution ω to the pseudo-Boolean constraint C in (3) yields the pseudo-Boolean constraint $C \upharpoonright_\omega \doteq \sum_i a_i \omega(\ell_i) \geq A$. This is extended to formulas by defining $F \upharpoonright_\omega \doteq \bigwedge_i C_i \upharpoonright_\omega$. The pseudo-Boolean constraint C is satisfied by an assignment ω if $\sum_{i: \omega(\ell_i)=1} a_i \geq A$. A pseudo-Boolean formula F is satisfied by ω if ω satisfies every constraint in F . If there is no assignment that satisfies F , then F is *unsatisfiable*.

We use the notation $F(\vec{x})$ to stress that the formula is defined over the list of variables $\vec{x} = x_1, \dots, x_n$, where we syntactically highlight a partitioning of the list of variables by writing $F(\vec{y}, \vec{z})$ or $F(\vec{a}, \vec{b}, \vec{c})$ meaning $\vec{x} = \vec{y}, \vec{z}$ or $\vec{x} = \vec{a}, \vec{b}, \vec{c}$, respectively (denoting concatenation of the lists of variables). To apply a substitution ω element-wise to a list of literals we write $\vec{\ell} \upharpoonright_\omega = \omega(\ell_1), \dots, \omega(\ell_n)$. For a formula $F(\vec{x})$ and a list of literals and truth values $\vec{y} = y_1, \dots, y_n$, the notation $F(\vec{y})$ is syntactic sugar for $F \upharpoonright_\omega$ with the implicitly defined substitution $\omega(x_i) = y_i$ for $i = 1, \dots, n$. Finally, we write $\text{var}(F)$ for the set of variables in a formula F .

2.1 The VERIPB Proof System

The proof system introduced by Bogaerts et al. (2023) (which we will refer to as the *original system*) can prove optimal values for *optimization problems* (F, f) , where F is a pseudo-Boolean formula, and f is an integer linear *objective* function over literals to be minimized subject to satisfying F . The *satisfiability (SAT) problem* is a special case by having $f = 0$ and F being a CNF formula. Proving the unsatisfiability of F then corresponds to proving that ∞ is a lower bound for (F, f) . For clarity of exposition, we focus on decision problems, i.e., problems with objective function $f = 0$, but the results can easily be extended to optimization problems as in Bogaerts et al. (2023).

A proof in this proof system consists of a sequence of rule applications, each deriving a new constraint. For implicational reasoning, the *cutting planes* proof system (Cook, Coullard, and Turán 1987) is used, which provides sound reasoning rules to derive pseudo-Boolean constraints implied by a pseudo-Boolean formula F , e.g., taking positive integer linear combinations or dividing by an integer and rounding up. We write $F \vdash C$ if there is a cutting planes proof deriving C from F . A set of constraints F' is derivable from another set F , denoted by $F \vdash F'$, if $F \vdash C$ for all $C \in F'$.

The proof system also has rules for deriving constraints which are not implied. To do this, the original system keeps track of two pseudo-Boolean formulas (i.e., sets of constraints), called *core* \mathcal{C} and *derived* \mathcal{D} , which Järvisalo, Heule, and Biere (2012) call *irredundant* and *redundant* clauses, respectively. In addition, we need a pseudo-Boolean formula $\mathcal{O}_{\preceq}(\vec{u}, \vec{v})$, encoding a preorder, i.e., a reflexive and transitive relation. This preorder is used to compare assignments α, β over the literals in a list \vec{z} and we write $\alpha \preceq \beta$ if $\mathcal{O}_{\preceq}(\vec{z} \upharpoonright_\alpha, \vec{z} \upharpoonright_\beta)$ evaluates to true. For a preorder \preceq , we define the strict order \prec such that $\alpha \prec \beta$ holds if $\alpha \preceq \beta$ and $\beta \not\preceq \alpha$.

We call the tuple $(\mathcal{C}, \mathcal{D}, \mathcal{O}_{\preceq}, \vec{z})$ a *configuration*. Formally, proof rules incrementally modify the configuration. To han-

dle optimization problems, the configuration of Bogaerts et al. (2023) also contains the current upper bound on f , which we can omit for decision problems.

The proof system maintains two invariants: (1) \mathcal{C} is satisfiable if F is satisfiable, and (2) for any assignments α satisfying \mathcal{C} there exists an assignment α' satisfying \mathcal{C}, \mathcal{D} , and $\alpha' \preceq \alpha$. Starting with the configuration $(F, \emptyset, \emptyset, \emptyset)$, any valid derivation of a configuration $(\mathcal{C}, \mathcal{D}, \mathcal{O}_{\preceq}, \vec{z})$ with $\perp \in \mathcal{C} \cup \mathcal{D}$ proves that F is unsatisfiable.

Proof Rules. We list the satisfiability version of the proof rules from Bogaerts et al. (2023) our work modifies; all other rules in the original proof system remain unchanged.

- The *redundance-based strengthening rule* (or *redundance rule* for short) allows transitioning from $(\mathcal{C}, \mathcal{D}, \mathcal{O}_{\preceq}, \vec{z})$ to $(\mathcal{C}, \mathcal{D} \cup \{C\}, \mathcal{O}_{\preceq}, \vec{z})$ if a substitution ω and cutting planes proofs are provided showing that

$$\mathcal{C} \cup \mathcal{D} \cup \{\neg C\} \vdash (\mathcal{C} \cup \mathcal{D} \cup \{C\}) \upharpoonright_\omega \cup \mathcal{O}_{\preceq}(\vec{z} \upharpoonright_\omega, \vec{z}). \quad (4)$$

- The *dominance-based strengthening rule* (or *dominance rule* for short) allows transitioning from $(\mathcal{C}, \mathcal{D}, \mathcal{O}_{\preceq}, \vec{z})$ to $(\mathcal{C}, \mathcal{D} \cup \{C\}, \mathcal{O}_{\preceq}, \vec{z})$ if a substitution ω and cutting planes proofs are provided showing that

$$\mathcal{C} \cup \mathcal{D} \cup \{\neg C\} \vdash \mathcal{C} \upharpoonright_\omega \cup \mathcal{O}_{\preceq}(\vec{z} \upharpoonright_\omega, \vec{z}) \quad (5)$$

$$\mathcal{C} \cup \mathcal{D} \cup \{\neg C\} \cup \mathcal{O}_{\preceq}(\vec{z}, \vec{z} \upharpoonright_\omega) \vdash \perp. \quad (6)$$

We now briefly explain why the redundance rule preserves the second invariant. Let α be an assignment satisfying \mathcal{C} . Since the invariant holds for $(\mathcal{C}, \mathcal{D}, \mathcal{O}_{\preceq}, \vec{z})$, there exists an assignment α' satisfying $\mathcal{C} \cup \mathcal{D}$ and $\alpha' \preceq \alpha$. If α' happens to satisfy C , we are done. Otherwise, the derivation (4) guarantees that $\alpha' \circ \omega$ satisfies $\mathcal{C} \cup \mathcal{D} \cup \{C\}$ and $\mathcal{O}_{\preceq}(\vec{z} \upharpoonright_{\alpha' \circ \omega}, \vec{z} \upharpoonright_{\alpha'})$, i.e., $\alpha' \circ \omega \preceq \alpha'$. By transitivity we get $\alpha' \circ \omega \preceq \alpha$. For the dominance rule, α' might have to be composed with ω repeatedly, but the process is guaranteed to eventually satisfy C , since the composed assignment strictly decreases with respect to the order, which is encoded by (6).

Preorders. Before using a preorder \mathcal{O}_{\preceq} , it needs to be proven within the proof system that \mathcal{O}_{\preceq} is indeed reflexive and transitive. For this, the original system requires cutting planes proofs for $\emptyset \vdash \mathcal{O}_{\preceq}(\vec{u}, \vec{u})$ and $\mathcal{O}_{\preceq}(\vec{u}, \vec{v}) \cup \mathcal{O}_{\preceq}(\vec{v}, \vec{w}) \vdash \mathcal{O}_{\preceq}(\vec{u}, \vec{w})$, where \vec{w} is of the same size as \vec{u} and \vec{v} .

2.2 Symmetry Breaking

We briefly review symmetry breaking as it is used in practice, which we want to certify. Typically, symmetry breaking considers permutations σ between literals with $\sigma(\ell) = \overline{\sigma(\ell)}$ for all literals ℓ , and finite support $\text{supp}(\sigma)$. Practical symmetry breaking algorithms only detect *syntactic* symmetries of a formula F , i.e., permutations σ with $F \upharpoonright_\sigma \doteq F$.

To encode an ordering of assignments, typically the *lex-leader* constraint is used. Let S be a set of detected symmetries in a formula F , and z_1, \dots, z_n be variables with $\text{supp}(\sigma) \subseteq \{z_1, \dots, z_n\}$ for all $\sigma \in S$. Then we define the *lexicographic order* \preceq_{lex} over assignments α, β and the *lex-leader constraint* B_σ for a symmetry $\sigma \in S$ as

$$\alpha \preceq_{\text{lex}} \beta \text{ iff } \sum_{i=1}^n 2^{n-i} \alpha(z_i) \leq \sum_{i=1}^n 2^{n-i} \beta(z_i) \quad (7)$$

$$B_\sigma \doteq \sum_{i=1}^n 2^{n-i} (\sigma(x_i) - x_i) \geq 0. \quad (8)$$

Intuitively, (8) constrains assignments to be smaller w.r.t. the preorder in (7) than their symmetric counterpart. Symmetry breaking introduces the constraints B_σ for $\sigma \in S$ such that if F is satisfiable, then $F \cup \bigcup_{\sigma \in S} B_\sigma$ is satisfiable.

When doing proof logging for symmetry breaking, the dominance rule in the original system can derive the lex-leader constraints B_σ as follows. Suppose that the symmetry breaker detect symmetries $\sigma_1, \dots, \sigma_m$ of \mathcal{C} . To log these symmetries, we use the order defined by

$$\mathcal{O}_{\preceq_{\text{lex}}}(\vec{x}, \vec{y}) = \left\{ \sum_{i=1}^n 2^{n-i} (y_i - x_i) \geq 0 \right\}. \quad (9)$$

To add a constraint B_{σ_i} to \mathcal{D} , we use the dominance rule with witness σ_i . The application of this rule is justified by $\mathcal{C} \vdash \mathcal{C} \upharpoonright_{\sigma_i}$ (trivial, because σ_i is a symmetry of \mathcal{C}), and the fact that $\neg B_{\sigma_i}$ implies both $\mathcal{O}_{\preceq_{\text{lex}}}(\vec{z} \upharpoonright_{\sigma_i}, \vec{z})$ and $\neg \mathcal{O}_{\preceq_{\text{lex}}}(\vec{z}, \vec{z} \upharpoonright_{\sigma_i})$.

When using symmetry breaking for the SAT problem, the symmetry breaker instead encodes $\vec{x} \preceq_{\text{lex}} \sigma(\vec{x})$ as the clauses

$$s_1 + \bar{x}_1 \geq 1, \quad s_{i+1} + \bar{s}_i + \bar{x}_{i+1} \geq 1, \quad (10a)$$

$$s_1 + \sigma(x_1) \geq 1, \quad s_{i+1} + \bar{s}_i + \sigma(x_{i+1}) \geq 1, \quad (10b)$$

$$\sigma(x_1) + \bar{x}_1 \geq 1, \quad \bar{s}_i + \sigma(x_{i+1}) + \bar{x}_{i+1} \geq 1, \quad (10c)$$

where s_i encodes that $(x_1, \dots, x_i) = (\sigma(x_1), \dots, \sigma(x_i))$. These clauses can be derived from (8) using redundancy.

3 Strengthening with Auxiliary Variables

While the method presented in Section 2.2 enables proof logging for symmetry breaking, encoding the coefficients in (8) and (9) grows quadratically in the size of \vec{z} , which often includes all variables in the formula, making proof logging for large symmetries infeasible in practice. For proof checking, the situation is even more dire, as the proof checker has to reason internally with arbitrary-precision integer arithmetic to handle the coefficients in (8) and (9).

One way to avoid these big integers would be to represent the order as a set of clauses as in Equation (10a)–(10c), using a list of extension variables \vec{s} . However, this leads to challenges when defining the actual preorder \preceq . For an order without extension variables, we define $\alpha \preceq \beta$ to hold if $\mathcal{O}_{\preceq}(\vec{z} \upharpoonright_\alpha, \vec{z} \upharpoonright_\beta)$ is true. However, for a formula $\mathcal{O}_{\preceq}(\vec{x}, \vec{y}, \vec{s})$ containing extension variables \vec{s} this does not work, since the variables \vec{s} in $\mathcal{O}_{\preceq}(\vec{z} \upharpoonright_\alpha, \vec{z} \upharpoonright_\beta, \vec{s})$ are unassigned and in general $\mathcal{O}_{\preceq}(\vec{z} \upharpoonright_\alpha, \vec{z} \upharpoonright_\beta, \vec{s})$ will not hold for all assignments to \vec{s} .

Instead, what we are trying to capture is that $\alpha \preceq \beta$ holds precisely when $\mathcal{O}_{\preceq}(\vec{z} \upharpoonright_\alpha, \vec{z} \upharpoonright_\beta, \vec{s})$ holds, *provided that* the extension variables \vec{s} are set in the right way. Equivalently, we want to say that $\alpha \preceq \beta$ holds precisely when there exists an assignment ρ to \vec{s} such that $\mathcal{O}_{\preceq}(\vec{z} \upharpoonright_\alpha, \vec{z} \upharpoonright_\beta, \vec{s} \upharpoonright_\rho)$ holds.

However, just adding extension variables to the proof obligations $\mathcal{O}_{\preceq}(\vec{z} \upharpoonright_\omega, \vec{z})$ in the redundancy and dominance rules would not work. What we would need to show is that *some* assignment to \vec{s} exists such that $\mathcal{O}_{\preceq}(\vec{z} \upharpoonright_\omega, \vec{z}, \vec{s})$ holds, but the proof system cannot express existential quantification. While the proof rules could specify the value of all extension variables \vec{s} , this would be very cumbersome. However, in all applications we have in mind, the preorder with extension variables already contains the information how to set the extension variables \vec{s} , since the extension variables are *defined* (functionally) in terms of the other variables.

To make this precise, let $\mathcal{S}_{\preceq}(\vec{x}, \vec{y}, \vec{s})$ be a *definition* of \vec{s} in terms of the other variables (i.e., each assignment to the \vec{x} and \vec{y} can uniquely be extended to an assignment to \vec{s} that satisfies $\mathcal{S}_{\preceq}(\vec{x}, \vec{y}, \vec{s})$). We now redefine \preceq such that $\alpha \preceq \beta$ holds precisely when there exists an assignment ρ to \vec{s} such that $\mathcal{S}_{\preceq}(\vec{z} \upharpoonright_\alpha, \vec{z} \upharpoonright_\beta, \vec{s} \upharpoonright_\rho) \wedge \mathcal{O}_{\preceq}(\vec{z} \upharpoonright_\alpha, \vec{z} \upharpoonright_\beta, \vec{s} \upharpoonright_\rho)$ holds.

In this case, whenever we need to show that $\alpha \preceq \beta$, because of the definitional nature of \mathcal{S}_{\preceq} we can *assume* that $\mathcal{S}_{\preceq}(\vec{z} \upharpoonright_\alpha, \vec{z} \upharpoonright_\beta, \vec{s} \upharpoonright_\rho)$ holds and derive $\mathcal{O}_{\preceq}(\vec{z} \upharpoonright_\alpha, \vec{z} \upharpoonright_\beta, \vec{s} \upharpoonright_\rho)$ from this, thereby completely eliminating the need for providing an assignment to \vec{s} in every rule application. In our actual proof system, we relax the condition that \mathcal{S}_{\preceq} is definitional slightly, but intuitively, \mathcal{S}_{\preceq} is best thought of as a circuit defining the value of \vec{s} in terms of the other variables.

An important restriction for this to be sound is that the extension variables \vec{s} in the preorder, which we call *auxiliary variables*, do not appear outside the preorder.

We now formalize this. As mentioned in Section 2.1, we focus on decision problems, and refer to Appendix B for the extension to optimization problems and proofs of all results.

3.1 Specifications

Let \vec{a} be a list of variables. A pseudo-Boolean formula $\mathcal{S}(\vec{x}, \vec{a})$ is a *specification over the variables \vec{a}* , if it is derivable from the empty formula \emptyset by the redundancy rule, where each application only witnesses over variables in \vec{a} .

Definition 1. A formula $\mathcal{S}(\vec{x}, \vec{a}) = \{C_1, C_2, \dots, C_n\}$ is a *specification over the variables \vec{a}* , if there is a list

$$(C_1, \omega_1), (C_2, \omega_2), \dots, (C_n, \omega_n)$$

which satisfies the following:

1. The constraint C_1 can be obtained from the empty formula \emptyset using the redundancy rule with witness ω_1 .
2. For each $i \in \{2, \dots, n\}$ we have that C_i can be added by the redundancy rule to $\bigcup_{j=1}^{i-1} \{C_j\}$ with the witness ω_i .
3. For every witness ω_i , $\text{supp}(\omega_i) \subseteq \vec{a}$ holds.

A crucial property of specifications is that we can recover an assignment of the auxiliary variables from the assignment of the non-auxiliary variables. We state this property below.

Lemma 1. Let $\mathcal{S}(\vec{x}, \vec{a})$ be a specification over \vec{a} . Let α be any assignment of the variables \vec{x} . Then, α can be extended to an assignment α' , such that

1. α' satisfies \mathcal{S} , and
2. $\alpha(x) = \alpha'(x)$ holds for every $x \in \vec{x}$.

To explain why Lemma 1 holds, recall from Section 2.1 that the redundancy rule satisfies the following: if a constraint C is added by redundancy with witness ω , then given an assignment α satisfying all other constraints, either α or $\alpha \circ \omega$ also satisfies C . Hence, defining α' by composing α with the witnesses ω_i corresponding to the constraints that do not already hold ensures that α' satisfies \mathcal{S} , and the fact that each witness ω_i is the identity on \vec{x} (since $\text{supp}(\omega_i) \subseteq \vec{a}$) ensures that $\alpha(x) = \alpha'(x)$ for $x \in \vec{x}$.

3.2 Orders with Auxiliary Variables

Next, we explain how two pseudo-Boolean formulas $\mathcal{O}_{\preceq}(\vec{u}, \vec{v}, \vec{a})$ and $\mathcal{S}_{\preceq}(\vec{u}, \vec{v}, \vec{a})$, together with the two disjoint lists of variables \vec{z} and \vec{a} , define a preorder.

Definition 2. Let \vec{u} and \vec{v} be disjoint lists of variables of size n and let \vec{a} be a list of auxiliary variables. Let $\mathcal{O}_{\preceq}(\vec{u}, \vec{v}, \vec{a})$ and $\mathcal{S}_{\preceq}(\vec{u}, \vec{v}, \vec{a})$ be two pseudo-Boolean formulas such that \mathcal{S}_{\preceq} is a specification over \vec{a} .

Then we define the relation \preceq over the domain of total assignments to a list of variables \vec{z} of size n as follows: For assignments α, β we let $\alpha \preceq \beta$ hold, if and only if there exists an assignment ρ to the variables \vec{a} , such that

$$\mathcal{S}_{\preceq}(\vec{z}|_{\alpha}, \vec{z}|_{\beta}, \vec{a}|_{\rho}) \wedge \mathcal{O}_{\preceq}(\vec{z}|_{\alpha}, \vec{z}|_{\beta}, \vec{a}|_{\rho})$$

evaluates to true.

To ensure that \mathcal{O}_{\preceq} and \mathcal{S}_{\preceq} actually define a preorder, we require cutting planes proofs that show reflexivity, i.e., $\emptyset \vdash \alpha \preceq \alpha$, and transitivity, i.e., $\alpha \preceq \beta \wedge \beta \preceq \gamma \vdash \alpha \preceq \gamma$. To write these proof obligations using the cutting planes proof system, which cannot handle an existentially quantified conclusion, we can use the specification as a premise. The specification premise essentially tells us which auxiliary variables the existential quantifier should pick. In particular, for reflexivity, the proof obligation is

$$\mathcal{S}_{\preceq}(\vec{x}, \vec{x}, \vec{a}) \vdash \mathcal{O}_{\preceq}(\vec{x}, \vec{x}, \vec{a}). \quad (11)$$

For transitivity, the proof obligation is

$$\begin{aligned} &\mathcal{S}_{\preceq}(\vec{x}, \vec{y}, \vec{a}) \cup \mathcal{O}_{\preceq}(\vec{x}, \vec{y}, \vec{a}) \cup \mathcal{S}_{\preceq}(\vec{y}, \vec{z}, \vec{b}) \\ &\cup \mathcal{O}_{\preceq}(\vec{y}, \vec{z}, \vec{b}) \cup \mathcal{S}_{\preceq}(\vec{x}, \vec{z}, \vec{c}) \vdash \mathcal{O}_{\preceq}(\vec{x}, \vec{z}, \vec{c}). \end{aligned} \quad (12)$$

Intuitively, (12) says that if the circuits defining the auxiliary variables are correctly evaluated, which is encoded by the premises $\mathcal{S}_{\preceq}(\vec{x}, \vec{y}, \vec{a}) \cup \mathcal{O}_{\preceq}(\vec{x}, \vec{y}, \vec{a}) \cup \mathcal{S}_{\preceq}(\vec{x}, \vec{z}, \vec{c})$, then transitivity should hold, i.e., $\mathcal{O}_{\preceq}(\vec{x}, \vec{y}, \vec{a}) \cup \mathcal{O}_{\preceq}(\vec{y}, \vec{z}, \vec{b}) \vdash \mathcal{O}_{\preceq}(\vec{x}, \vec{z}, \vec{c})$. However, if the auxiliary variables are not correctly set, then no claims are made.

These proof obligations ensure that \preceq is a preorder:

Lemma 2. If \mathcal{O}_{\preceq} and \mathcal{S}_{\preceq} satisfy Equations (11) and (12), then \preceq as defined by \mathcal{O}_{\preceq} and \mathcal{S}_{\preceq} is a preorder.

3.3 Validity

We extend the configurations of the proof system to $(\mathcal{C}, \mathcal{D}, \mathcal{O}_{\preceq}, \mathcal{S}_{\preceq}, \vec{z}, \vec{a})$. In particular, we extend the notion of weak- (F, f) -validity from Bogaerts et al. (2023) to our new configurations, focusing on decision problems:

Definition 3. A configuration $(\mathcal{C}, \mathcal{D}, \mathcal{O}_{\preceq}, \mathcal{S}_{\preceq}, \vec{z}, \vec{a})$ is weakly F -valid if the following conditions hold:

1. If F is satisfiable, then \mathcal{C} is satisfiable.
2. For every assignment α satisfying \mathcal{C} , there exists an assignment α' satisfying $\mathcal{C} \cup \mathcal{D}$ and $\alpha' \preceq \alpha$.

In the following, we furthermore assume that for any configuration $(\mathcal{C}, \mathcal{D}, \mathcal{O}_{\preceq}, \mathcal{S}_{\preceq}, \vec{z}, \vec{a})$, the following hold:

1. \mathcal{O}_{\preceq} and \mathcal{S}_{\preceq} refer to formulas for which Equations (11) and (12) have been successfully proven.

2. \mathcal{S}_{\preceq} is a specification over \vec{a} .

3. The variables \vec{a} only occur in \mathcal{O}_{\preceq} and \mathcal{S}_{\preceq} , and these variables are disjoint from \vec{z} .

Observe that due to these invariants, satisfying assignments for $\mathcal{C} \cup \mathcal{D}$ do not need to assign the variables \vec{a} . In the following, we assume that such assignments are indeed defined only over the domain $\text{var}(\mathcal{C} \cup \mathcal{D})$.

3.4 Dominance-Based Strengthening Rule

As in the original system, the dominance rule allows adding a constraint C to the derived set using witness ω if from the premises $\mathcal{C} \cup \mathcal{D} \cup \{-C\}$ we can derive $\mathcal{C}|_{\omega}$ and show that $\alpha \circ \omega \prec \alpha$ holds for all assignments α satisfying $\mathcal{C} \cup \mathcal{D} \cup \{-C\}$. To show $\alpha \circ \omega \prec \alpha$, we separately show that $\alpha \circ \omega \preceq \alpha$ and $\alpha \not\preceq \alpha \circ \omega$. To show that $\alpha \circ \omega \preceq \alpha$, we have to show that $\mathcal{O}_{\preceq}(\vec{z}|_{\alpha \circ \omega}, \vec{z}|_{\alpha}, \vec{a})$, assuming that the circuit defining the auxiliary variables \vec{a} has been evaluated correctly, which is encoded by the specification $\mathcal{S}_{\preceq}(\vec{z}|_{\alpha \circ \omega}, \vec{z}|_{\alpha}, \vec{a})$. This leads to the proof obligation

$$\mathcal{C} \cup \mathcal{D} \cup \{-C\} \cup \mathcal{S}_{\preceq}(\vec{z}|_{\alpha \circ \omega}, \vec{z}|_{\alpha}, \vec{a}) \vdash \mathcal{O}_{\preceq}(\vec{z}|_{\alpha \circ \omega}, \vec{z}|_{\alpha}, \vec{a}). \quad (13)$$

To show that $\alpha \not\preceq \alpha \circ \omega$, we have to show that $\neg \mathcal{O}_{\preceq}(\vec{z}|_{\alpha}, \vec{z}|_{\alpha \circ \omega}, \vec{a})$ assuming $\mathcal{S}_{\preceq}(\vec{z}|_{\alpha}, \vec{z}|_{\alpha \circ \omega}, \vec{a})$. However, since $\neg \mathcal{O}_{\preceq}(\vec{z}|_{\alpha}, \vec{z}|_{\alpha \circ \omega}, \vec{a})$ is not necessarily a pseudo-Boolean formula (due to the negation), we instead show that we can derive contradiction from $\mathcal{O}_{\preceq}(\vec{z}|_{\alpha}, \vec{z}|_{\alpha \circ \omega}, \vec{a})$, leading to the proof obligation:

$$\mathcal{C} \cup \mathcal{D} \cup \{-C\} \cup \mathcal{S}_{\preceq}(\vec{z}|_{\alpha}, \vec{z}|_{\alpha \circ \omega}, \vec{a}) \cup \mathcal{O}_{\preceq}(\vec{z}|_{\alpha}, \vec{z}|_{\alpha \circ \omega}, \vec{a}) \vdash \perp. \quad (14)$$

The following lemma shows that these proof obligations indeed imply $\alpha \circ \omega \preceq \alpha$ and $\alpha \not\preceq \alpha \circ \omega$, respectively:

Lemma 3. Let G be a formula and ω a witness with $\text{supp}(\omega) \subseteq \text{var}(G)$. Furthermore, let $\vec{a} \cap \text{var}(G) = \emptyset$ and \mathcal{S}_{\preceq} be a specification over \vec{a} . Also, let \mathcal{O}_{\preceq} and \mathcal{S}_{\preceq} define a pre-order \preceq . Then the following hold:

1. If $G \cup \mathcal{S}_{\preceq}(\vec{z}|_{\omega}, \vec{z}, \vec{a}) \vdash \mathcal{O}_{\preceq}(\vec{z}|_{\omega}, \vec{z}, \vec{a})$ holds, then for each assignment α satisfying G , $\alpha \circ \omega \preceq \alpha$ holds.
2. If $G \cup \mathcal{S}_{\preceq}(\vec{z}|_{\omega}, \vec{z}, \vec{a}) \cup \mathcal{O}_{\preceq}(\vec{z}|_{\omega}, \vec{z}, \vec{a}) \vdash \perp$ holds, then for each assignment α satisfying G , $\alpha \not\preceq \alpha \circ \omega$ holds.

Hence, we define the dominance rule as follows:

Definition 4 (Dominance-based strengthening with specification). We can transition from the configuration $(\mathcal{C}, \mathcal{D}, \mathcal{O}_{\preceq}, \mathcal{S}_{\preceq}, \vec{z}, \vec{a})$ to $(\mathcal{C}, \mathcal{D} \cup \{C\}, \mathcal{O}_{\preceq}, \mathcal{S}_{\preceq}, \vec{z}, \vec{a})$ using the dominance rule if the following conditions are met:

1. The constraint C does not contain variables in \vec{a} .
2. There is a witness ω for which $\text{image}(\omega) \cap \vec{a} = \emptyset$ holds.
3. We have cutting planes proofs for the following:

$$\mathcal{C} \cup \mathcal{D} \cup \{-C\} \cup \mathcal{S}_{\preceq}(\vec{z}|_{\omega}, \vec{z}, \vec{a}) \vdash \mathcal{C}|_{\omega} \cup \mathcal{O}_{\preceq}(\vec{z}|_{\omega}, \vec{z}, \vec{a}) \quad (15)$$

$$\mathcal{C} \cup \mathcal{D} \cup \{-C\} \cup \mathcal{S}_{\preceq}(\vec{z}|_{\omega}, \vec{z}, \vec{a}) \cup \mathcal{O}_{\preceq}(\vec{z}|_{\omega}, \vec{z}, \vec{a}) \vdash \perp. \quad (16)$$

Using Lemma 3, we can show that the dominance rule preserves the invariants required by weak F -validity:

Lemma 4. If we can transition from $(\mathcal{C}, \mathcal{D}, \mathcal{O}_{\preceq}, \mathcal{S}_{\preceq}, \vec{z}, \vec{a})$ to $(\mathcal{C}, \mathcal{D} \cup \{C\}, \mathcal{O}_{\preceq}, \mathcal{S}_{\preceq}, \vec{z}, \vec{a})$ by the dominance rule, and $(\mathcal{C}, \mathcal{D}, \mathcal{O}_{\preceq}, \mathcal{S}_{\preceq}, \vec{z}, \vec{a})$ is weakly F -valid, then the configuration $(\mathcal{C}, \mathcal{D} \cup \{C\}, \mathcal{O}_{\preceq}, \mathcal{S}_{\preceq}, \vec{z}, \vec{a})$ is also weakly F -valid.

3.5 Redundance-Based Strengthening Rule

We also modify the redundance rule to work in our extended proof system. Similarly to the dominance rule, we can use $\mathcal{S}_\leq(\vec{z}|\omega, \vec{z}, \vec{a})$ as an extra premise in our proof obligations.

Definition 5 (Redundance-based strengthening with specification). We can transition from the configuration $(\mathcal{C}, \mathcal{D}, \mathcal{O}_\leq, \mathcal{S}_\leq, \vec{z}, \vec{a})$ to $(\mathcal{C}, \mathcal{D} \cup \{C\}, \mathcal{O}_\leq, \mathcal{S}_\leq, \vec{z}, \vec{a})$ using the redundance rule if the following conditions are met:

1. The constraint C does not contain variables in \vec{a} .
2. There is a witness ω for which $\text{image}(\omega) \cap \vec{a} = \emptyset$ holds.
3. We have cutting planes proof that the following holds:

$$\begin{aligned} & \mathcal{C} \cup \mathcal{D} \cup \{\neg C\} \cup \mathcal{S}_\leq(\vec{z}|\omega, \vec{z}, \vec{a}) \\ & \vdash (\mathcal{C} \cup \mathcal{D} \cup \{C\})|_\omega \cup \mathcal{O}_\leq(\vec{z}|\omega, \vec{z}, \vec{a}). \end{aligned} \quad (17)$$

The redundance rule preserves weak F -validity:

Lemma 5. *If we can transition from $(\mathcal{C}, \mathcal{D}, \mathcal{O}_\leq, \mathcal{S}_\leq, \vec{z}, \vec{a})$ to $(\mathcal{C}, \mathcal{D} \cup \{C\}, \mathcal{O}_\leq, \mathcal{S}_\leq, \vec{z}, \vec{a})$ by the redundance rule, and $(\mathcal{C}, \mathcal{D}, \mathcal{O}_\leq, \mathcal{S}_\leq, \vec{z}, \vec{a})$ is weakly F -valid, then the configuration $(\mathcal{C}, \mathcal{D} \cup \{C\}, \mathcal{O}_\leq, \mathcal{S}_\leq, \vec{z}, \vec{a})$ is also weakly F -valid.*

4 Efficient Proof Logging in SATSUMA

Using our extended proof system, we implement proof logging in the state-of-the-art symmetry breaker SATSUMA¹. The details can be found in Appendix C and a worked-out example is given in Appendix D. Like in the original system (as explained in Section 2.2), the (negation of the) symmetry breaking constraints can be used to show the proof obligations for the order. However, in the extended system this is more complicated, because we need to relate two different sets of extension variables (those in the symmetry breaking constraints and the auxiliary variables in the specification).

Our new method achieves an asymptotic speedup over the old method. Defining the lexicographical order over n variables can be done in time $O(n)$ with our new method (for both checking and logging), while the old method requires time $O(n^2)$. Breaking a symmetry σ over $k = |\text{supp}(\sigma)|$ variables ($k \leq n$) takes time $O(k)$ for logging and $O(n)$ for checking with our new method, while the old method requires time $O(nk)$ for logging and time $O(n^2 + nk^2)$ for checking. Therefore, the new method is in each case asymptotically at least a factor n faster for both logging and checking than the old method used by Bogaerts et al. (2023).

5 Proof Checker Implementation

We implemented checking for our extended proof system in the proof checker VERIPB² and the formally verified proof checker CAKEPB³. Several optimizations are necessary to handle orders with many specification constraints efficiently.

Lazy Constraint Loading and Evaluation. When checking cutting planes derivations for the dominance or redundance rule (15)–(17), the proof checkers load the specification constraints from \mathcal{S}_\leq only when they are used in the proof. More specifically, the constraints in \mathcal{S}_\leq are not even computed until loaded explicitly, which improves the checking performance by a linear factor if the specification \mathcal{S}_\leq is not required for a cutting planes derivation.

Implicit Reflexivity Proof. Since the loaded order is always proven to be reflexive (11), the cutting planes derivation for $\mathcal{O}_\leq(\vec{z}|\omega, \vec{z}, \vec{a})$ can be skipped for the redundance rule (17) if the domain of the witness ω does not contain a variable in \vec{z} . Requiring an explicit cutting planes derivation for $\mathcal{O}_\leq(\vec{z}|\omega, \vec{z}, \vec{a})$ would again involve computation over all constraints in the specification \mathcal{S}_\leq , which incurs a linear overhead for proof logging and checking.

Formal Verification. We updated CAKEPB in two phases, yielding the same end-to-end verification guarantees for proof checking as discussed in Gocht et al. (2024). First, we formally verified soundness of all updates to the proof system (including Lemmas 1–5). Second, we implemented and verified these changes in the CAKEPB codebase, including soundness for the optimizations described above.

6 Experimental Evaluation

From the analysis in Section 4, we know that our new proof system is asymptotically better in theory. We now show that it indeed enables much faster proof logging and checking in practice, on both crafted and real-world problem instances. The experiments in this section are performed on machines with dual AMD EPYC 7643 processors, 2TBytes of RAM, and local solid state hard drives, running Ubuntu 22.04.2. We limit each individual process to 32GBytes RAM, and run up to 16 processes in parallel (having checked that this does not make a measurable difference to runtimes). We give SATSUMA a time limit of 1,000s per instance, and VERIPB and CAKEPB 10,000s. We remark that runtimes involving writing proofs are often bound by disk I/O performance; nevertheless, our general experimental trends are valid. In each case, when we run VERIPB, we run it in elaboration mode. This means that, in addition to checking a proof, it also outputs a simplified proof that is suitable for giving to CAKEPB. This also means that any instance that fails for VERIPB due to limits cannot be run through CAKEPB.

The aim of our experiments is not to determine whether symmetry breaking is a good idea, or how it should be done. Indeed, SATSUMA produces the same CNF (modulo a potential sorting of the constraints) regardless of whether it is outputting proofs using the old method or the new method, or not outputting proofs at all. Thus, we limit our experiments to checking that the proofs produced by SATSUMA are in fact valid, rather than reporting times for checking the entire solving process. This allows us to precisely measure the effects of our changes.

We perform experiments across two sets of instances, with different purposes. Our first set consists of five families of crafted benchmarks which have well-understood symmetries, generated using CNFGEN (Lauria et al. 2017). We note

¹SATSUMA code: <https://doi.org/10.5281/zenodo.17607863>

²VERIPB code: <https://doi.org/10.5281/zenodo.17608873>

³CAKEPB code: <https://doi.org/10.5281/zenodo.17609070>

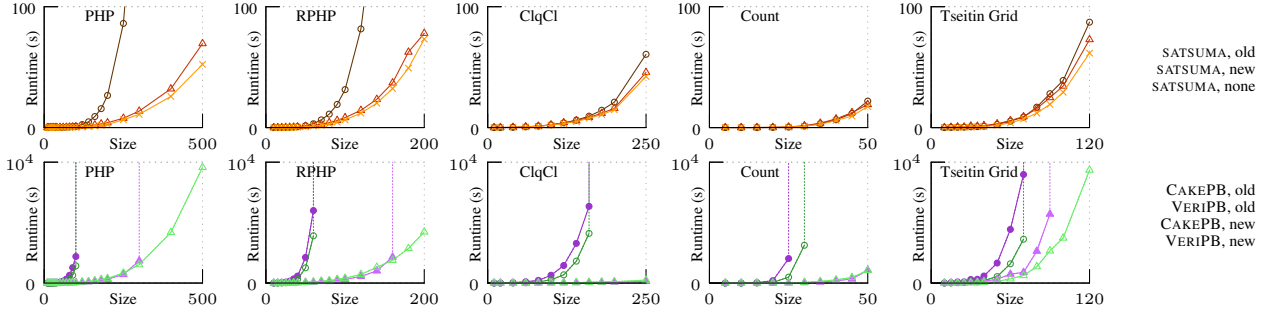


Figure 1: On top, the cost of running SATSUMA with or without proof logging, on crafted benchmark instances, as the instance size grows. In each case logging with the new method scales similarly to not doing logging, whilst the old method exhibits worse scaling for several families. On the bottom, the cost of checking these proofs: the new method exhibits better scaling.

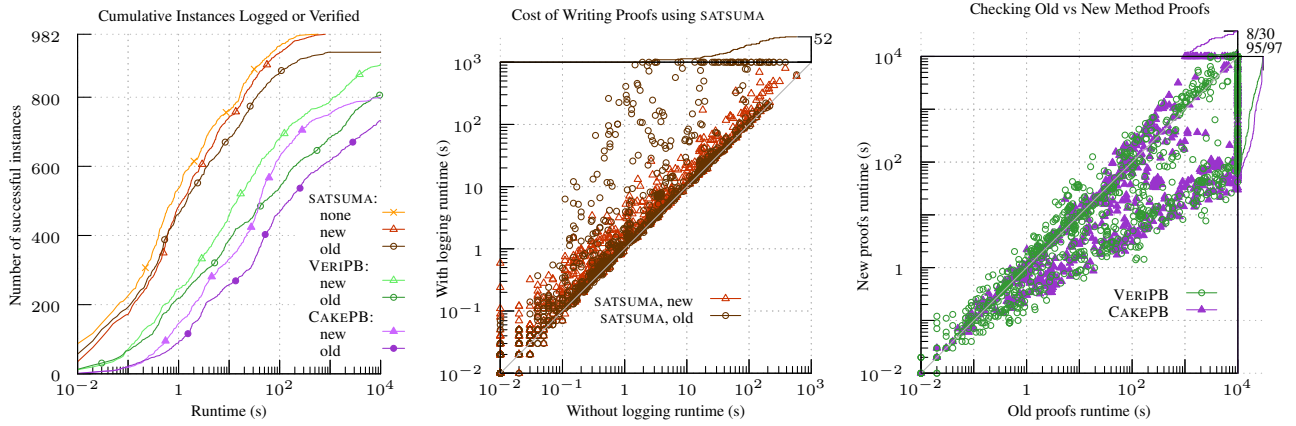


Figure 2: On the left, the cumulative number of “interesting” SAT competition instances broken, logged, and checked over time. The centre plot compares the added cost of writing proofs with the old and new methods, compared to not writing proofs; 52 instances reached time or memory limits with the old method. The right-hand plot compares the cost of checking proofs using the old and new methods, with points below the diagonal line showing where both methods succeeded but the new method was faster; additionally, 8 and 30 instances reach limits with VERIPB and CAKEPB respectively with the new method where the old method succeeded, compared to 95 and 97 respectively with the old method where the new method succeeded.

that the number of variables grows quadratically or cubically in the instance size. Appendix E provides more details about the crafted benchmarks.

We show the results in Figure 1. For each of the five families, on the top row we plot the time needed to run SATSUMA to produce symmetry breaking constraints, without proof logging and with both kinds of proof logging enabled. In each case, the new method scales similarly to not doing proof logging, although there is a cost to be paid to output the proofs to disk. However, particularly for the PHP and RPHP families, it is clear that even writing the proofs is both asymptotically and practically much more expensive using the old method. On the bottom row of the figure, we plot the checking times. We see much better scaling from the new method in all five cases. Formally verified proof checking using CAKEPB is slightly slower than with VERIPB, which is not surprising—for the families where CAKEPB’s curve

stops on smaller instances, this is due to CAKEPB hitting memory limits when VERIPB did not. The new method is particularly helpful for CAKEPB as its formally verified arbitrary precision arithmetic library is known to be less efficient than other (unverified) libraries (Tan et al. 2019).

Our second set of instances are taken from the SAT competition (Iser and Jabs 2024). Because we are only interested in instances where we can measure something interesting about symmetry breaking, we selected the 982 instances from the main competition tracks from 2020 to 2024 where SATSUMA was able to run to completion and identify at least one symmetry. In the left-hand plot of Figure 2, we show the cumulative number of instances successfully logged or checked over time. The leftmost (“best”) curve is to run SATSUMA with no proof logging, and this is closely followed by running SATSUMA with proof logging using the new method, where we could produce proofs for all 982 in-

stances. When producing proofs using the old method, in contrast, we were only able to produce proofs for 930 instances before limits were reached. We were able to check the correctness of the symmetry breaking constraints for the new method for 893 instances (799 with CAKEPB), and with the old method for only 806 instances (732 with CAKEPB). The two scatter plots in the figure give a more detailed comparison of the added cost of running SATSUMA with proof-logging enabled, and comparing the checking costs of the old and new proof methods. In both cases it is clear that the new method is never more than a small constant factor worse than the old method, and that it is often many orders of magnitude faster.

7 Concluding Remarks

We have presented a substantial redesign of the VERIPB proof system (Bogaerts et al. 2023; Gocht and Nordström 2021; Gocht 2022) in order to support faster certified symmetry breaking. Central to our redesign is support for using auxiliary variables to encode ordering constraints over assignments. Theoretically, the use of orders with auxiliary variables allows us to avoid encoding lexicographical orders using big integers, that are prohibitive for problems with large symmetries; this improves on the previous state-of-the-art proof logging approach (Bogaerts et al. 2023) by at least a linear factor. To evaluate this in practice, we implemented proof logging using our new method in the state-of-the-art symmetry breaking tool SATSUMA (Anders, Brenner, and Rattan 2024), and proof checking for our extended system in VERIPB and the formally verified checker CAKEPB (Gocht et al. 2024); our experimental evaluation shows orders-of-magnitude improvement for proof logging and checking compared to the old approach.

Although proof logging is now asymptotically as fast as symmetry breaking, enabling proof logging can still incur a constant factor overhead. However, improving this would be mostly an engineering effort—we are already able to produce proofs for all of our benchmark instances within reasonable time. Checking the proof can still be asymptotically slower than symmetry breaking in theory, which leaves room to significantly improve the performance of proof checking for symmetry breaking. The key challenge here is that the proof checker currently has to reason about all variables in the order for each symmetry broken, while the symmetry breaker does this once at a higher level. Future work could investigate proof logging for conditional and dynamic symmetry breaking during search (Gent et al. 2005) or other dynamic methods of exploiting symmetries (Devriendt, Bogaerts, and Bruynooghe 2017), in contrast to the static symmetry breaking we presented here.

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A The Cutting Planes Proof System

In this appendix, the cutting planes proof system is explained in sufficient detail to understand the rest of the appendix. We will also present the syntax in the VERIPB format used to write the proofs to a file and for the example in Appendix D. The cutting planes proof system was first introduced by Cook, Coullard, and Turán (1987) and more details about recent results about cutting planes can be found in Buss and Nordström (2021).

As mentioned in Section 2, a *Boolean variable* x can take values 0 (false) and 1 (true). A *literal* is either the variable x itself or its negation $\bar{x} = 1 - x$. A *pseudo-Boolean constraint* is an integer linear inequality over literals ℓ_i

$$C \doteq \sum_i a_i \ell_i \geq A, \quad (18)$$

where without loss of generality the coefficients a_i and the right-hand side A are non-negative and the literals ℓ_i are over distinct variables. A (partial) assignment is a (partial) function mapping variables to 0 and 1, which is extended to literals in the natural way. The *cutting planes* proof system provides rules to derive a constraint C that is implied by a formula F , i.e., F and $F \cup \{C\}$ have the same set of satisfying assignments.

In the VERIPB syntax a constraint is referred to by a positive integer called its *constraint ID*. The constraint IDs are consecutively assigned to the constraints in the order in which they are derived and start with the original formula F getting IDs $1, \dots, |F|$. A negative constraint ID can be used as a relative reference, e.g., the ID -1 means the previously derived constraint. A cutting planes step starts with the keyword `pol` and ends with a semicolon. After each cutting planes step, the derived constraint is assigned the next ID and added to the derived constraints in the checker. The derivation is written in reverse Polish notation (or postfix notation), hence the checker maintains a stack of operands. Each of the following rule pops its operands from the stack and pushed on the stack its result.

Let $C \doteq \sum_i a_i \ell_i \geq A$ and $D \doteq \sum_i b_i \ell_i \geq B$ be constraints from the formula F or derived in previous steps. The constraint C can always be derived by the *constraint axiom* rule, which is just the ID of the constraint in the syntax. *Literal axioms* $x_1 \geq 0$ and $\bar{x}_1 \geq 0$ are derivable by using the syntax `x1` and `~x1`, respectively. The constraint C can be *multiplied* by an integer m by multiplying each coefficient and the right-hand side by m , resulting in $\sum_i (ma_i) \ell_i \geq mA$. This is denoted by `*` in the syntax and assumes the last operand on the stack is an integer and the second to last is a constraint. The constraints C and D can be *added* to derive $\sum_i (a_i + b_i) \ell_i \geq A + B$, which is denoted by `+` in the syntax, which assumes that the last two operands on the stack are constraints. The constraint C can be *saturated* to derive $\sum_i \min\{a_i, A\} \ell_i \geq A$, which is denoted by `s` in the syntax and expects that the last operand on the stack is a constraint. *Weakening* is syntactic sugar for adding literal axioms to the constraint C to eliminate the term $a_j \ell_j$ in C resulting $\sum_{i \neq j} a_i \ell_i \geq A - a_j$. This is denoted in the syntax by `w`, which assumes that the last operand on the stack is the variable over which ℓ is over and the second to last operand is a constraint.

VERIPB also supports the cutting planes rule of *division*, which we will mention for completeness, but we will not need it for the example in Appendix D. Considering the constraint C and a positive integer d , each coefficient and the right-hand side is divided by d and rounded up, resulting in $\sum_i \lceil a_i/d \rceil \ell_i \geq \lceil A/d \rceil$. This is denoted by `d` in the syntax and assumes that the last operand on the stack is an integer and the second to last is a constraint.

To better illustrate how the syntax relates to rules, suppose we have the example constraint

$$2\bar{x}_1 + 3x_2 + 2x_3 \geq 5, \quad (19)$$

with constraint ID 19. The cutting planes proof line

```
pol 19 x2 2 * + x1 w s 2 d ;
```

first pushes the constraint with ID 19 to the stack, then the literal axiom $x_2 \geq 0$ is pushed on the stack, which is immediately afterwards multiplied by 2 resulting on $2x_2 \geq 0$ being pushed on the stack. Then the last two constraints on the stack, which are $2\bar{x}_1 + 3x_2 + 2x_3 \geq 5$ and $2x_2 \geq 0$, are added, resulting in $2\bar{x}_1 + 5x_2 + 2x_3 \geq 5$. This constraint is then weakened on the variable x_1 resulting in the constraint $5x_2 + 2x_3 \geq 3$, which is saturated to yield $3x_2 + 2x_3 \geq 3$. Finally, dividing this constraint by 2 yields $2x_2 + x_3 \geq 2$, which is the result of this step and added to the derived constraints.

The *slack* of the pseudo-Boolean constraint (18) with respect to the (partial) assignment ρ is

$$\text{slack}(\sum_i a_i \ell_i \geq A; \rho) := \sum_{i: \rho(\ell_i) \neq 0} a_i - A.$$

If the slack is negative, then the constraint can not be satisfied by any extension of ρ , as there are not enough literals assigned to 1 or unassigned. If there is a literal ℓ_i in the constraint C that is unassigned and $0 \leq a_i < \text{slack}(C; \rho)$, then ℓ_i has to be assigned to 1, as assigning ℓ_i to 0 would lead to a negative slack. This assignment is then added to ρ , and we say that C *propagated* ℓ_i to 1 under ρ . This process is repeated until there are no further propagations by any constraint or a constraint is falsified, i.e., negative slack. We refer to the latter case as a *conflict*.

The negation of the pseudo-Boolean constraint C in (18) is $\neg C \doteq \sum_i a_i \bar{\ell}_i \geq \sum_i a_i - A + 1$. A formula F implies a constraint C by *reverse unit propagation* (RUP) (Goldberg and Novikov 2003) if $F \cup \{\neg C\}$ propagates to a conflict. This shows that $\neg C$ is falsified by all satisfying assignments to F , thus any assignment satisfying F also satisfies $F \cup \{C\}$.

In the VERIPB syntax, a RUP step contains the constraint to be derived in a modified OPB format (Roussel and Manquinho 2016). Additionally, the syntax also allows for hints similar to LRAT (Cruz-Filipe et al. 2017) that specify which constraints have to be propagated to reach a conflict. For instance, the RUP step

```
rup 2 x2 1 x3 >= 2 : 19 ;
```

derives the constraint $2x_2 + x_3 \geq 2$ with the hint that only the constraint with ID 19 above ($2\bar{x}_1 + 3x_2 + 2x_3 \geq 5$) is required in addition to the negation ($2\bar{x}_2 + \bar{x}_3 \geq 2$) of the constraint to propagate from the empty assignment to a conflict.

B Full Description of Extended Proof System

In this appendix, we provide further details on the extended proof system introduced in Sections 3.1 up to 3.5, including the statement of all results in the more general context of optimization problems, and all proofs. For a discussion on the intuition behind the proof system, we refer the start of Section 3. To make it easier to read the appendix as its own section, we also include all content that was already in Sections 3.1 up to 3.5 in the main paper.

B.1 Specifications

Let \vec{a} be a list of variables. A pseudo-Boolean formula $\mathcal{S}(\vec{x}, \vec{a})$ is a *specification over the variables \vec{a}* , if it is derivable from the empty formula \emptyset by the redundancy rule, where each application only witnesses over variables in \vec{a} .

Definition 6. A formula $\mathcal{S}(\vec{x}, \vec{a}) = \{C_1, C_2, \dots, C_n\}$ is a *specification over the variables \vec{a}* , if there is a list

$$(C_1, \omega_1), (C_2, \omega_2), \dots, (C_n, \omega_n)$$

which satisfies the following:

1. The constraint C_1 can be obtained from the empty formula \emptyset using the redundancy rule with witness ω_1 .
2. For each $i \in \{2, \dots, n\}$ we have that C_i can be added by the redundancy rule to $\bigcup_{j=1}^{i-1} \{C_j\}$ with the witness ω_i . In other words, it should hold that

$$\bigcup_{j=1}^{i-1} \{C_j\} \cup \{-C_i\} \vdash \bigcup_{j=1}^i \{C_j \upharpoonright \omega_i\}. \quad (20)$$

3. For every witness ω_i , $\text{supp}(\omega_i) \subseteq \vec{a}$ holds.

In terms of configurations, we can say that a formula $\mathcal{S}(\vec{x}, \vec{a}) = \{C_1, C_2, \dots, C_n\}$ is a specification over \vec{a} if we can transition from the configuration $(\emptyset, \emptyset, \emptyset, \emptyset)$ to the configuration $(\emptyset, \mathcal{S}(\vec{x}, \vec{a}), \emptyset, \emptyset)$ using only the redundancy rule with witnesses ω_i such that $\text{supp}(\omega_i) \subseteq \vec{a}$ holds.

Remark 1. Note that the definition of the specification uses the redundancy rule, while the redundancy rule in our extended proof system is only defined later. This is not a problem since the applications of the redundancy rule in a specification require that the loaded order is the trivial preorder $\mathcal{O}_\top \doteq \emptyset$ relating all assignments, for which it is irrelevant whether we use auxiliary variables or not.

A crucial property of specifications is that we can recover an assignment of the auxiliary variables from the assignment of the non-auxiliary variables. We state this property below.

Lemma 6. Let $\mathcal{S}(\vec{x}, \vec{a})$ be a specification over \vec{a} . Let α be any assignment of the variables \vec{x} . Then, α can be extended to an assignment α' , such that

1. α' satisfies \mathcal{S} , and
2. $\alpha(x) = \alpha'(x)$ holds for every $x \in \vec{x}$.

To explain why Lemma 6 holds, recall from Section 2.1 that the redundancy rule satisfies the following: if a constraint C is added by redundancy with witness ω , then given an assignment α satisfying all other constraints, either α or $\alpha \circ \omega$ also satisfies C . Hence, defining α' by composing α

with the witnesses ω_i corresponding to the constraints that do not already hold ensures that α' satisfies \mathcal{S} , and the fact that each witness ω_i is the identity on \vec{x} (since $\text{supp}(\omega_i) \subseteq \vec{a}$) ensures that $\alpha(x) = \alpha'(x)$ for $x \in \vec{x}$.

We now give a complete proof of Lemma 6.

Proof of Lemma 6. Write $\mathcal{S}(\vec{x}, \vec{a}) = \{C_1, C_2, \dots, C_n\}$. We show by induction on $i \geq 0$ that we can extend α to an assignment α_i such that

1. α_i satisfies $F_i = \{C_1, C_2, \dots, C_i\}$, and
2. $\alpha(x) = \alpha_i(x)$ holds for every $x \in \vec{x}$.

For the base case $i = 0$, we extend α to an assignment α_0 by setting all variables of \vec{a} to false. Then α_0 trivially satisfies $F_0 = \emptyset$, and $\alpha(x) = \alpha_0(x)$ for every $x \in \vec{x}$ since we only change the assignment of \vec{a} .

We proceed with the induction step. By the induction hypothesis, we can extend α to an assignment α_i such that α_i satisfies F_i and $\alpha(x) = \alpha_i(x)$ holds for every $x \in \vec{x}$.

If α_i indeed happens to satisfy C_{i+1} , we set $\alpha_{i+1} = \alpha_i$. Then α_{i+1} indeed satisfies $F_{i+1} = F_i \cup \{C_{i+1}\}$ and we have $\alpha(x) = \alpha_i(x) = \alpha_{i+1}(x)$ for every $x \in \vec{x}$.

If α_i does not satisfy C_{i+1} , we set $\alpha_{i+1} = \alpha_i \circ \omega_{i+1}$. Since then α_i satisfies $\bigcup_{j=1}^i \{C_j\} \cup \{-C_{i+1}\}$, Equation (20) then implies that α_i satisfies $F_{i+1} \upharpoonright \omega_{i+1} = \bigcup_{j=1}^{i+1} \{C_j \upharpoonright \omega_{i+1}\}$, which in turn implies that $\alpha_{i+1} = \alpha_i \circ \omega_{i+1}$ satisfies F_{i+1} . Since $\text{supp}(\omega_{i+1}) \subseteq \vec{a}$, we have $\omega_{i+1}(x) = x$ for every $x \in \vec{x}$, so $\alpha(x) = \alpha_i(x) = \alpha_i(\omega_{i+1}(x)) = (\alpha_i \circ \omega_{i+1})(x) = \alpha_{i+1}(x)$ for every $x \in \vec{x}$. This completes the induction.

Since $F_n = \mathcal{S}$ holds by definition, the claim follows. \square

B.2 Orders with Auxiliary Variables

Next, we explain how two pseudo-Boolean formulas $\mathcal{O}_\preceq(\vec{u}, \vec{v}, \vec{a})$ and $\mathcal{S}_\preceq(\vec{u}, \vec{v}, \vec{a})$, together with the two disjoint lists of variables \vec{z} and \vec{a} , define a preorder.

Definition 7. Let \vec{u} and \vec{v} be disjoint lists of variables of size n and let \vec{a} be a list of auxiliary variables. Let $\mathcal{O}_\preceq(\vec{u}, \vec{v}, \vec{a})$ and $\mathcal{S}_\preceq(\vec{u}, \vec{v}, \vec{a})$ be two pseudo-Boolean formulas such that \mathcal{S}_\preceq is a specification over \vec{a} .

Then we define the relation \preceq over the domain of total assignments to a list of variables \vec{z} of size n as follows: For assignments α, β we let $\alpha \preceq \beta$ hold, if and only if there exists an assignment ρ to the variables \vec{a} , such that

$$\mathcal{S}_\preceq(\vec{z} \upharpoonright \alpha, \vec{z} \upharpoonright \beta, \vec{a} \upharpoonright \rho) \wedge \mathcal{O}_\preceq(\vec{z} \upharpoonright \alpha, \vec{z} \upharpoonright \beta, \vec{a} \upharpoonright \rho)$$

evaluates to true.

To ensure that \mathcal{O}_\preceq and \mathcal{S}_\preceq actually define a preorder, we require cutting planes proofs that show reflexivity, i.e., $\emptyset \vdash \alpha \preceq \alpha$, and transitivity, i.e., $\alpha \preceq \beta \wedge \beta \preceq \gamma \vdash \alpha \preceq \gamma$. To write these proof obligations using the cutting planes proof system, which cannot handle an existentially quantified conclusion, we can use the specification as a premise. The specification premise essentially tells us which auxiliary variables the existential quantifier should pick. In particular, for reflexivity, the proof obligation is

$$\mathcal{S}_\preceq(\vec{x}, \vec{x}, \vec{a}) \vdash \mathcal{O}_\preceq(\vec{x}, \vec{x}, \vec{a}). \quad (21)$$

For transitivity, the proof obligation is

$$\begin{aligned} \mathcal{S}_{\preceq}(\vec{x}, \vec{y}, \vec{a}) \cup \mathcal{O}_{\preceq}(\vec{x}, \vec{y}, \vec{a}) \cup \mathcal{S}_{\preceq}(\vec{y}, \vec{z}, \vec{b}) \\ \cup \mathcal{O}_{\preceq}(\vec{y}, \vec{z}, \vec{b}) \cup \mathcal{S}_{\preceq}(\vec{x}, \vec{z}, \vec{c}) \vdash \mathcal{O}_{\preceq}(\vec{x}, \vec{z}, \vec{c}). \end{aligned} \quad (22)$$

Intuitively, (22) says that if the circuits defining the auxiliary variables are correctly evaluated, which is encoded by the premises $\mathcal{S}_{\preceq}(\vec{x}, \vec{y}, \vec{a}) \cup \mathcal{S}_{\preceq}(\vec{y}, \vec{z}, \vec{b}) \cup \mathcal{S}_{\preceq}(\vec{x}, \vec{z}, \vec{c})$, then transitivity should hold, i.e., $\mathcal{O}_{\preceq}(\vec{x}, \vec{y}, \vec{a}) \cup \mathcal{O}_{\preceq}(\vec{y}, \vec{z}, \vec{b}) \vdash \mathcal{O}_{\preceq}(\vec{x}, \vec{z}, \vec{c})$. However, if the auxiliary variables are not correctly set, then no claims are made.

These proof obligations ensure that \preceq is a preorder:

Lemma 7. *If \mathcal{O}_{\preceq} and \mathcal{S}_{\preceq} satisfy Equations (21) and (22), then \preceq as defined by \mathcal{O}_{\preceq} and \mathcal{S}_{\preceq} is a preorder.*

Proof. We need to show that \preceq is reflexive and transitive.

(*Reflexivity.*) Let α be a total assignment to the variables \vec{z} . We have to show that $\alpha \preceq \alpha$ holds, that is, there is an assignment ρ to the variables \vec{a} such that

$$\mathcal{S}_{\preceq}(\vec{z} \upharpoonright \alpha, \vec{z} \upharpoonright \alpha, \vec{a} \upharpoonright \rho) \wedge \mathcal{O}_{\preceq}(\vec{z} \upharpoonright \alpha, \vec{z} \upharpoonright \alpha, \vec{a} \upharpoonright \rho) \quad (23)$$

is satisfied. From Lemma 1, it follows that we can choose an extension ρ to α , such that $\mathcal{S}_{\preceq}(\vec{z} \upharpoonright \alpha, \vec{z} \upharpoonright \alpha, \vec{a} \upharpoonright \rho)$ is satisfied. Since this suffices to satisfy the preconditions of Equation 21, it follows that also $\mathcal{O}_{\preceq}(\vec{z} \upharpoonright \alpha, \vec{z} \upharpoonright \alpha, \vec{a} \upharpoonright \rho)$ is satisfied, which suffices to conclude that $\alpha \preceq \alpha$ holds.

(*Transitivity.*) Let α, β, γ be total assignments to the variables \vec{z} . We have to show that if $\alpha \preceq \beta$ and $\beta \preceq \gamma$ hold, then $\alpha \preceq \gamma$ holds. That is we can assume that there are assignments ρ_1, ρ_2 to \vec{a} such that

$$\begin{aligned} \mathcal{S}_{\preceq}(\vec{z} \upharpoonright \alpha, \vec{z} \upharpoonright \beta, \vec{a} \upharpoonright \rho_1) \cup \mathcal{O}_{\preceq}(\vec{z} \upharpoonright \alpha, \vec{z} \upharpoonright \beta, \vec{a} \upharpoonright \rho_1) \\ \cup \mathcal{S}_{\preceq}(\vec{z} \upharpoonright \beta, \vec{z} \upharpoonright \gamma, \vec{a} \upharpoonright \rho_2) \cup \mathcal{O}_{\preceq}(\vec{z} \upharpoonright \beta, \vec{z} \upharpoonright \gamma, \vec{a} \upharpoonright \rho_2) \end{aligned} \quad (24)$$

is satisfied. From Lemma 1, it follows that we can choose an extension ρ_3 , such that $\mathcal{S}_{\preceq}(\vec{z} \upharpoonright \alpha, \vec{z} \upharpoonright \gamma, \vec{a} \upharpoonright \rho_3)$ is satisfied. Since this suffices to satisfy the preconditions of Equation (22), it follows that also $\mathcal{O}_{\preceq}(\vec{z} \upharpoonright \alpha, \vec{z} \upharpoonright \gamma, \vec{a} \upharpoonright \rho_3)$ is satisfied, from which we conclude that $\alpha \preceq \gamma$ holds. \square

B.3 Validity

We extend the configurations of the proof system to $(\mathcal{C}, \mathcal{D}, \mathcal{O}_{\preceq}, \mathcal{S}_{\preceq}, \vec{z}, \vec{a}, v)$, where, as explained in Bogaerts et al. (2023), v denotes the current upper bound on the objective f . In particular, we extend the notion of *weak-(F, f)-validity* from Bogaerts et al. (2023) to our new configurations. For this, we first define the relation \preceq_f as follows:

$$\alpha \preceq_f \beta \quad \text{iff} \quad \alpha \preceq \beta \wedge f \upharpoonright \alpha \leq f \upharpoonright \beta. \quad (25)$$

We define the strict order \prec_f by defining $\alpha \prec_f \beta$ to mean that both $\alpha \preceq_f \beta$ and $\beta \not\preceq_f \alpha$ hold.

Definition 8. A configuration $(\mathcal{C}, \mathcal{D}, \mathcal{O}_{\preceq}, \mathcal{S}_{\preceq}, \vec{z}, \vec{a}, v)$ is *weakly-(F, f)-valid* if the following conditions hold:

1. For every $v' < v$, it holds that if $F \cup \{f \leq v'\}$ is satisfiable, then $\mathcal{C} \cup \{f \leq v'\}$ is satisfiable.
2. For every total assignment α satisfying the constraints $\mathcal{C} \cup \{f \leq v - 1\}$, there exists a total assignment $\alpha' \preceq_f \alpha$ satisfying $\mathcal{C} \cup \mathcal{D} \cup \{f \leq v - 1\}$.

In the following, we furthermore assume that for any configuration $(\mathcal{C}, \mathcal{D}, \mathcal{O}_{\preceq}, \mathcal{S}_{\preceq}, \vec{z}, \vec{a}, v)$, the following hold:

1. \mathcal{O}_{\preceq} and \mathcal{S}_{\preceq} refer to formulas for which Equations (21) and (22) have been successfully proven.
2. \mathcal{S}_{\preceq} is a specification over \vec{a} .
3. The variables \vec{a} only occur in \mathcal{O}_{\preceq} and \mathcal{S}_{\preceq} , and these variables are disjoint from \vec{z} .

These invariants only concern \mathcal{O}_{\preceq} , \mathcal{S}_{\preceq} , \vec{a} and \vec{z} . Whenever the order is changed, it is ensured that these invariants hold, and other rules do not change these components of the configuration.

Observe that due to these invariants, satisfying assignments for $\mathcal{C} \cup \mathcal{D}$ do not need to assign the variables \vec{a} . In the following, we assume that such assignments are indeed defined only over the domain $\text{var}(\mathcal{C} \cup \mathcal{D})$.

B.4 Dominance-Based Strengthening Rule

As in the original system, the dominance rule allows adding a constraint C to the derived set using witness ω if from the premises $\mathcal{C} \cup \mathcal{D} \cup \{\neg C\}$ we can derive $\mathcal{C} \upharpoonright \omega$ and show that $\alpha \circ \omega \prec \alpha$ holds for all assignments α satisfying $\mathcal{C} \cup \mathcal{D} \cup \{\neg C\}$. To show $\alpha \circ \omega \prec \alpha$, we separately show that $\alpha \circ \omega \preceq \alpha$ and $\alpha \not\preceq \alpha \circ \omega$. To show that $\alpha \circ \omega \preceq \alpha$, we have to show that $\mathcal{O}_{\preceq}(\vec{z} \upharpoonright \omega, \vec{z}, \vec{a})$, assuming that the circuit defining the auxiliary variables \vec{a} has been evaluated correctly, which is encoded by the specification $\mathcal{S}_{\preceq}(\vec{z} \upharpoonright \omega, \vec{z}, \vec{a})$. This leads to the proof obligation

$$\begin{aligned} \mathcal{C} \cup \mathcal{D} \cup \{\neg C\} \cup \{f \leq v - 1\} \\ \cup \mathcal{S}_{\preceq}(\vec{z} \upharpoonright \omega, \vec{z}, \vec{a}) \vdash \mathcal{O}_{\preceq}(\vec{z} \upharpoonright \omega, \vec{z}, \vec{a}). \end{aligned} \quad (26)$$

To show that $\alpha \not\preceq \alpha \circ \omega$, we have to show that $\neg \mathcal{O}_{\preceq}(\vec{z}, \vec{z} \upharpoonright \omega, \vec{a})$ assuming $\mathcal{S}_{\preceq}(\vec{z}, \vec{z} \upharpoonright \omega, \vec{a})$. However, since $\neg \mathcal{O}_{\preceq}(\vec{z}, \vec{z} \upharpoonright \omega, \vec{a})$ is not necessarily a pseudo-Boolean formula (due to the negation), we instead show that we can derive contradiction from $\mathcal{O}_{\preceq}(\vec{z}, \vec{z} \upharpoonright \omega, \vec{a})$, leading to the proof obligation:

$$\begin{aligned} \mathcal{C} \cup \mathcal{D} \cup \{\neg C\} \cup \{f \leq v - 1\} \\ \cup \mathcal{S}_{\preceq}(\vec{z}, \vec{z} \upharpoonright \omega, \vec{a}) \cup \mathcal{O}_{\preceq}(\vec{z}, \vec{z} \upharpoonright \omega, \vec{a}) \vdash \perp. \end{aligned} \quad (27)$$

The following lemma shows that these proof obligations indeed imply $\alpha \circ \omega \preceq \alpha$ and $\alpha \not\preceq \alpha \circ \omega$, respectively:

Lemma 8. *Let G be a formula and ω a witness with $\text{supp}(\omega) \subseteq \text{var}(G)$. Furthermore, let $\vec{a} \cap \text{var}(G) = \emptyset$ and \mathcal{S}_{\preceq} be a specification over \vec{a} . Also, let \mathcal{O}_{\preceq} and \mathcal{S}_{\preceq} define a preorder \preceq . Then the following hold:*

1. *If $G \cup \mathcal{S}_{\preceq}(\vec{z} \upharpoonright \omega, \vec{z}, \vec{a}) \vdash \{f \upharpoonright \omega \leq f\} \cup \mathcal{O}_{\preceq}(\vec{z} \upharpoonright \omega, \vec{z}, \vec{a})$, then for each assignment α satisfying G , $\alpha \circ \omega \preceq_f \alpha$ holds.*
2. *If $G \cup \mathcal{S}_{\preceq}(\vec{z}, \vec{z} \upharpoonright \omega, \vec{a}) \cup \mathcal{O}_{\preceq}(\vec{z}, \vec{z} \upharpoonright \omega, \vec{a}) \vdash \perp$ holds, then for each assignment α satisfying G , $\alpha \not\preceq_f \alpha \circ \omega$ holds.*

Proof. 1. Let α be an assignment satisfying G . By Lemma 1, we can extend α to ρ such that $\mathcal{S}_{\preceq}(\vec{z} \upharpoonright \alpha \circ \omega, \vec{z} \upharpoonright \alpha, \vec{a} \upharpoonright \rho)$ is satisfied. This means that ρ satisfies the preconditions of

$$G \cup \mathcal{S}_{\preceq}(\vec{z} \upharpoonright \omega, \vec{z}, \vec{a}) \vdash \{f \upharpoonright \omega \leq f\} \cup \mathcal{O}_{\preceq}(\vec{z} \upharpoonright \omega, \vec{z}, \vec{a}), \quad (28)$$

so also $\mathcal{O}_{\preceq}(\vec{z} \upharpoonright \rho \circ \omega, \vec{z} \upharpoonright \rho, \vec{a} \upharpoonright \rho)$ holds. Since α and ρ coincide on \vec{z} (which holds since \vec{a} and \vec{z} are disjoint), it follows that

$\mathcal{O}_{\preceq}(\vec{z}|_{\alpha \circ \omega}, \vec{z}|_{\alpha}, \vec{a}|_{\rho})$ holds, which is sufficient to conclude that $\alpha \circ \omega \preceq \alpha$ holds. In addition, ρ satisfies $\{f|_{\omega} \leq f\}$. Since ρ and α coincide on the variables of f , this implies that $f|_{\alpha \circ \omega} \leq f|_{\alpha}$, so together with $\alpha \circ \omega \preceq \alpha$ we conclude that $\alpha \circ \omega \preceq_f \alpha$ holds, as required.

2. Let α be an assignment satisfying G . We argue by contradiction. Suppose that $\alpha \preceq \alpha \circ \omega$ holds. Then there exists an assignment ρ to \vec{a} such that

$$\mathcal{S}_{\preceq}(\vec{z}|_{\alpha}, \vec{z}|_{\alpha \circ \omega}, \vec{a}|_{\rho}) \cup \mathcal{O}_{\preceq}(\vec{z}|_{\alpha}, \vec{z}|_{\alpha \circ \omega}, \vec{a}|_{\rho}).$$

Define α' to coincide with ρ on \vec{a} , and with α otherwise. Since \vec{a} and \vec{z} are disjoint, α' then coincides with α on \vec{z} . Since $\vec{a} \cap \text{var}(G) = \emptyset$, α' also coincides with α on $\text{var}(G)$. Hence, α' satisfies G and

$$\mathcal{S}_{\preceq}(\vec{z}|_{\alpha'}, \vec{z}|_{\alpha' \circ \omega}, \vec{a}|_{\alpha'}) \cup \mathcal{O}_{\preceq}(\vec{z}|_{\alpha'}, \vec{z}|_{\alpha' \circ \omega}, \vec{a}|_{\alpha'}),$$

so α' satisfies the preconditions of $G \cup \mathcal{S}_{\preceq}(\vec{z}, \vec{z}|_{\omega}, \vec{a}) \cup \mathcal{O}_{\preceq}(\vec{z}, \vec{z}|_{\omega}, \vec{a}) \vdash \perp$, which is clearly contradictory. Hence, we have $\alpha \not\preceq_f \alpha \circ \omega$, which implies $\alpha \not\preceq_f \alpha \circ \omega$. \square

Hence, we define the dominance rule as follows:

Definition 9 (Dominance-based strengthening with specification). We can transition from the configuration $(\mathcal{C}, \mathcal{D}, \mathcal{O}_{\preceq}, \mathcal{S}_{\preceq}, \vec{z}, \vec{a}, v)$ to $(\mathcal{C}, \mathcal{D} \cup \{C\}, \mathcal{O}_{\preceq}, \mathcal{S}_{\preceq}, \vec{z}, \vec{a}, v)$ using the dominance rule if the following conditions are met:

1. The constraint C does not contain variables in \vec{a} .
2. There is a witness ω for which $\text{image}(\omega) \cap \vec{a} = \emptyset$ holds.
3. We have cutting planes proofs for the following:

$$\begin{aligned} &\mathcal{C} \cup \mathcal{D} \cup \{\neg C\} \cup \{f \leq v-1\} \cup \mathcal{S}_{\preceq}(\vec{z}|_{\omega}, \vec{z}, \vec{a}) \\ &\vdash \mathcal{C}|_{\omega} \cup \{f|_{\omega} \leq f\} \cup \mathcal{O}_{\preceq}(\vec{z}|_{\omega}, \vec{z}, \vec{a}), \end{aligned} \quad (29)$$

$$\begin{aligned} &\mathcal{C} \cup \mathcal{D} \cup \{\neg C\} \cup \{f \leq v-1\} \\ &\cup \mathcal{S}_{\preceq}(\vec{z}, \vec{z}|_{\omega}, \vec{a}) \cup \mathcal{O}_{\preceq}(\vec{z}, \vec{z}|_{\omega}, \vec{a}) \vdash \perp. \end{aligned} \quad (30)$$

Using Lemma 8, we can show that the dominance rule preserves the invariants required by weak validity:

Lemma 9. *If we can transition from $(\mathcal{C}, \mathcal{D}, \mathcal{O}_{\preceq}, \mathcal{S}_{\preceq}, \vec{z}, \vec{a}, v)$ to $(\mathcal{C}, \mathcal{D} \cup \{C\}, \mathcal{O}_{\preceq}, \mathcal{S}_{\preceq}, \vec{z}, \vec{a}, v)$ by the dominance rule, and the configuration $(\mathcal{C}, \mathcal{D}, \mathcal{O}_{\preceq}, \mathcal{S}_{\preceq}, \vec{z}, \vec{a}, v)$ is weakly- (F, f) -valid, then the configuration $(\mathcal{C}, \mathcal{D} \cup \{C\}, \mathcal{O}_{\preceq}, \mathcal{S}_{\preceq}, \vec{z}, \vec{a}, v)$ is also weakly- (F, f) -valid.*

Proof. Since F, f, \mathcal{C} , and v are not affected, item 1. in Definition 8 still holds. Hence, it remains to show that item 2. holds, i.e., that for every total assignment α that satisfies $\mathcal{C} \cup \{f \leq v-1\}$ there exists a total assignment $\alpha' \preceq_f \alpha$ that satisfies $\mathcal{C} \cup \mathcal{D} \cup \{C\} \cup \{f \leq v-1\}$.

Assume towards a contradiction that it does not hold. Let S denote the set of total assignments α that

1. satisfy $\mathcal{C} \cup \{f \leq v-1\}$ and
2. admit no $\alpha' \preceq_f \alpha$ satisfying $\mathcal{C} \cup \mathcal{D} \cup \{C\} \cup \{f \leq v-1\}$.

By our assumption, S is non-empty. Since \prec_f is a strict order and S is finite and non-empty, S contains a \prec_f -minimal element. Let α be some \prec_f -minimal assignment in S . Since $(\mathcal{C}, \mathcal{D}, \mathcal{O}_{\preceq}, \mathcal{S}_{\preceq}, \vec{z}, \vec{a}, v)$ is weakly- (F, f) -valid, there exists some $\alpha_1 \preceq_f \alpha$ that satisfies $\mathcal{C} \cup \mathcal{D} \cup \{f \leq v-1\}$. We know that α_1 cannot satisfy C since $\alpha \in S$. Hence, α_1 satisfies

$$G = \mathcal{C} \cup \mathcal{D} \cup \{\neg C\} \cup \{f \leq v-1\}.$$

Since G does not contain variables of \vec{a} , both points of Lemma 8 apply by (29) and (30), respectively. Hence, it follows that $\alpha_1 \circ \omega \preceq_f \alpha_1$ and $\alpha_1 \not\preceq_f \alpha_1 \circ \omega$ hold, which together imply that $\alpha_1 \circ \omega \prec_f \alpha_1$. Note that (29) also implies that α_1 satisfies $\mathcal{C}|_{\omega}$, so $\alpha_1 \circ \omega$ satisfies \mathcal{C} . Moreover, α_1 satisfies $f \leq v-1$ and $f|_{\omega} \leq f$, which together imply $f|_{\omega} \leq v-1$. Hence, $\alpha_1 \circ \omega$ satisfies $f \leq v-1$.

Let $\alpha_2 = \alpha_1 \circ \omega$. Then α_2 satisfies $\mathcal{C} \cup \{f \leq v-1\}$. Since $\alpha_2 \prec_f \alpha_1$ and $\alpha_1 \preceq_f \alpha$, we have $\alpha_2 \prec_f \alpha$. Since α is a \prec_f -minimal element of S , this implies that $\alpha_2 \notin S$. Hence, since α_2 does satisfy $\mathcal{C} \cup \{f \leq v-1\}$, it follows that α_2 does admit an $\alpha' \preceq_f \alpha_2$ satisfying $\mathcal{C} \cup \mathcal{D} \cup \{C\} \cup \{f \leq v-1\}$. But it also holds that $\alpha' \preceq_f \alpha_2 \prec_f \alpha$, so $\alpha' \preceq_f \alpha$. This contradicts $\alpha \in S$, finishing the proof. \square

B.5 Redundance-Based Strengthening Rule

We also modify the redundance rule to work in our extended proof system. Similarly to the dominance rule, we can use $\mathcal{S}_{\preceq}(\vec{z}|_{\omega}, \vec{z}, \vec{a})$ as an extra premise in our proof obligations.

Definition 10 (Redundance-based strengthening with specification). We can transition from $(\mathcal{C}, \mathcal{D}, \mathcal{O}_{\preceq}, \mathcal{S}_{\preceq}, \vec{z}, \vec{a}, v)$ to $(\mathcal{C}, \mathcal{D} \cup \{C\}, \mathcal{O}_{\preceq}, \mathcal{S}_{\preceq}, \vec{z}, \vec{a}, v)$ using the redundance rule if the following conditions are met:

1. The constraint C does not contain variables in \vec{a} .
2. There is a witness ω for which $\text{image}(\omega) \cap \vec{a} = \emptyset$ holds.
3. We have cutting planes proof that the following holds:

$$\begin{aligned} &\mathcal{C} \cup \mathcal{D} \cup \{\neg C\} \cup \{f \leq v-1\} \cup \mathcal{S}_{\preceq}(\vec{z}|_{\omega}, \vec{z}, \vec{a}) \\ &\vdash (\mathcal{C} \cup \mathcal{D} \cup \{C\})|_{\omega} \cup \{f|_{\omega} \leq f\} \cup \mathcal{O}_{\preceq}(\vec{z}|_{\omega}, \vec{z}, \vec{a}). \end{aligned} \quad (31)$$

The redundance rule preserves weak validity:

Lemma 10. *If we can transition from $(\mathcal{C}, \mathcal{D}, \mathcal{O}_{\preceq}, \mathcal{S}_{\preceq}, \vec{z}, \vec{a}, v)$ to $(\mathcal{C}, \mathcal{D} \cup \{C\}, \mathcal{O}_{\preceq}, \mathcal{S}_{\preceq}, \vec{z}, \vec{a}, v)$ by the redundance rule, and the configuration $(\mathcal{C}, \mathcal{D}, \mathcal{O}_{\preceq}, \mathcal{S}_{\preceq}, \vec{z}, \vec{a}, v)$ is weakly- (F, f) -valid, then the configuration $(\mathcal{C}, \mathcal{D} \cup \{C\}, \mathcal{O}_{\preceq}, \mathcal{S}_{\preceq}, \vec{z}, \vec{a}, v)$ is also weakly- (F, f) -valid.*

Proof. As for the dominance rule, F, f, \mathcal{C} , and v are not affected, so item 1. in Definition 8 still holds. Hence, it remains to show that item 2. holds, i.e., that for every total assignment α that satisfies $\mathcal{C} \cup \{f \leq v-1\}$ there exists a total assignment $\alpha' \preceq_f \alpha$ that satisfies

$$\mathcal{C} \cup \mathcal{D} \cup \{C\} \cup \{f \leq v-1\}.$$

Let α be an assignment that satisfies $\mathcal{C} \cup \{f \leq v-1\}$. Since $(\mathcal{C}, \mathcal{D}, \mathcal{O}_{\preceq}, \mathcal{S}_{\preceq}, \vec{z}, \vec{a}, v)$ is weakly- (F, f) -valid, there exists some $\alpha_1 \preceq_f \alpha$ that satisfies $\mathcal{C} \cup \mathcal{D} \cup \{f \leq v-1\}$. If α_1 also satisfies C , then we are done. Otherwise, α_1 satisfies

$$G = \mathcal{C} \cup \mathcal{D} \cup \{\neg C\} \cup \{f \leq v-1\}.$$

Since G does not contain variables of \vec{a} , point 1 of Lemma 8 apply by (31). Hence, it follows that $\alpha_1 \circ \omega \preceq_f \alpha_1$. Equation (31) also implies that α_1 satisfies $(\mathcal{C} \cup \mathcal{D} \cup \{C\})|_{\omega}$, so $\alpha_1 \circ \omega$ satisfies $\mathcal{C} \cup \mathcal{D} \cup \{C\}$. Finally, α_1 satisfies $f \leq v-1$ and $f|_{\omega} \leq f$, which together imply $f|_{\omega} \leq v-1$. Hence, $\alpha_1 \circ \omega$ satisfies $f \leq v-1$. Then $\alpha_1 \circ \omega \preceq_f \alpha_1 \preceq_f \alpha$ and $\alpha_1 \circ \omega$ satisfies $\mathcal{C} \cup \mathcal{D} \cup \{C\} \cup \{f \leq v-1\}$, so $\alpha_1 \circ \omega$ is an assignment showing that item 2. in Definition 8 holds for α . \square

C Proof Logging in Satsuma

In this appendix, we show how our proof system using auxiliary variables is used to log the symmetry-breaking clauses generated by SATSUMA. This is done in two steps:

1. Defining the lexicographical order using auxiliary variables and proving its transitivity and reflexivity.
2. Deriving the symmetry-breaking clauses using the dominance rule and the loaded lexicographical order.

C.1 Defining the Lexicographical Order

We first discuss how to define the lexicographical order on sequences of length n using auxiliary variables. In Lemma 11, we provide an encoding of the lexicographical order using auxiliary variables, and show that this indeed encodes the lexicographical order. In Lemma 12, we explain how to translate this encoding to pseudo-Boolean constraints. In Lemma 13, we then show that the reflexivity follows by reverse unit propagation (RUP), while Lemma 14 shows that transitivity can be shown using a cutting planes derivation of size $O(n)$. Finally, this implies Theorem 1, which states that we can define the lexicographical order in VERIPB using a derivation of size $O(n)$.

Throughout this section, $n \in \mathbb{N}^+$ denotes the length of the sequences over which we define lexicographical order, x_i and y_i are Boolean variables, and a_i and d_i are (auxiliary) Boolean variables.

Lemma 11. *Let x_1, \dots, x_n and y_1, \dots, y_n be given sequences of Boolean variables. Define the Boolean variables a_1, \dots, a_{n-1} and d_1, \dots, d_n inductively by*

$$a_1 \iff (x_1 \geq y_1), \quad (32)$$

$$a_{i+1} \iff (a_i \wedge x_{i+1} \geq y_{i+1}), \quad (33)$$

$$d_1 \iff (y_1 \geq x_1), \quad (34)$$

$$d_{i+1} \iff (d_i \wedge (\bar{a}_i \vee y_{i+1} \geq x_{i+1})). \quad (35)$$

Then d_n holds if and only if $(x_1, \dots, x_n) \preceq_{\text{lex}} (y_1, \dots, y_n)$.

Proof. We show by induction on $i \geq 1$ that d_i holds if and only if $(x_1, \dots, x_i) \preceq_{\text{lex}} (y_1, \dots, y_i)$ (for $i \leq n$) and that $d_i \wedge a_i$ holds if and only if $(x_1, \dots, x_i) = (y_1, \dots, y_i)$ (for $i \leq n-1$). The base case is immediate from the definitions of a_1 and d_1 .

We proceed with the induction step. Note that we have to show two equivalences, so four implications.

d_{i+1} **implies** $(x_1, \dots, x_{i+1}) \preceq_{\text{lex}} (y_1, \dots, y_{i+1})$. Suppose that d_{i+1} holds. Then (35) implies that d_i and $\bar{a}_i \vee y_{i+1} \geq x_{i+1}$ hold. Since d_i holds, the induction hypothesis implies that

$$(x_1, \dots, x_i) \preceq_{\text{lex}} (y_1, \dots, y_i). \quad (36)$$

We split cases using $\bar{a}_i \vee y_{i+1} \geq x_{i+1}$:

- If \bar{a}_i , then $d_i \wedge a_i$ does not hold, so the induction hypothesis implies $(x_1, \dots, x_i) \neq (y_1, \dots, y_i)$, which combined with (36) implies $(x_1, \dots, x_{i+1}) \preceq_{\text{lex}} (y_1, \dots, y_{i+1})$.
- If $y_{i+1} \geq x_{i+1}$, then (36) also implies $(x_1, \dots, x_{i+1}) \preceq_{\text{lex}} (y_1, \dots, y_{i+1})$.

Hence, we have $(x_1, \dots, x_{i+1}) \preceq_{\text{lex}} (y_1, \dots, y_{i+1})$.

$(x_1, \dots, x_{i+1}) \preceq_{\text{lex}} (y_1, \dots, y_{i+1})$ **implies** d_{i+1} . Conversely, suppose that $(x_1, \dots, x_{i+1}) \preceq_{\text{lex}} (y_1, \dots, y_{i+1})$. Then also $(x_1, \dots, x_i) \preceq_{\text{lex}} (y_1, \dots, y_i)$, so the induction hypothesis implies that d_i holds. We consider two cases:

- If $(x_1, \dots, x_i) \neq (y_1, \dots, y_i)$, then $d_i \wedge a_i$ does not hold (by the induction hypothesis), so \bar{a}_i holds (since d_i holds).
- If $(x_1, \dots, x_i) = (y_1, \dots, y_i)$, then $(x_1, \dots, x_{i+1}) \preceq_{\text{lex}} (y_1, \dots, y_{i+1})$ implies $y_{i+1} \geq x_{i+1}$.

Hence, $\bar{a}_i \vee y_{i+1} \geq x_{i+1}$ holds, which implies that d_{i+1} holds.

$d_{i+1} \wedge a_{i+1}$ **implies** $(x_1, \dots, x_{i+1}) = (y_1, \dots, y_{i+1})$. Suppose that $d_{i+1} \wedge a_{i+1}$ holds. Then also $d_i \wedge a_i$ holds (by (33) and (35)), so the induction hypothesis implies that $(x_1, \dots, x_i) = (y_1, \dots, y_i)$. It remains to show that $x_{i+1} = y_{i+1}$, which we show by showing both inequalities:

- a_{i+1} implies $x_{i+1} \geq y_{i+1}$ (by (33)).
- d_{i+1} implies $\bar{a}_i \vee y_{i+1} \geq x_{i+1}$ (by (35)), which using that a_i holds implies $y_{i+1} \geq x_{i+1}$.

Hence, we conclude that $(x_1, \dots, x_{i+1}) = (y_1, \dots, y_{i+1})$.

$(x_1, \dots, x_{i+1}) = (y_1, \dots, y_{i+1})$ **implies** $d_{i+1} \wedge a_{i+1}$. Conversely, suppose that $(x_1, \dots, x_{i+1}) = (y_1, \dots, y_{i+1})$. Then also $(x_1, \dots, x_i) = (y_1, \dots, y_i)$, so the induction hypothesis implies that $d_i \wedge a_i$ holds. Moreover, we have $x_{i+1} = y_{i+1}$. Then $y_{i+1} \geq x_{i+1}$ together with d_i implies d_{i+1} (by (35)), while $x_{i+1} \geq y_{i+1}$ together with a_i implies a_{i+1} (by (33)). We conclude that $d_{i+1} \wedge a_{i+1}$ holds.

This completes the induction and hence the proof. \square

The constraints given in Lemma 11 are not written as pseudo-Boolean constraints, but they can be expressed using pseudo-Boolean constraints. For this, note that the constraint $r \iff C$, where r is a Boolean variable and $C \doteq \sum_i p_i x_i \geq P$ is a pseudo-Boolean constraint in normalized form, can be expressed using the pair of pseudo-Boolean constraints $P\bar{r} + \sum_i p_i x_i \geq P$ and $(\sum_i p_i - P + 1)r + \sum_i p_i \bar{x}_i \geq \sum_i p_i - P + 1$. We call a constraint of the form $r \iff C$ a *reification*, and the process of rewriting a reification to a pair of pseudo-Boolean constraints *expanding the reification*.

Lemma 12. *We can write the constraints (32) up to (35) as follows as pseudo-Boolean constraints:*

$$\bar{a}_1 + x_1 + \bar{y}_1 \geq 1, \quad (37)$$

$$2a_1 + \bar{x}_1 + y_1 \geq 2, \quad (38)$$

$$3\bar{a}_{i+1} + 2a_i + x_{i+1} + \bar{y}_{i+1} \geq 3, \quad (39)$$

$$2a_{i+1} + 2\bar{a}_i + \bar{x}_{i+1} + y_{i+1} \geq 2, \quad (40)$$

$$\bar{d}_1 + y_1 + \bar{x}_1 \geq 1, \quad (41)$$

$$2d_1 + \bar{y}_1 + x_1 \geq 2, \quad (42)$$

$$4\bar{d}_{i+1} + 3d_i + \bar{a}_i + y_{i+1} + \bar{x}_{i+1} \geq 4, \quad (43)$$

$$4d_{i+1} + 3\bar{d}_i + a_i + \bar{y}_{i+1} + x_{i+1} \geq 3. \quad (44)$$

Proof. For constraint (32), we rewrite $x_1 \geq y_1$ as $x_1 + \bar{y}_1 \geq 1$, and then expand the reification.

For constraint (33), we note that $a_i \wedge x_{i+1} \geq y_{i+1}$ is equivalent to $a_i \wedge x_{i+1} + \bar{y}_{i+1} \geq 1$, which in turn is equivalent to $2a_i + x_{i+1} + \bar{y}_{i+1} \geq 3$. Then we expand the reification.

For constraint (34), we rewrite $y_1 \geq x_1$ as $y_1 + \bar{x}_1 \geq 1$, and then expand the reification.

For constraint (35), we first note that $\bar{a}_i \vee y_{i+1} \geq x_{i+1}$ is equivalent to $\bar{a}_i \vee y_{i+1} + \bar{x}_{i+1} \geq 1$, which is in turn equivalent to $\bar{a}_i + y_{i+1} + \bar{x}_{i+1} \geq 1$. Hence, $d_i \wedge (\bar{a}_i \vee y_{i+1} \geq x_{i+1})$ is equivalent to the pseudo-Boolean constraint $3d_i + \bar{a}_i + y_{i+1} + \bar{x}_{i+1} \geq 4$. Finally, we expand the reification. \square

We write $\mathcal{S}_{\preceq}(\vec{x}, \vec{y}, \vec{a}, \vec{d})$ for the constraints (37) up to (44) in Lemma 12. Note that \mathcal{S}_{\preceq} has four arguments, rather than three as in Section 3.2, since we use two different symbols for the auxiliary variables to emphasize their different functions. We write $\mathcal{O}_{\preceq}(\vec{d})$ for the single constraint $d_n \geq 1$. Throughout the remainder of this appendix, we write \preceq for the preorder with the specification $\mathcal{S}_{\preceq}(\vec{x}, \vec{y}, \vec{a}, \vec{d})$ and definition $\mathcal{O}_{\preceq}(\vec{d})$. Then Lemma 11 shows that the preorder \preceq defined by \mathcal{S}_{\preceq} and \mathcal{O}_{\preceq} is indeed the lexicographical order.

Next, we show how we can prove reflexivity and transitivity of this order in VERIPB.

Lemma 13. *We can prove reflexivity of the order \preceq using a single RUP step, requiring $O(n)$ propagations.*

Proof. Since proof goals are always shown by contradiction in VERIPB, we assume that $d_n \geq 1$ does not hold, i.e., that $\bar{d}_n \geq 1$ holds. For the reflexivity proof, we have access to the specification $\mathcal{S}_{\preceq}(\vec{x}, \vec{x}, \vec{a}, \vec{d})$. Under the substitution $y_i = x_i$, the constraints (42) and (44) simplify to

$$2d_1 \geq 1, \quad (45)$$

$$4d_{i+1} + 3\bar{d}_i + a_i \geq 2. \quad (46)$$

Then (45) propagates d_1 to 1, after which (46) inductively propagates d_{i+1} to 1 (for $1 \leq i \leq n-1$). After deriving d_n , we obtain a contradiction with $\bar{d}_n \geq 1$. \square

Lemma 14. *We can prove transitivity of the order \preceq using a cutting planes derivation of size $O(n)$.*

Proof. Let $\mathcal{S}_{\preceq}(\vec{x}, \vec{y}, \vec{a}, \vec{d})$, $\mathcal{S}_{\preceq}(\vec{y}, \vec{z}, \vec{b}, \vec{e})$ and $\mathcal{S}_{\preceq}(\vec{x}, \vec{z}, \vec{c}, \vec{f})$ be the specifications corresponding to the relations $(x_1, \dots, x_n) \preceq (y_1, \dots, y_n)$, $(y_1, \dots, y_n) \preceq (z_1, \dots, z_n)$, and $(x_1, \dots, x_n) \preceq (z_1, \dots, z_n)$, respectively. We also assume that $\mathcal{O}_{\preceq}(\vec{d})$ and $\mathcal{O}_{\preceq}(\vec{e})$ hold, which are the constraints $d_n \geq 1$ and $e_n \geq 1$. From this, we need to show that $\mathcal{O}_{\preceq}(\vec{f})$ holds, which is the constraint $f_n \geq 1$.

Overview. The proof consists of two main steps:

1. Deriving inductively that $d_i \geq 1$ for $1 \leq i \leq n$, and simplifying constraints (41) and (43) to account for the known values of the d_i -variables. Similarly, deriving that $e_i \geq 1$ for $1 \leq i \leq n$, and simplifying constraints using known values of the e_i -variables.
2. Deriving inductively that $f_i \geq 1$ for $1 \leq i \leq n$, for which we also inductively derive that c_i implies a_i and b_i . The intuition for why c_i implies a_i and b_i is as follows. We know that $(x_1, \dots, x_i) \preceq (y_1, \dots, y_i) \preceq (z_1, \dots, z_i)$, and c_i encodes that $(x_1, \dots, x_i) = (z_1, \dots, z_i)$. However, since (y_1, \dots, y_i) is ‘squeezed’ in between (x_1, \dots, x_i)

and (z_1, \dots, z_i) , this equality means that also the equalities $(x_1, \dots, x_i) = (y_1, \dots, y_i)$ and $(y_1, \dots, y_i) = (z_1, \dots, z_i)$ hold, which is in turn encoded by a_i and b_i .

Since we use a constant number of cutting planes steps in the inductive step of these inductive derivations, the total number of cutting planes steps is $O(n)$. Since each constraint has bounded size, the size of the derivation is also $O(n)$.

Deriving $d_i \geq 1$ and simplifying constraints. We start by deriving that all d_i hold and simplify the constraints containing the d_i . Weakening constraint (43) on a_i , y_{i+1} and x_i yields $4\bar{d}_{i+1} + 3d_i \geq 1$. Hence, from $d_n \geq 1$ we can inductively prove $d_i \geq 1$ for $1 \leq i \leq n$. This in turn allows us to derive the constraints $\bar{a}_i + y_{i+1} + \bar{x}_{i+1} \geq 1$ (by adding $4 \cdot (d_{i+1} \geq 1)$ to (43) and weakening on d_i), and the constraint $y_1 + \bar{x}_1 \geq 1$. Similarly, we derive $e_i \geq 1$, the constraints $\bar{b}_i + z_{i+1} + \bar{y}_{i+1} \geq 1$, and the constraint $z_1 + \bar{y}_1 \geq 1$.

Deriving $f_i \geq 1$ and c_i implies a_i and b_i . We inductively show that $f_i \geq 1$, and that c_i implies a_i and b_i , i.e., that $a_i + \bar{c}_i \geq 1$ and $b_i + \bar{c}_i \geq 1$. For the base case, adding $y_1 + \bar{x}_1 \geq 1$ and $z_1 + \bar{y}_1 \geq 1$ yields $z_1 + \bar{x}_1 \geq 1$, which in turn implies that $f_1 \geq 1$. Adding the three constraints

$$\begin{aligned} 2a_1 + \bar{x}_1 + y_1 &\geq 2, \\ \bar{c}_1 + x_1 + \bar{z}_1 &\geq 1, \\ z_1 + \bar{y}_1 &\geq 1, \end{aligned}$$

and saturating yields $a_1 + \bar{c}_1 \geq 1$. Similarly, we derive the implication $b_1 + \bar{c}_1 \geq 1$, which completes the base case.

We proceed with the inductive step. We first derive that f_{i+1} holds. Adding the constraints

$$\begin{aligned} \bar{a}_i + y_{i+1} + \bar{x}_{i+1} &\geq 1, \\ \bar{b}_i + z_{i+1} + \bar{y}_{i+1} &\geq 1, \\ a_i + \bar{c}_i &\geq 1, \\ b_i + \bar{c}_i &\geq 1, \end{aligned}$$

and saturating, we get $\bar{c}_i + z_{i+1} + \bar{x}_{i+1} \geq 1$. Adding this constraint, $3 \cdot (f_i \geq 1)$ and $3f_{i+1} + 3\bar{f}_i + c_i + \bar{z}_{i+1} + x_{i+1} \geq 3$ and saturating then allows us to derive $f_{i+1} \geq 1$.

Finally, we have to show that c_{i+1} implies a_{i+1} and b_{i+1} . First note that $c_{i+1} \Rightarrow (c_i \wedge x_{i+1} \geq z_{i+1})$ implies the two constraints $c_{i+1} \Rightarrow c_i$ and $c_{i+1} \Rightarrow (x_{i+1} \geq z_{i+1})$. Indeed, weakening $3\bar{c}_{i+1} + 2c_i + x_{i+1} + \bar{z}_{i+1} \geq 3$ on x_{i+1} and z_{i+1} and saturating yields $\bar{c}_{i+1} + c_i \geq 1$, while weakening on c_i and saturating yields $\bar{c}_{i+1} + x_{i+1} + \bar{z}_{i+1} \geq 1$. Adding $\bar{c}_{i+1} + c_i \geq 1$ to $a_i + \bar{c}_i \geq 1$ and $b_i + \bar{c}_i \geq 1$ respectively yields $a_i + \bar{c}_{i+1} \geq 1$ and $b_i + \bar{c}_{i+1} \geq 1$. Adding the constraints

$$\begin{aligned} \bar{c}_{i+1} + x_{i+1} + \bar{z}_{i+1} &\geq 1, \\ 2a_{i+1} + 2\bar{a}_i + \bar{x}_{i+1} + y_{i+1} &\geq 2, \\ \bar{b}_i + z_{i+1} + \bar{y}_{i+1} &\geq 1, \end{aligned}$$

and saturating yields $\bar{c}_{i+1} + a_{i+1} + \bar{a}_i + \bar{b}_i \geq 1$. Then adding $a_i + \bar{c}_{i+1} \geq 1$ and $b_i + \bar{c}_{i+1} \geq 1$ and saturating yields $\bar{c}_{i+1} + a_{i+1} \geq 1$. We similarly derive $\bar{c}_{i+1} + b_{i+1} \geq 1$, which completes the inductive step and hence the proof. \square

Together, this shows the following result:

Theorem 1. *We can define the lexicographical order over n variables in VERIPB using auxiliary variables and prove its transitivity and reflexivity using a VERIPB proof of size $O(n)$ that can be checked in time $O(n)$.*

C.2 Deriving the Symmetry-Breaking Clauses

Next, we show how to use our proof system in applications of the dominance rule where the witness is a symmetry σ of the input formula. Let $\text{supp}(\sigma) = \{x_{i_1}, \dots, x_{i_k}\}$ be the support of σ (where $i_1 \leq \dots \leq i_k$). We want to derive symmetry breaking clauses encoding the lex-leader constraint

$$(x_{i_1}, \dots, x_{i_k}) \preceq_{\text{lex}} (\sigma(x_{i_1}), \dots, \sigma(x_{i_k})).$$

Throughout this appendix, we assume that the lexicographical order \preceq over n variables is loaded. We first show how to define a circuit over variables s_l and t_l defining the lexicographical order over just $\text{supp}(\sigma)$. Then we show how to do the actual symmetry breaking using the dominance rule. In particular, Lemma 15 and Lemma 16 show how we can prove the order proof goals corresponding to the dominance rule application. Lemma 17 then shows how to derive the symmetry breaking clauses. Together, this leads to Theorem 2, which shows that we can justify a symmetry σ of the input formula with $k = |\text{supp}(\sigma)|$ using a VERIPB proof of size $O(k)$ that can be checked in time $O(n)$.

Defining a circuit. We first define at top level (using the redundancy rule) a circuit similarly to the circuit defined in Lemma 11, using the following equations:

$$s_1 \Leftrightarrow (x_{i_1} \geq \sigma(x_{i_1})), \quad (47)$$

$$s_{l+1} \Leftrightarrow (s_l \wedge x_{i_{l+1}} \geq \sigma(x_{i_{l+1}})), \quad (48)$$

$$t_1 \Leftrightarrow (\sigma(x_{i_1}) \geq x_{i_1}), \quad (49)$$

$$t_{l+1} \Leftrightarrow (t_l \wedge (\overline{s_l} \vee \sigma(x_{i_{l+1}}) \geq x_{i_{l+1}})), \quad (50)$$

where we have constraint (48) for $1 \leq l \leq k-2$ and constraint (50) for $1 \leq l \leq k-1$.

Intuitively, the variables s_l and t_l have the same role as the auxiliary variables a_{i_l} and d_{i_l} in the specification, respectively. As in the proof of Lemma 11, t_l hold if and only if $(x_{i_1}, \dots, x_{i_l}) \preceq_{\text{lex}} (\sigma(x_{i_1}), \dots, \sigma(x_{i_l}))$, while $s_l \wedge t_l$ holds if and only if $(x_{i_1}, \dots, x_{i_l}) = (\sigma(x_{i_1}), \dots, \sigma(x_{i_l}))$.

As in Lemma 12, we can write these constraints as follows as pseudo-Boolean constraints:

$$\overline{s_1} + x_{i_1} + \overline{\sigma(x_{i_1})} \geq 1, \quad (51)$$

$$2s_1 + \overline{x_{i_1}} + \sigma(x_{i_1}) \geq 2, \quad (52)$$

$$3\overline{s_{l+1}} + 2s_l + x_{i_{l+1}} + \overline{\sigma(x_{i_{l+1}})} \geq 3, \quad (53)$$

$$2s_{l+1} + 2\overline{s_l} + \overline{x_{i_{l+1}}} + \sigma(x_{i_{l+1}}) \geq 2, \quad (54)$$

$$\overline{t_1} + \sigma(x_{i_1}) + \overline{x_{i_1}} \geq 1, \quad (55)$$

$$2t_1 + \overline{\sigma(x_{i_1})} + x_{i_1} \geq 2, \quad (56)$$

$$4\overline{t_{l+1}} + 3t_l + \overline{s_l} + \sigma(x_{i_{l+1}}) + \overline{x_{i_{l+1}}} \geq 4, \quad (57)$$

$$3t_{l+1} + 3\overline{t_l} + s_l + \overline{\sigma(x_{i_{l+1}})} + x_{i_{l+1}} \geq 3. \quad (58)$$

In total, this circuit consists of $4k-2$ pseudo-Boolean constraints, and can be derived using the redundancy rule in time $O(1)$ per constraint. Note that for this the autoproving of the

order proof goal of the redundancy rule and the explicit loading of the specification discussed in Section 5 are essential to avoid a factor n overhead. Namely, these applications of the redundancy rule witness over fresh variables only, so in particular not over the variables over which the order is loaded. Hence, the order proof goal follows directly from the reflexivity of the order, and these checker improvements allow us to detect this in time $O(1)$, instead of wasting time $O(n)$ on loading the specification and repeating the reflexivity proof.

Overview: the symmetry breaking clauses. We first state the symmetry breaking clauses:

$$s_1 + \overline{x_{i_1}} \geq 1, \quad (59)$$

$$s_{l+1} + \overline{s_l} + \overline{x_{i_{l+1}}} \geq 1, \quad (60)$$

$$s_1 + \sigma(x_{i_1}) \geq 1, \quad (61)$$

$$s_{l+1} + \overline{s_l} + \sigma(x_{i_{l+1}}) \geq 1, \quad (62)$$

$$\sigma(x_{i_1}) + \overline{x_{i_1}} \geq 1, \quad (63)$$

$$\overline{s_l} + \sigma(x_{i_{l+1}}) + \overline{x_{i_{l+1}}} \geq 1. \quad (64)$$

Note that a constraint whose coefficients and whose right hand side is 1 corresponds to a clause: for example, the constraint $s_1 + \overline{x_{i_1}} \geq 1$ is equivalent to the clause $s_1 \vee \overline{x_{i_1}}$.

In the symmetry breaking clauses, the variable s_l corresponds to the equality $(x_{i_1}, \dots, x_{i_l}) = (\sigma(x_{i_1}), \dots, \sigma(x_{i_l}))$. Clauses (59) up to (62) ensure that s_l propagates to true if this equality holds, while (63) and (64) provide the symmetry breaking constraint $\sigma(x_{i_1}) \geq x_{i_1}$ and that $\sigma(x_{i_{l+1}}) \geq x_{i_{l+1}}$ should hold if $(x_{i_1}, \dots, x_{i_l}) = (\sigma(x_{i_1}), \dots, \sigma(x_{i_l}))$.

Intuitively, to break a symmetry we will show using dominance that t_k and hence all t_l hold, and then we derive these clauses from the circuit.

Deriving the symmetry breaking clauses. Using the dominance rule, we now derive that $t_k \geq 1$. Let $C \doteq t_k \geq 1$. To apply the dominance rule, we have to show that

$$\mathcal{C} \cup \mathcal{D} \cup \{\neg C\} \cup \mathcal{S}_{\preceq}(\vec{x} \upharpoonright_{\sigma}, \vec{x}, \vec{a}, \vec{d}) \vdash \mathcal{C} \upharpoonright_{\sigma} \cup \mathcal{O}_{\preceq}(\vec{d}), \quad (65)$$

$$\mathcal{C} \cup \mathcal{D} \cup \{\neg C\} \cup \mathcal{S}_{\preceq}(\vec{x}, \vec{x} \upharpoonright_{\sigma}, \vec{a}, \vec{d}) \cup \mathcal{O}_{\preceq}(\vec{d}) \vdash \perp, \quad (66)$$

Since the witness σ is a symmetry of the formula, the proof goals corresponding to $\mathcal{C} \upharpoonright_{\sigma}$ follow directly from $\mathcal{C} \upharpoonright_{\sigma} = \mathcal{C}$. The next two lemmas explain how to prove the other two proof goals.

Note that within the subproof of the dominance rule, we can use the negation $\neg C \doteq \overline{t_k} \geq 1$, which intuitively means that $(x_{i_1}, \dots, x_{i_k}) \not\preceq_{\text{lex}} (\sigma(x_{i_1}), \dots, \sigma(x_{i_k}))$.

Lemma 15. *Assume that the lexicographical order \preceq over n variables is loaded. Let σ be a symmetry of the input formula and let $k = |\text{supp}(\sigma)|$. Then we can show the proof goal*

$$\mathcal{C} \cup \mathcal{D} \cup \{\neg C\} \cup \mathcal{S}_{\preceq}(\vec{x} \upharpoonright_{\sigma}, \vec{x}, \vec{a}, \vec{d}) \vdash \mathcal{O}_{\preceq}(\vec{d}),$$

using $O(k)$ RUP steps and cutting planes steps, where the RUP steps require $O(n)$ propagations in total.

Proof. Note that $\mathcal{O}_{\preceq}(\vec{d})$ is the single constraint $d_n \geq 1$. Since proof goals are always shown by contradiction, we assume that $\overline{d_n} \geq 1$ holds. This means that $(\sigma(x_{i_1}), \dots, \sigma(x_{i_k})) \not\preceq_{\text{lex}} (x_{i_1}, \dots, x_{i_k})$, but since \preceq_{lex} is a complete order and we also have $(x_{i_1}, \dots, x_{i_k}) \not\preceq_{\text{lex}} (\sigma(x_{i_1}), \dots, \sigma(x_{i_k}))$ via the constraint $\neg C \doteq \overline{t_k} \geq 1$, we should be able to derive contradiction.

Overview. The proof consists of four main steps:

1. Rewriting the specification $\mathcal{S}_\leq(x \upharpoonright_\sigma, x, a, d)$ to only $O(k)$ constraints involving auxiliary variables a_i and d_i and variables from $\text{supp}(\sigma)$ only.
2. Showing that $s_l \Rightarrow d_{i_l}$ and $a_{i_l} \Rightarrow t_l$.
3. Showing that $t_{l+1} + \bar{t}_l + d_{i_l} \geq 1$ and $d_{i_{l+1}} + \bar{d}_{i_l} + t_l \geq 1$.
4. Showing $d_{i_l} + t_l \geq 1$ and $t_{l+1} + d_{i_l} \geq 1$ and $d_{i_{l+1}} + t_l \geq 1$.

The first of these steps requires $O(k)$ RUP steps for which $O(n)$ propagations are needed in total, and $O(k)$ cutting planes steps. The remaining steps require $O(k)$ RUP steps for which $O(k)$ propagations are needed.

Rewriting the specification constraints. Write $S = \{i_1, \dots, i_k\}$. There are two cases which are slightly different, depending on whether $1 \in S$ or not. Here we consider the case $1 \notin S$. Under the substitution σ , the specification constraints $\mathcal{S}_\leq(x \upharpoonright_\sigma, x, a, d)$ are (semantically) given by

$$a_1 \geq 1, \quad (67)$$

$$a_i \Leftrightarrow (a_{i-1} \wedge \sigma(x_i) \geq x_i), \quad (i \in S) \quad (68)$$

$$a_i \Leftrightarrow a_{i-1}, \quad (i \notin S) \quad (69)$$

$$d_1 \geq 1, \quad (70)$$

$$d_i \Leftrightarrow (d_{i-1} \wedge (\bar{a}_{i-1} \vee x_i \geq \sigma(x_i))), \quad (i \in S) \quad (71)$$

$$d_i \Leftrightarrow d_{i-1}. \quad (i \notin S) \quad (72)$$

Using the constraints (67) and (69), it follows by RUP that $a_{i_{l-1}} \geq 1$ and $a_{i_{l-1}} \Leftrightarrow a_{i_{l-1}}$ for $2 \leq l \leq k$. Similarly, using the constraints (70) and (72), it follows by RUP that $d_{i_{l-1}} \geq 1$ and $d_{i_{l-1}} \Leftrightarrow d_{i_{l-1}}$ for $2 \leq l \leq k$.

Using these constraints, we can (using cutting planes steps) rewrite constraints (68) and (71) as

$$a_{i_l} \Leftrightarrow (\sigma(x_{i_l}) \geq x_{i_l}), \quad (73)$$

$$a_{i_{l+1}} \Leftrightarrow (a_{i_l} \wedge \sigma(x_{i_{l+1}}) \geq x_{i_{l+1}}), \quad (74)$$

$$d_{i_l} \Leftrightarrow (x_{i_l} \geq \sigma(x_{i_l})), \quad (75)$$

$$d_{i_{l+1}} \Leftrightarrow (d_{i_l} \wedge (\bar{a}_{i_l} \vee x_{i_{l+1}} \geq \sigma(x_{i_{l+1}}))). \quad (76)$$

Showing that $s_l \Rightarrow d_{i_l}$ and $a_{i_l} \Rightarrow t_l$. We inductively derive using RUP that $s_l \Rightarrow d_{i_l}$, i.e., $\bar{s}_l + d_{i_l} \geq 1$. When showing this by RUP, the negation propagates s_l and \bar{d}_{i_l} .

For the base case, note that \bar{d}_{i_1} propagates that the inequality $x_{i_1} \geq \sigma(x_{i_1})$ is false (i.e., that \bar{x}_{i_1} and $\sigma(x_{i_1})$ hold), which yields together with s_1 a conflict in (51).

For the inductive step, note that s_{l+1} propagates s_l using (53), which by the induction hypothesis propagates d_{i_l} . Then $\bar{d}_{i_{l+1}}$ propagates that $\bar{a}_{i_l} \vee (x_{i_{l+1}} \geq \sigma(x_{i_{l+1}}))$ is false, which in particular propagates that $x_{i_{l+1}} \geq \sigma(x_{i_{l+1}})$ is false (i.e., that $\bar{x}_{i_{l+1}}$ and $\sigma(x_{i_{l+1}})$ hold), which yields together with s_{l+1} a conflict in (53).

In the same way, we derive using RUP that $a_{i_l} \Rightarrow t_l$.

Showing that $t_{l+1} + \bar{t}_l + d_{i_l} \geq 1$ and $d_{i_{l+1}} + \bar{d}_{i_l} + t_l \geq 1$. Next, we derive using RUP that $t_{l+1} + \bar{t}_l + d_{i_l} \geq 1$. Its negation propagates \bar{t}_{l+1} , t_l , and \bar{d}_{i_l} . This propagates s_l using (58). But then we have s_l and \bar{d}_{i_l} , which is a conflict in $s_l \Rightarrow d_{i_l}$. In the same way, we derive using RUP that $d_{i_{l+1}} + \bar{d}_{i_l} + t_l \geq 1$.

Showing $d_{i_l} + t_l \geq 1$ and $t_{l+1} + d_{i_l} \geq 1$ and $d_{i_{l+1}} + t_l \geq 1$. Next, we derive using RUP inductively that $d_{i_l} + t_l \geq 1$ and $t_{l+1} + d_{i_l} \geq 1$ and $d_{i_{l+1}} + t_l \geq 1$. The negated constraint propagates \bar{d}_{i_l} and \bar{t}_l . For the base case, note that \bar{d}_{i_1} propagates that the inequality $x_{i_1} \geq \sigma(x_{i_1})$ is false, which yields together with \bar{t}_1 a conflict in (56).

For the inductive step, we first show how to derive $t_{l+1} + d_{i_l} \geq 1$ and $d_{i_{l+1}} + t_l \geq 1$ using $d_{i_l} + t_l \geq 1$. The negation of $t_{l+1} + d_{i_l} \geq 1$ propagates \bar{t}_{l+1} and \bar{d}_{i_l} , which propagates t_l using $t_l + d_{i_l} \geq 1$, which then yields a conflict in the constraint $t_{l+1} + \bar{t}_l + d_{i_l} \geq 1$.

The derivation of $d_{i_{l+1}} + t_l \geq 1$ is similar.

Next, we show how to derive $d_{i_{l+1}} + t_{l+1} \geq 1$. The negation of $d_{i_{l+1}} + t_{l+1} \geq 1$ propagates \bar{t}_{l+1} and $\bar{d}_{i_{l+1}}$, which propagates t_l and d_{i_l} using $t_{l+1} + d_{i_l} \geq 1$ and $d_{i_{l+1}} + t_l \geq 1$. Then $\bar{d}_{i_{l+1}}$ and d_{i_l} propagate that $x_{i_{l+1}} \geq \sigma(x_{i_{l+1}})$ is false, but \bar{t}_{l+1} and t_l propagate that $\sigma(x_{i_{l+1}}) \geq x_{i_{l+1}}$ is false, which is a conflict. Hence, we derive $d_{i_{l+1}} + t_{l+1} \geq 1$ by RUP.

Finally, using $d_{i_k} + t_k \geq 1$, constraints (72) which yield $d_{i_k} \Leftrightarrow d_n$, and the two constraints $\bar{d}_n \geq 1$ and $\bar{t}_k \geq 1$, we propagate to conflict, which completes the proof. \square

Alternative derivation using cutting planes steps. For step 2, 3 and 4 in the proof of Lemma 15, we can also provide a derivation using cutting planes steps instead.

Showing that $s_l \Rightarrow d_{i_l}$ and $a_{i_l} \Rightarrow t_l$. We show this by induction. For the base case, adding constraint (51), i.e., $\bar{s}_1 + x_{i_1} + \sigma(x_{i_1}) \geq 1$, and $2d_{i_1} + \bar{x}_{i_1} + \sigma(x_{i_1}) \geq 2$ (from (75)), and saturating yields $\bar{s}_1 + d_{i_1} \geq 1$.

For the inductive step, we first add

$$3 \cdot (3\bar{s}_{l+1} + 2s_l + x_{i_{l+1}} + \sigma(x_{i_{l+1}}) \geq 3),$$

$$2 \cdot (3d_{i_{l+1}} + 3\bar{d}_{i_l} + a_{i_l} + \sigma(x_{i_{l+1}}) + x_{i_{l+1}} \geq 3),$$

and then weaken on a_{i_l} , $\sigma(x_{i_{l+1}})$, and $x_{i_{l+1}}$, which yields

$$9\bar{s}_{l+1} + 6s_l + 6d_{i_{l+1}} + 6\bar{d}_{i_l} \geq 7.$$

Then we add $6 \cdot (\bar{s}_l + d_{i_l} \geq 1)$ and saturate, which finally yields the constraint $\bar{s}_{l+1} + d_{i_{l+1}} \geq 1$.

The derivation of $\bar{a}_{i_l} + t_l \geq 1$ is similar.

Showing that $t_{l+1} + \bar{t}_l + d_{i_l} \geq 1$ and $d_{i_{l+1}} + \bar{d}_{i_l} + t_l \geq 1$. To show the constraint $t_{l+1} + \bar{t}_l + d_{i_l} \geq 1$, we start from constraint (58), i.e.,

$$3t_{l+1} + 3\bar{t}_l + s_l + \sigma(x_{i_{l+1}}) + x_{i_{l+1}} \geq 3,$$

add $\bar{s}_l + d_{i_l} \geq 1$, weaken on $\sigma(x_{i_{l+1}})$ and $x_{i_{l+1}}$, and then saturate. The derivation of $d_{i_{l+1}} + \bar{d}_{i_l} + t_l \geq 1$ is similar.

Showing $d_{i_l} + t_l \geq 1$ and $t_{l+1} + d_{i_l} \geq 1$ and $d_{i_{l+1}} + t_l \geq 1$. We show this by induction. For the base case, adding constraint (56), i.e., $2t_{i_1} + x_{i_1} + \sigma(x_{i_1}) \geq 2$, and $2d_{i_1} + \bar{x}_{i_1} + \sigma(x_{i_1}) \geq 2$, and dividing by 2 yields $\bar{s}_1 + d_{i_1} \geq 1$.

For the inductive step, note that adding $d_{i_l} + t_l \geq 1$ to $t_{l+1} + \bar{t}_l + d_{i_l} \geq 1$ and saturating yields $t_{l+1} + d_{i_l} \geq 1$.

Similarly, adding $d_{i_l} + t_l \geq 1$ to $d_{i_{l+1}} + \bar{d}_{i_l} + t_l \geq 1$ and saturating yields $d_{i_{l+1}} + t_l \geq 1$.

To derive $d_{i_{l+1}} + t_{l+1} \geq 1$, we start by adding

$$\begin{aligned} 3t_{l+1} + 3\bar{t}_l + s_l + \overline{\sigma(x_{i_{l+1}})} + x_{i_{l+1}} &\geq 3, \\ 3d_{i_{l+1}} + 3\bar{d}_l + a_l + \sigma(x_{i_{l+1}}) + \bar{x}_{i_{l+1}} &\geq 3, \end{aligned}$$

and weakening on a_i and s_l , which yields

$$3t_{l+1} + 3\bar{t}_l + 3d_{i_{l+1}} + 3\bar{d}_l \geq 2.$$

Then adding $3 \cdot (t_{l+1} + d_{i_l} \geq 1)$ and $3 \cdot (d_{i_{l+1}} + t_l \geq 1)$ and saturating yields $d_{i_{l+1}} + t_{l+1} \geq 1$. \square

We proceed by showing the second proof goal.

Lemma 16. *Assume that the lexicographical order \preceq over n variables is loaded. Let σ be a symmetry of the input formula and let $k = |\text{supp}(\sigma)|$. Then we can show the proof goal*

$$\mathcal{C} \cup \mathcal{D} \cup \{-C\} \cup \mathcal{S}_{\preceq}(\vec{x}, \vec{x} \upharpoonright \sigma, \vec{a}, \vec{d}) \cup \mathcal{O}_{\preceq}(\vec{d}) \vdash \perp,$$

using $O(k)$ RUP steps and cutting planes steps, where the RUP steps require $O(n)$ propagations in total.

Proof. Note that $\neg C \doteq \bar{t}_k \geq 1$ still has the interpretation that $(x_{i_1}, \dots, x_{i_k}) \not\preceq_{\text{lex}} (\sigma(x_{i_1}), \dots, \sigma(x_{i_k}))$, while the premises $\mathcal{S}_{\preceq}(\vec{x}, \vec{x} \upharpoonright \sigma, \vec{a}, \vec{d}) \cup \mathcal{O}_{\preceq}(\vec{d})$ have the interpretation that $(x_{i_1}, \dots, x_{i_k}) \preceq_{\text{lex}} (\sigma(x_{i_1}), \dots, \sigma(x_{i_k}))$, so we should be able to derive contradiction.

The idea of the proof is to show that a_{i_l} and s_l have the same interpretation and that d_{i_l} and t_l have the same interpretation. This then gives a contradiction since we have the premises $d_n \geq 1$ but $\bar{t}_k \geq 1$.

Overview. The proof consists of four main steps:

1. Rewriting the specification $\mathcal{S}_{\preceq}(x, x \upharpoonright \sigma, a, d)$ to only $O(k)$ constraints involving auxiliary variables a_{i_l} and d_{i_l} and variables from $\text{supp}(\sigma)$ only.
2. Showing that $d_{i_l} \geq 1$.
3. Showing that $a_{i_l} + \bar{s}_l \geq 1$.
4. Showing that $t_l \geq 1$.

The first of these steps requires $O(k)$ RUP steps for which $O(n)$ propagations are needed in total, and $O(k)$ cutting planes steps. The remaining steps require $O(k)$ RUP steps for which $O(k)$ propagations are needed.

Rewriting the specification constraints. Write $S = \{i_1, \dots, i_k\}$. We again consider the case $1 \notin S$. Under the substitution σ , the specification constraints $\mathcal{S}_{\preceq}(x, x \upharpoonright \sigma, a, d)$ are (semantically) given by

$$a_1 \geq 1, \tag{77}$$

$$a_i \Leftrightarrow (a_{i-1} \wedge x_i \geq \sigma(x_i)), \quad (i \in S) \tag{78}$$

$$a_i \Leftrightarrow a_{i-1}, \quad (i \notin S) \tag{79}$$

$$d_1 \geq 1, \tag{80}$$

$$d_i \Leftrightarrow (d_{i-1} \wedge (\bar{a}_{i-1} \vee \sigma(x_i) \geq x_i)), \quad (i \in S) \tag{81}$$

$$d_i \Leftrightarrow d_{i-1}. \quad (i \notin S) \tag{82}$$

In the same way as in Lemma 15, we can rewrite this to

$$a_{i_1} \Leftrightarrow (x_{i_1} \geq \sigma(x_{i_1})), \tag{83}$$

$$a_{i_{l+1}} \Leftrightarrow (a_{i_l} \wedge x_{i_{l+1}} \geq \sigma(x_{i_{l+1}})), \tag{84}$$

$$d_{i_1} \Leftrightarrow (\sigma(x_{i_1}) \geq x_{i_1}), \tag{85}$$

$$d_{i_{l+1}} \Leftrightarrow (d_{i_l} \wedge (\bar{a}_{i_l} \vee \sigma(x_{i_{l+1}}) \geq x_{i_{l+1}})). \tag{86}$$

We also show by RUP using (82) that $d_{i_k} \Leftrightarrow d_n$.

When replacing a_{i_l} with s_l and d_{i_l} with t_l , constraints (83) up to (86) exactly match constraints (47) up to (50).

Showing that $d_{i_l} \geq 1$. Since we have the constraint $d_n \geq 1$, first $d_{i_k} \Leftrightarrow d_n$ propagates that $d_{i_k} \geq 1$, and then (86) inductively propagates $d_{i_l} \geq 1$ for all $1 \leq l \leq k$, starting with the high indices.

Showing that $a_{i_l} + \bar{s}_l \geq 1$. We show this inductively. The negation of $a_{i_l} + \bar{s}_l \geq 1$ propagates \bar{a}_{i_l} and s_l .

For the base case, we note that \bar{a}_{i_1} propagates that $x_{i_1} \geq \sigma(x_{i_1})$ is false (i.e., that \bar{x}_{i_1} and $\sigma(x_{i_1})$ hold) using (83), which then yields together with s_1 a conflict in (47).

For the inductive step, we note that s_{l+1} propagates s_l using (48), which by the induction hypothesis propagates a_{i_l} . Together with $\bar{a}_{i_{l+1}}$ and (84) this then propagates the inequality $x_{i_{l+1}} \geq \sigma(x_{i_{l+1}})$ to false, which then yields together with s_{l+1} a conflict in (48).

Showing that $t_l \geq 1$. We show this inductively. For the base case, the negation $\bar{t}_1 \geq 1$ propagates the inequality $\sigma(x_{i_1}) \geq x_{i_1}$ to false, which propagates d_{i_1} to false, which is a contradiction.

For the inductive step, the negation $\bar{t}_{l+1} \geq 1$ propagates together with $t_l \geq 1$ using (50) that $\bar{s}_l \vee (\sigma(x_{i_{l+1}}) \geq x_{i_{l+1}})$ is false, i.e., that s_l , $\sigma(x_{i_{l+1}})$ and $x_{i_{l+1}}$. Then s_l propagates a_{i_l} , which together means that $\bar{a}_{i_l} \vee (\sigma(x_{i_{l+1}}) \geq x_{i_{l+1}})$ is false. This is a conflict in (86), since we know that $d_{i_{l+1}}$ holds.

We finish the proof by noting that this shows that $t_k \geq 1$, which directly conflicts with the premise $\bar{t}_k \geq 1$. \square

Alternative derivation using cutting planes steps. For step 3 and 4 in the proof of Lemma 16, we can also provide a derivation using cutting planes steps instead.

Showing that $a_{i_l} + \bar{s}_l \geq 1$. For the base case, we add the constraint $2a_{i_1} + \bar{x}_{i_1} + \sigma(x_{i_1}) \geq 2$ (from (83)) and the constraint $\bar{s}_1 + x_{i_1} + \sigma(x_{i_1}) \geq 1$ (from (47)) and saturate.

For the inductive step, we add the constraints

$$2a_{i_{l+1}} + 2\bar{a}_{i_l} + \bar{x}_{i_{l+1}} + \sigma(x_{i_{l+1}}) \geq 2$$

$$3\bar{s}_{l+1} + 2s_l + x_{i_{l+1}} + \sigma(x_{i_{l+1}}) \geq 3$$

$$2 \cdot (a_{i_l} + \bar{s}_l \geq 1)$$

from (84), (48) and the induction hypothesis respectively, and then saturate, which yields $a_{i_{l+1}} + \bar{s}_{l+1} \geq 1$.

Showing that $t_l \geq 1$. For the base case, we add the constraint $2t_1 + \sigma(x_{i_1}) + x_{i_1} \geq 2$ (from (49)) and the constraint $\bar{d}_{i_1} + \bar{x}_{i_1} + \sigma(x_{i_1}) \geq 1$ (from (85)) and the constraint $d_{i_1} \geq 1$ and saturate, which yields $t_1 \geq 1$.

For the inductive step, we add the constraints

$$\begin{aligned} 4\overline{d_{i_{l+1}}} + 3d_{i_l} + \overline{a_{i_l}} + \sigma(x_{i_{l+1}}) + \overline{x_{i_{l+1}}} &\geq 4 \\ 3t_{l+1} + 3\overline{t_l} + s_l + \overline{\sigma(x_{i_{l+1}})} + x_{i_{l+1}} &\geq 3 \\ a_{i_l} + \overline{s_l} &\geq 1 \end{aligned}$$

from (50), (86), and earlier derivations. This yields

$$4\overline{d_{i_{l+1}}} + 3d_{i_l} + 3t_{l+1} + 3\overline{t_l} \geq 4$$

Then we add the constraints $4 \cdot (d_{i_{l+1}} \geq 1)$ and $3 \cdot (t_l \geq 1)$ (from the induction hypothesis) and weaken away d_{i_l} and saturate, which yields $t_{l+1} \geq 1$. \square

Finally, we show how to derive the symmetry breaking clauses (59) up to (64) from the constraint $t_k \geq 1$ that we derived using dominance.

Lemma 17. *Let σ be a symmetry of the input formula and let $k = |\text{supp}(\sigma)|$. Then we can derive the symmetry breaking clauses (59) up to (64) from the constraint $t_k \geq 1$ using $O(k)$ RUP steps and cutting planes steps.*

Proof. We can derive constraint (59), i.e., $s_1 + \overline{x_{i_1}} \geq 1$, by weakening constraint (52), i.e., $2s_1 + \overline{x_{i_1}} + \sigma(x_{i_1}) \geq 2$, on $\sigma(x_{i_1})$ and saturating. Similarly, we can derive (61), i.e., $s_1 + \sigma(x_{i_1}) \geq 1$, by weakening (52) on x_{i_1} and saturating.

For constraint (60), i.e., $s_{l+1} + \overline{s_l} + \overline{x_{i_{l+1}}} \geq 1$, we weaken (54), i.e., $2s_{l+1} + 2\overline{s_l} + \overline{x_{i_{l+1}}} + \sigma(x_{i_{l+1}}) \geq 2$, on $\sigma(x_{i_{l+1}})$ and saturate. Similarly, for (62), i.e., $s_{l+1} + \overline{s_l} + \sigma(x_{i_{l+1}}) \geq 1$, we weaken (54) on $x_{i_{l+1}}$ and saturate.

It remains to prove constraints (63), i.e., $\sigma(x_{i_l}) + \overline{x_{i_l}} \geq 1$, and (64), i.e., $\overline{s_l} + \sigma(x_{i_{l+1}}) + \overline{x_{i_{l+1}}} \geq 1$. For this, we first derive using RUP from $t_k \geq 1$ and (57) that $t_l \geq 1$ for $1 \leq l \leq k$ (starting with the high indices). Then adding $t_l \geq 1$ to (55) yields (63), while adding $4 \cdot (t_{l+1} \geq 1)$ to (57), weakening on t_l and saturating yields (64). \square

Together this shows

Theorem 2. *Assume that the lexicographical order \preceq over n variables is loaded. Let σ be a symmetry of the input formula and let $k = |\text{supp}(\sigma)|$. Then we can derive the symmetry breaking clauses (59) up to (64) using a proof of size $O(k)$ that can be checked in time $O(n)$.*

D Example Of Symmetry Breaking Proof

We present an example of a proof in the VERIPB format using auxiliary preorder variables. In order to keep the size of the proof somewhat manageable, we consider a toy example: the pigeonhole principle with 3 pigeons and 2 holes. This formula contains variables x_i , where x_1 and x_2 encode that pigeon 1 flies into hole 1 and 2, x_3 and x_4 encode that pigeon 2 flies into hole 1 and 2, and x_5 and x_6 encode that pigeon 3 flies into hole 1 and 2, respectively. This unsatisfiable formula claims that, first, each pigeon is in some hole:

$$x_1 \vee x_2 \quad x_3 \vee x_4 \quad x_5 \vee x_6,$$

and second, that each hole contains at most one pigeon:

$$\begin{aligned} \overline{x_1} \vee \overline{x_3} & \quad \overline{x_1} \vee \overline{x_5} & \quad \overline{x_3} \vee \overline{x_5} \\ \overline{x_2} \vee \overline{x_4} & \quad \overline{x_2} \vee \overline{x_6} & \quad \overline{x_4} \vee \overline{x_6}. \end{aligned}$$

Our proof example breaks two symmetries of this formula:

$$\begin{aligned} \sigma &= \{x_1 \mapsto x_3, x_2 \mapsto x_4, x_3 \mapsto x_1, x_4 \mapsto x_2\}, \\ \tau &= \{x_1 \mapsto x_6, x_2 \mapsto x_5, x_3 \mapsto x_2, \\ & \quad x_4 \mapsto x_1, x_5 \mapsto x_4, x_6 \mapsto x_3\}, \end{aligned}$$

where σ swaps pigeons 1 and 2, and τ moves every pigeon to the other hole, and renames it to the previous pigeon.

The proof starts by defining a preorder lex6 , corresponding to the lexicographic order over a list of 6 variables.

```
pseudo-Boolean proof version 3.0
def_order lex6
vars
left u1 u2 u3 u4 u5 u6;
right v1 v2 v3 v4 v5 v6;
aux $a1 $a2 $a3 $a4 $a5 $d1 $d2 $d3 $d4 $d5 $d6;
end vars;
```

Here, variables u_i and v_i are declared as left-hand and right-hand variables for the preorder lex6 . Furthermore, auxiliary preorder variables a_i and d_i are declared. Next, the specification is declared, following Lemma 12.

```
spec
red +1 ~$a1 +1 u1 +1 ~v1 >= 1 : $a1 -> 0;
red +2 $a1 +1 ~u1 +1 v1 >= 2 : $a1 -> 1;
red +3 ~$a2 +2 $a1 +1 u2 +1 ~v2 >= 3 : $a2 -> 0;
red +2 $a2 +2 ~$a1 +1 ~u2 +1 v2 >= 2 : $a2 -> 1;
```

Here, the constraint $\overline{a_1} + u_1 + \overline{v_1} \geq 1$ is first introduced in the specification by redundancy with witness $\{a_1 \mapsto 0\}$. Since a_1 is a fresh variable at this point, this is correct. Next, the constraint $a_1 + \overline{u_1} + v_1 \geq 2$ is introduced by redundancy with witness $\{a_1 \mapsto 1\}$. Note that this witness satisfies the previous constraint, so this is correct too. The specification then introduces constraints $3\overline{a_2} + 2a_1 + u_2 + \overline{v_2} \geq 3$ and $2a_2 + \overline{a_1} + \overline{u_2} + v_2 \geq 2$ with witnesses $\{a_2 \mapsto 0\}$ and $\{a_2 \mapsto 1\}$, which are correct for similar reasons. The rest of the specification is introduced in a similar way:

```
red +3 ~$a3 +2 $a2 +1 u3 +1 ~v3 >= 3 : $a3 -> 0;
red +2 $a3 +2 ~$a2 +1 ~u3 +1 v3 >= 2 : $a3 -> 1;
red +3 ~$a4 +2 $a3 +1 u4 +1 ~v4 >= 3 : $a4 -> 0;
red +2 $a4 +2 ~$a3 +1 ~u4 +1 v4 >= 2 : $a4 -> 1;
red +3 ~$a5 +2 $a4 +1 u5 +1 ~v5 >= 3 : $a5 -> 0;
red +2 $a5 +2 ~$a4 +1 ~u5 +1 v5 >= 2 : $a5 -> 1;
red +1 ~$d1 +1 ~u1 +1 v1 >= 1 : $d1 -> 0;
red +2 $d1 +1 u1 +1 ~v1 >= 2 : $d1 -> 1;
red +4 ~$d2 +3 $d1 +1 ~$a1 +1 ~u2 +1 v2 >= 4 : $d2 -> 0;
red +3 $d2 +3 ~$d1 +1 $a1 +1 u2 +1 ~v2 >= 3 : $d2 -> 1;
red +4 ~$d3 +3 $d2 +1 ~$a2 +1 ~u3 +1 v3 >= 4 : $d3 -> 0;
red +3 $d3 +3 ~$d2 +1 $a2 +1 u3 +1 ~v3 >= 3 : $d3 -> 1;
red +4 ~$d4 +3 $d3 +1 ~$a3 +1 ~u4 +1 v4 >= 4 : $d4 -> 0;
red +3 $d4 +3 ~$d3 +1 $a3 +1 u4 +1 ~v4 >= 3 : $d4 -> 1;
red +4 ~$d5 +3 $d4 +1 ~$a4 +1 ~u5 +1 v5 >= 4 : $d5 -> 0;
red +3 $d5 +3 ~$d4 +1 $a4 +1 u5 +1 ~v5 >= 3 : $d5 -> 1;
red +4 ~$d6 +3 $d5 +1 ~$a5 +1 ~u6 +1 v6 >= 4 : $d6 -> 0;
red +3 $d6 +3 ~$d5 +1 $a5 +1 u6 +1 ~v6 >= 3 : $d6 -> 1;
end spec;
```

Finally, the preorder constraints for lex6 are introduced. As per Section C.1, these are a single constraint $d_6 \geq 1$.

```
def
+1 $d6 >= 1;
end def;
```

This section is followed by a transitivity proof. The proof starts by declaring some new variables.

```
transitivity
vars
fresh_right w1 w2 w3 w4 w5 w6 ;
fresh_aux_1 $b1 $b2 $b3 $b4 $b5 $e1 $e2 $e3 $e4 $e5 $e6;
fresh_aux_2 $c1 $c2 $c3 $c4 $c5 $f1 $f2 $f3 $f4 $f5 $f6;
end vars;
```

These are needed to account for a new set of preorder variables \vec{w} , and two new sets of preorder auxiliary variables \vec{b}, \vec{e} and \vec{c}, \vec{f} . These are given by the lines `fresh_right`, `fresh_aux_1` and `fresh_aux_2`, respectively. This proof will show the following implication:

$$\begin{aligned} \mathcal{S}_{\leq}(\vec{u}, \vec{v}, \vec{a}, \vec{d}) \cup \mathcal{O}_{\leq}(\vec{u}, \vec{v}, \vec{a}, \vec{d}) \\ \cup \mathcal{S}_{\leq}(\vec{v}, \vec{w}, \vec{b}, \vec{e}) \cup \mathcal{O}_{\leq}(\vec{v}, \vec{w}, \vec{b}, \vec{e}) \\ \cup \mathcal{S}_{\leq}(\vec{u}, \vec{w}, \vec{c}, \vec{f}) \vdash \mathcal{O}_{\leq}(\vec{u}, \vec{w}, \vec{c}, \vec{f}) \end{aligned}$$

Copies of the specification and preorder constraints are implicitly brought into scope and assigned numerical identifiers. In this particular case, the specification contains 22 constraints, and the preorder contains a single constraint. Then, a total of 68 constraints are implicitly introduced in the proof scope, and given identifiers in the same order as they were defined in the `spec` and `ord` sections:

- $\mathcal{S}_{\leq}(\vec{u}, \vec{v}, \vec{a}, \vec{d})$ is mapped into identifiers 1 through 22.
- $\mathcal{S}_{\leq}(\vec{v}, \vec{w}, \vec{b}, \vec{e})$ is mapped into identifiers 23 through 44.
- $\mathcal{S}_{\leq}(\vec{u}, \vec{w}, \vec{c}, \vec{f})$ is mapped into identifiers 45 through 66.
- The single constraints in $\mathcal{O}_{\leq}(\vec{u}, \vec{v}, \vec{a}, \vec{d})$ and $\mathcal{O}_{\leq}(\vec{v}, \vec{w}, \vec{b}, \vec{e})$ are mapped into identifiers 67 and 68, respectively.

```
proof
proofgoal #1
```

Next, the `proof` section shows the implication above. Each `proofgoal` line provides a proof for one constraint in $\mathcal{O}_{\leq}(\vec{u}, \vec{w}, \vec{c}, \vec{f})$, in order of definition in the `def` section. In our case, we only must prove $f_6 \geq 1$, so only one `proofgoal` appears in the proof. Within this context, the negation of the goal, $\overline{f_6} \geq 1$, is automatically assigned the next available identifier, in this case 69; to satisfy the proof-goal we must reach a contradiction.

```
pol 67 4 * 21 +;
rup +1 $d5 >= 1 : -1;
pol -2 $d5 w;
pol -2 4 * 19 +;
rup +1 $d4 >= 1 : -1;
pol -2 $d4 w;
```

The proof roughly follows Lemma 14. First, constraints

$$3d_i + \overline{a_i} + u_{i+1} + \overline{v_{i+1}} \geq 4 \quad (87)$$

$$d_i \geq 1 \quad (88)$$

$$\overline{a_i} + \overline{u_{i+1}} + v_{i+1} \geq 1 \quad (89)$$

are derived for $i = 5, \dots, 1$. We can see this in the displayed fragment for $i = 5$. Constraint (87) is derived through the line `pol 67 4 * 21 +`. Lines starting with `pol` derive the result of a cutting planes proof by providing explicit operations in

reverse Polish notation. This line first fetches constraint 67 (which is $d_6 \geq 1$), then scales it by 4, then adds the result with constraint 21 (i.e. $4d_6 + 3d_5 + \overline{a_5} + u_6 + \overline{v_6} \geq 4$). Constraint (88) is derived through the line `rup +1 $d5 >= 1 : -1`. Lines starting with `rup` apply a form of cutting planes proof search called *reverse unit propagation* (RUP) that can automatically identify some inferences. After the colon, a list of constraint identifiers can be provided to use as premises for proof search. Negative identifiers refer to identifiers relative to the current constraint, so -1 in this case refers to the constraint we just derived, namely (87). For premises F and a conclusion C , RUP tries to identify a contradiction in $F \cup \{C\}$ by constraint propagation. In this case, this formula contains the constraints

$$3d_5 + \overline{a_5} + u_6 + \overline{v_6} \geq 4 \quad \overline{d_5} \geq 1$$

which constraint propagation proves inconsistent. Finally, constraint (89) is derived through the line `pol -2 $d5 w`, which is reverse Polish notation for weakening constraint (87) on variable d_5 .

```
pol -2 4 * 17 +;
rup +1 $d3 >= 1 : -1;
pol -2 $d3 w;
pol -2 4 * 15 +;
rup +1 $d2 >= 1 : -1;
pol -2 $d2 w;
pol -2 4 * 13 +;
rup +1 $d1 >= 1 : -1;
pol -2 $d1 w;
pol -2 11 +;
```

This process iterates on i until the following constraints are derived

$$\begin{aligned} d_5 \geq 1 & \quad (71), & \overline{a_5} + \overline{u_6} + v_6 \geq 1 & \quad (72), \\ d_4 \geq 1 & \quad (74), & \overline{a_4} + \overline{u_5} + v_5 \geq 1 & \quad (75), \\ d_3 \geq 1 & \quad (77), & \overline{a_3} + \overline{u_4} + v_4 \geq 1 & \quad (78), \\ d_2 \geq 1 & \quad (80), & \overline{a_2} + \overline{u_3} + v_3 \geq 1 & \quad (81), \\ d_1 \geq 1 & \quad (83), & \overline{a_1} + \overline{u_2} + v_2 \geq 1 & \quad (84), \\ \overline{u_1} + v_1 \geq 1 & \quad (85), \end{aligned}$$

where the last constraint is derived through the line `pol -2 11 +`. The proof then repeats the process above for variables b_i, e_i .

```
pol 68 4 * 43 +;
rup +1 $e5 >= 1 : -1;
pol -2 $e5 w;
pol -2 4 * 41 +;
rup +1 $e4 >= 1 : -1;
pol -2 $e4 w;
pol -2 4 * 39 +;
rup +1 $e3 >= 1 : -1;
pol -2 $e3 w;
pol -2 4 * 37 +;
rup +1 $e2 >= 1 : -1;
pol -2 $e2 w;
pol -2 4 * 35 +;
rup +1 $e1 >= 1 : -1;
pol -2 $e1 w;
pol -2 33 +;
```

This yields constraints

$$e_5 \geq 1 \quad (87), \quad \overline{b_5} + \overline{v_6} + w_6 \geq 1 \quad (88),$$

$$e_4 \geq 1 \quad (90), \quad \overline{b_4} + \overline{v_5} + w_5 \geq 1 \quad (91),$$

$$e_3 \geq 1 \quad (93), \quad \overline{b_3} + \overline{v_4} + w_4 \geq 1 \quad (94),$$

$$e_2 \geq 1 \quad (96), \quad \overline{b_2} + \overline{v_3} + w_3 \geq 1 \quad (97),$$

$$e_1 \geq 1 \quad (99), \quad \overline{b_1} + \overline{v_2} + w_2 \geq 1 \quad (100),$$

$$\overline{v_1} + w_1 \geq 1 \quad (101).$$

Next, the proof derives constraints A_i and B_i given by $a_i + \overline{c_i} \geq 1$ and $b_i + \overline{c_i} \geq 1$ for $i = 1, \dots, 6$, respectively. This follows the inductive process detailed in Lemma 14.

```
pol 24 45 + 85 + s;
pol 47 u2 w w2 w s;
```

In the base case, the line `pol 2 45 + -1 + s` derives $a_1 + \overline{c_1} \geq 1$ by adding and saturating constraints

$$\begin{aligned} \overline{u_1} + v_1 + 2a_1 &\geq 2 \\ u_1 + \overline{w_1} + \overline{c_1} &\geq 1 \\ \overline{v_1} + w_1 &\geq 1 \end{aligned}$$

where the two constraints above come from the specifications, and the one below is 101. Similarly, the line `pol 24 45 + 85 + s` derives $b_1 + \overline{c_1} \geq 1$.

```
pol -1 -3 +;
pol -2 -3 +;
pol 47 $c1 w s;
pol -2 100 + -1 + -3 2 * + 4 + s;
pol -4 84 + -2 + -3 2 * + 26 + s;
```

The induction case needs to derive some intermediate lemmas at each step.

- First, the constraint C_i given by $c_i + c_{i+1} \geq 1$ is obtained by weakening the specification constraint $u_{i+1} + \overline{w_{i+1}} + 2c_i + 3\overline{c_{i+1}} \geq 3$ on variables u_{i+1} and w_{i+1} and then saturating; in the fragment displayed, this is done for $i = 1$ by the line `pol 47 u2 w w2 w s`.
- Then, the constraints A'_i and B'_i given by $a_i + \overline{c_{i+1}} \geq 1$ and $b_i + \overline{c_{i+1}} \geq 1$ are derived by adding C_i with A_i and with B_i , respectively. In the displayed fragment, this is done for $i = 1$ by the lines `pol -1 -3 +` and `pol -2 -3 +`.
- The constraint L_i given by $u_{i+1} + \overline{w_{i+1}} + \overline{c_{i+1}} \geq 1$ is derived by weakening the specification constraint $u_{i+1} + \overline{w_{i+1}} + 2c_i + 3\overline{c_{i+1}} \geq 3$ on variable c_i and then saturating. In the displayed fragment, this is done for $i = 1$ by the line `pol 47 $c1 w s`.
- Finally, A_{i+1} is derived by adding the constraints $2 \cdot A'_i + B'_i + L_i$ with $\overline{b_i} + \overline{v_{i+1}} + w_{i+1} \geq 1$ (which was derived in the previous proof fragment) and $\overline{u_{i+1}} + \overline{v_{i+1}} + 2\overline{a_i} + 2a_{i+1} \geq 2$ (which is a specification constraint), and then saturating. The constraint B_{i+1} is analogously derived as well. In the displayed fragment, this is done for $i = 1$ by the lines `pol -2 100 + -1 + -3 2 * + 4 + s` and `pol -4 84 + -2 + -3 2 * + 26 + s`.

```
pol 49 u3 w w3 w s;
pol -1 -3 +;
pol -2 -3 +;
```

```
pol 49 $c2 w s;
pol -2 97 + -1 + -3 2 * + 6 + s;
pol -4 81 + -2 + -3 2 * + 28 + s;
pol 51 u4 w w4 w s;
pol -1 -3 +;
pol -2 -3 +;
pol 51 $c3 w s;
pol -2 94 + -1 + -3 2 * + 8 + s;
pol -4 78 + -2 + -3 2 * + 30 + s;
pol 53 u5 w w5 w s;
pol -1 -3 +;
pol -2 -3 +;
pol 53 $c4 w s;
pol -2 91 + -1 + -3 2 * + 10 + s;
pol -4 75 + -2 + -3 2 * + 32 + s;
pol 85 101 +;
```

Last, the proof derives the constraint F_i given by $f_i \geq 1$ for $i = 1, \dots, 6$.

```
pol -1 56 + s;
pol 84 100 + 102 + 103 + s -1 3 * +;
pol -1 58 + s;
pol 81 97 + 108 + 109 + s -1 3 * +;
pol -1 60 + s;
pol 78 94 + 114 + 115 + s -1 3 * +;
pol -1 62 + s;
pol 75 91 + 120 + 121 + s -1 3 * +;
pol -1 64 + s;
pol 72 88 + 126 + 127 + s -1 3 * +;
pol -1 66 + s;
```

For the base case, can derive F_1 by adding constraints $\overline{u_1} + v_1 \geq 1$ and $\overline{v_1} + w_1 \geq 1$, which we derived as 85 and 101; this is the line `pol 85 101 +`. In the induction case, we first derive a lemma F'_i given by $\overline{u_{i+1}} + w_{i+1} + \overline{c_i} + 3f_i \geq 4$. This is obtained by first and then saturating the constraints

$$\begin{aligned} \overline{a_i} + \overline{u_{i+1}} + v_{i+1} &\geq 1, \\ \overline{b_i} + \overline{u_{i+1}} + v_{i+1} &\geq 1, \\ a_i + \overline{c_i} &\geq 1, \\ b_i + \overline{c_i} &\geq 1, \end{aligned}$$

resulting in $\overline{c_i} + \overline{u_{i+1}} + \overline{w_{i+1}}$; the two constraints above come from the first transitivity proof fragment; the two constraints below are A_i and B_i . The result is added to $3 \cdot F_i$, which yields F'_i . This is done for $i = 1$ by line `pol 84 100 + 102 + 103 + s -1 3 * +`. The constraint F_{i+1} can then be derived by adding F'_i to the specification constraint $u_{i+1} + \overline{w_{i+1}} + c_i + 3\overline{f_i} + 3\overline{f_{i+1}} \geq 3$ and saturating; this is done for $i = 1$ by the line `pol -1 58 + s`.

```
pol -1 69 +;
qed #1 : -1;
qed proof;
end transitivity;
```

Once F_6 has been derived, the line `pol -1 69 +` adds it to the negated proof goal $\overline{f_6} \geq 1$, creating a contradiction $0 \geq 1$. The line `qed #1 : -1` communicates this contradiction to the proof checker, finishing the transitivity proof. The reflexivity proof follows.

```
reflexivity
proof
proofgoal #1
```

```

rup >= 1;
qed #1 : -1;
qed proof;
end reflexivity;

```

The reflexivity section automatically identifies the constraints in $\mathcal{S}_{\leq}(\vec{u}, \vec{u}, \vec{a}, \vec{d})$ as constraints 1 through 22. We must provide a proof of $\mathcal{O}_{\leq}(\vec{u}, \vec{u}, \vec{a}, \vec{d})$, which contains a single constraint $d_6 \geq 1$, identified by the proofgoal #1. Similar to the transitivity proof, entering the `proofgoal #1` section negates this goal as $\overline{d_6} \geq 1$ and adds it to the premises as constraint 23. As shown by Lemma 13, this proof derives a contradiction in a single RUP step in line `rup >= 1`.

```

end def_order;
load_order lex6 x5 x6 x1 x2 x3 x4;

```

After this, the preorder `lex6` has been correctly defined, and the refutation of the pigeonhole principle starts, assigning its constraints identifiers 1 through 9. The line `load_order lex6 x5 x6 x1 x2 x3 x4` applies an order change rule with the preorder `lex6`. The list of variables there tells the proof checker to map formula variables to preorder variables as:

$$\begin{array}{lll} x_5 \mapsto u_1, v_1 & x_6 \mapsto u_2, v_2 & x_1 \mapsto u_3, v_3 \\ x_2 \mapsto u_4, v_4 & x_3 \mapsto u_5, v_5 & x_3 \mapsto u_6, v_6 \end{array}$$

The proof immediately proceeds to break the symmetry σ . First, the symmetry breaker circuit is introduced.

```

red +1 ~s1 +1 x1 +1 ~x3 >= 1 : s1 -> 0;
red +2 s1 +1 ~x1 +1 x3 >= 2 : s1 -> 1;
red +3 ~s2 +2 s1 +1 x2 +1 ~x4 >= 3 : s2 -> 0;
red +2 s2 +2 ~s1 +1 ~x2 +1 x4 >= 2 : s2 -> 1;
red +3 ~s3 +2 s2 +1 x3 +1 ~x1 >= 3 : s3 -> 0;
red +2 s3 +2 ~s2 +1 ~x3 +1 x1 >= 2 : s3 -> 1;
red +1 ~t1 +1 ~x1 +1 x3 >= 1 : t1 -> 0;
red +2 t1 +1 x1 +1 ~x3 >= 2 : t1 -> 1;
red +4 ~t2 +3 t1 +1 ~s1 +1 ~x2 +1 x4 >= 4 : t2 -> 0;
red +3 t2 +3 ~t1 +1 s1 +1 x2 +1 ~x4 >= 3 : t2 -> 1;
red +4 ~t3 +3 t2 +1 ~s2 +1 ~x3 +1 x1 >= 4 : t3 -> 0;
red +3 t3 +3 ~t2 +1 s2 +1 x3 +1 ~x1 >= 3 : t3 -> 1;
red +4 ~t4 +3 t3 +1 ~s3 +1 ~x4 +1 x2 >= 4 : t4 -> 0;
red +3 t4 +3 ~t3 +1 s3 +1 x4 +1 ~x2 >= 3 : t4 -> 1;

```

These correspond to constraints (51) through (58). Each constraint is introduced by redundance-based strengthening. First, the constraint $C_1 = \overline{s_1} + x_1 + \overline{x_3} \geq 1$ is derived with witness $\omega_1 = \{s_1 \mapsto 0\}$; this corresponds to constraint (51). The check $\mathcal{C} \cup \mathcal{D} \cup \{C_1\} \vdash (\mathcal{C} \cup \mathcal{D} \cup \{C_1\}) \upharpoonright_{\omega_1}$ succeeds trivially, since s_1 does not occur in $\mathcal{C} \cup \mathcal{D}$ and $C_1 \upharpoonright_{\omega_1}$ is a tautology. For the same reason, the check $\mathcal{S}_{\leq}(\vec{x} \upharpoonright_{\omega_1}, \vec{x}, \vec{a}, \vec{d}) \vdash \mathcal{O}_{\leq}(\vec{x} \upharpoonright_{\omega_1}, \vec{x}, \vec{a}, \vec{d})$ can be skipped, since this is the same as the reflexivity check.

Next, the constraint $C_2 = 2s_1 + \overline{x_1} + x_3 \geq 2$ is derived with witness $\omega_2 = \{s_1 \mapsto 1\}$; this corresponds to constraint (52). The check $\mathcal{C} \cup \mathcal{D} \cup \{C_2\} \vdash (\mathcal{C} \cup \mathcal{D} \cup \{C_2\}) \upharpoonright_{\omega_2}$ succeeds trivially. The reasons are the same as above for all constraints except the recently introduced C_1 ; in that case simply observe that C_1 is the result of weakening $\overline{C_2}$ (i.e. $2\overline{s_1} + x_1 + \overline{x_3} \geq 3$) on variable x_3 .

A similar reasoning proves all remaining redundance-based strengthening steps. They receive constraint identifiers 10 through 23. Once these circuit constraints have

been introduced, the proof is ready to introduce the symmetry breaker by dominance-based strengthening.

```

dom +1 t4 >= 1 : x1 -> x3 x2 -> x4 x3 -> x1 x4 -> x2 :
subproof

```

This line starts the dominance proof to derive the constraint C given by $t_4 \geq 1$ with witness σ . Note that all the constraints derived above by redundance-based strengthening have been derived into \mathcal{D} ; this becomes relevant for the checks elicited by dominance-based strengthening:

$$\begin{aligned} \mathcal{C} \cup \mathcal{D} \cup \{\neg C\} \cup \mathcal{S}_{\leq}(\vec{x} \upharpoonright_{\sigma}, \vec{x}, \vec{a}, \vec{d}) &\vdash \mathcal{C} \upharpoonright_{\sigma} \cup \mathcal{O}_{\leq}(\vec{x} \upharpoonright_{\sigma}, \vec{x}, \vec{a}, \vec{d}) \\ \mathcal{C} \cup \mathcal{D} \cup \{\neg C\} \cup \mathcal{S}_{\leq}(\vec{x}, \vec{x} \upharpoonright_{\sigma}, \vec{a}, \vec{d}) \cup \mathcal{O}_{\leq}(\vec{x}, \vec{x} \upharpoonright_{\sigma}, \vec{a}, \vec{d}) &\vdash \perp \end{aligned}$$

The single constraint in $\mathcal{O}_{\leq}(\vec{x} \upharpoonright_{\sigma}, \vec{x}, \vec{a}, \vec{d})$ in the first check is assigned proof goal identifier #1, and the \perp in the second check is assigned proof goal identifier #2. The goals $\mathcal{C} \upharpoonright_{\sigma}$ in the first check are assigned as proof goal identifiers their constraint identifiers from \mathcal{C} , without a “#”; we actually do not need to prove these, since σ is a symmetry of \mathcal{C} , so $\mathcal{C} \upharpoonright_{\sigma} = \mathcal{C}$ holds. The `subproof` context above introduces the constraint $\neg C$ given by $\overline{t_4} \geq 1$ with identifier 24.

```

scope leq
proofgoal #1

```

The scope `leq` introduces the constraints $\mathcal{S}_{\leq}(\vec{x} \upharpoonright_{\sigma}, \vec{x}, \vec{a}, \vec{d})$ from the first check with constraint identifiers 25 through 46: and `proofgoal #1` selects the only constraint in $\mathcal{O}_{\leq}(\vec{x} \upharpoonright_{\sigma}, \vec{x}, \vec{a}, \vec{d})$, namely $d_6 \geq 1$, as a proofgoal; its negation is introduced with constraint identifier 47. First the proof tries to derive the simplified specification constraints:

$$\begin{array}{ll} \overline{a_3} + x_3 + \overline{x_1} \geq 1 & \overline{d_3} + x_1 + \overline{x_3} \geq 1 \\ 2a_3 + \overline{x_3} + x_1 \geq 2 & 2d_3 + \overline{x_1} + x_3 \geq 2 \\ 3\overline{a_4} + 2a_3 + x_4 + \overline{x_2} \geq 3 & 4\overline{d_4} + 3d_3 + \overline{a_3} + x_2 + \overline{x_4} \geq 4 \\ 2a_4 + 2\overline{a_3} + \overline{x_4} + x_2 \geq 2 & 3d_4 + 3\overline{d_3} + a_3 + \overline{x_2} + x_4 \geq 3 \\ 3\overline{a_5} + 2a_4 + x_1 + \overline{x_3} \geq 3 & 4\overline{d_5} + 3d_4 + \overline{a_4} + x_3 + \overline{x_1} \geq 4 \\ 2a_5 + 2\overline{a_4} + \overline{x_1} + x_3 \geq 2 & 3d_5 + 3\overline{d_4} + a_4 + \overline{x_3} + x_1 \geq 3 \\ & 4\overline{d_6} + 3d_5 + \overline{a_5} + x_4 + \overline{x_2} \geq 4 \\ & 3d_6 + 3\overline{d_5} + a_5 + \overline{x_4} + x_2 \geq 3. \end{array}$$

First the reification $a_3 \iff x_3 \geq x_1$ is obtained.

```

rup +1 ~$d6 >= 1;
rup +1 $a2 >= 1;
pol 29 ~$a2 2 * + s;
pol 30 -2 2 * +;

```

Since $\sigma(x_5) = x_5$ and $\sigma(x_6) = x_6$, $\mathcal{S}_{\leq}(\vec{x} \upharpoonright_{\sigma}, \vec{x}, \vec{a}, \vec{d})$ contains the constraints $a_1 \geq 1$ and $2a_2 + 2\overline{a_i} \geq 1$, from which we can derive the lemma $a_2 \geq 1$ by RUP. Then from the literal axiom $\overline{a_2} \geq 2$ and specification constraint $3\overline{a_3} + 2a_2 + x_3 + \overline{x_1} \geq 3$ we can derive $\overline{a_3} + x_3 + \overline{x_1} \geq 1$. Similarly, from the lemma $a_1 \geq 1$ and the specification constraint $2a_3 + 2\overline{a_2} + \overline{x_3} + x_1 \geq 2$ we derive $2a_3 + \overline{x_3} + x_1 \geq 2$.

Next, we obtain the reification $a_4 \iff a_3 \wedge (x_4 \geq x_2)$.

```

rup +1 ~$a3 +1 $a3 >= 1;
rup +1 $a3 +1 ~$a3 >= 1;
pol 31 -2 2 * +;
pol 32 -2 2 * +;

```

The proof first tries to derive $a_i \iff a_j$ where i is the index for a right after the previous support variable, and j is the index for a at the next support variable. Since in this case $i = j = 3$, the proof actually degenerates into deriving two tautologies by RUP. Next, the proof derives constraints $3\overline{a_4} + 2a_3 + x_4 + \overline{x_2} \geq 3$ and $2a_4 + 2\overline{a_3} + \overline{x_4} + x_2 \geq 2$ by operating over some specification constraints and the reification $a_i \iff a_j$. These constraints actually already appear in the specification $\mathcal{S}_{\leq}(\vec{x}|\sigma, \vec{x}, \vec{a}, \vec{d})$, so we do not really derive anything new. The proof then repeats this last step for the variable a_5 .

```
rup +1 ~$a4 +1 $a4 >= 1;
rup +1 $a4 +1 ~$a4 >= 1;
pol 33 -2 2 * +;
pol 34 -2 2 * +;
rup +1 ~$a5 +1 $a5 >= 1;
rup +1 $a5 +1 ~$a5 >= 1;
```

A very similar approach derives the simplified reifications of the d_i variables.

```
rup +1 $d2 >= 1;
pol 39 ~$d2 3 * + 49 + s;
pol 40 -2 3 * + ~$a2 +;
rup +1 ~$d3 +1 $d3 >= 1;
rup +1 $d3 +1 ~$d3 >= 1;
pol 41 -2 3 * + -14 +;
pol 42 -2 3 * + -16 +;
rup +1 ~$d4 +1 $d4 >= 1;
rup +1 $d4 +1 ~$d4 >= 1;
pol 43 -2 3 * + -14 +;
pol 44 -2 3 * + -16 +;
rup +1 ~$d5 +1 $d5 >= 1;
rup +1 $d5 +1 ~$d5 >= 1;
pol 45 -2 3 * + -14 +;
pol 46 -2 3 * + -16 +;
```

The proof now proves the following constraints

$$\begin{array}{ll} d_3 + \overline{s_1} \geq 1 & t_1 + \overline{a_3} \geq 1 \\ d_4 + \overline{s_2} \geq 1 & t_2 + \overline{a_4} \geq 1 \\ d_5 + \overline{s_3} \geq 1 & t_3 + \overline{a_5} \geq 1 \end{array}$$

by RUP.

```
rup +1 $d3 +1 ~s1 >= 1;
rup +1 $d4 +1 ~s2 >= 1;
rup +1 $d5 +1 ~s3 >= 1;
rup +1 t1 +1 ~$a3 >= 1;
rup +1 t2 +1 ~$a4 >= 1;
rup +1 t3 +1 ~$a5 >= 1;
```

Here, checking that $d_3 + \overline{s_1} \geq 1$ is implied by RUP can be done by assuming literals $\overline{d_3}$ and s_1 are true, and then propagating. The simplified specification constraint $2d_3 + \overline{x_1} + x_3 \geq 2$ then propagates $\overline{x_1}$ and x_3 . This in turn yields a conflict with the symmetry breaker circuit constraint $\overline{s_1} + x_1 + \overline{x_3} \geq 1$. To show that $d_4 + \overline{s_2} \geq 2$ is implied by RUP, we similarly assume literals $\overline{d_4}$ and s_2 are true. Then, the symmetry breaker circuit constraint $3\overline{s_2} + 2s_1 + x_2 + \overline{x_4} \geq 3$ propagates s_1 , and so the previously derived constraint propagates d_3 . Finally, the simplified specification constraint $3d_4 + 3\overline{d_3} + a_3 + x_2 + \overline{x_4} \geq 3$ propagates $\overline{x_2}$ and x_4 , which in turn conflict with the same symmetry breaker circuit constraint above. Similar RUP steps derive the proof of the constraints above, as well as the rest of the proof.

```
rup +1 $d3 +1 ~s1 >= 1;
rup +1 $d4 +1 ~s2 >= 1;
rup +1 $d5 +1 ~s3 >= 1;
rup +1 t1 +1 ~$a3 >= 1;
rup +1 t2 +1 ~$a4 >= 1;
rup +1 t3 +1 ~$a5 >= 1;
rup +1 t2 +1 ~t1 +1 $d3 >= 1;
rup +1 t3 +1 ~t2 +1 $d4 >= 1;
rup +1 t4 +1 ~t3 +1 $d5 >= 1;
rup +1 $d4 +1 ~$d3 +1 t1 >= 1;
rup +1 $d5 +1 ~$d4 +1 t2 >= 1;
rup +1 $d6 +1 ~$d5 +1 t3 >= 1;
rup +1 $d3 +1 t1 >= 1;
rup +1 $d3 +1 t2 >= 1;
rup +1 $d4 +1 t1 >= 1;
rup +1 $d4 +1 t2 >= 1;
rup +1 $d4 +1 t3 >= 1;
rup +1 $d5 +1 t2 >= 1;
rup +1 $d5 +1 t3 >= 1;
rup +1 $d5 +1 t4 >= 1;
rup +1 $d6 +1 t3 >= 1;
rup +1 $d6 +1 t4 >= 1;
rup >= 1;
qed #1 : -1;
end scope;
```

Finally, the proof derives $d_6 + t_4 \geq 1$, which immediately yields a contradiction: $\overline{t_4} \geq 1$ is assumed as the negation of the current dominance constraint, and $\overline{d_6} \geq 1$ is assumed as the negation of the current proof goal.

```
rup >= 1;
qed #1 : -1;
end scope;
```

Next, the proof shows the second dominance check.

```
scope geq
proofgoal #2
```

The scope `geq` reintroduces the constraints $\mathcal{S}_{\leq}(\vec{x}|\sigma, \vec{x}, \vec{a}, \vec{d})$ with constraint identifiers 100 through 121, just like `leq`. Additionally, the single constraint in $\mathcal{O}_{\leq}(\vec{x}|\sigma, \vec{x}, \vec{a}, \vec{d})$ is introduced as well with identifier 122. The proof itself proceeds similarly to the `leq` check, by deriving constraints that connect \vec{a} , \vec{d} , \vec{s} and \vec{t} . The RUP steps follow closely Lemma 16.

```
rup +1 $d5 >= 1;
rup +1 $d4 >= 1;
rup +1 $d3 >= 1;
rup +1 ~s1 +1 $a3 >= 1;
rup +1 ~s2 +1 $a4 >= 1;
rup +1 ~s3 +1 $a5 >= 1;
rup +1 t1 >= 1;
rup +1 t2 >= 1;
rup +1 t3 >= 1;
```

The proof eventually derives $t_3 \geq 1$ and $s_3 + \overline{a_5} \geq 1$. This leads to a contradiction, since the assumptions $\overline{t_4} \geq 1$ and $d_6 \geq 1$ then propagate x_4 , $\overline{x_2}$ and s_3 from the symmetry breaker circuit constraint $3t_4 + 3\overline{t_3} + s_3 + x_4 + \overline{x_2} \geq 3$. The constraint $s_3 + \overline{a_5} \geq 1$ then propagates a_5 , and then the constraint $4d_6 + 3d_5 + \overline{a_5} + x_2 + \overline{x_4} \geq 4$ yields a conflict. This concludes the dominance proof.

```
rup >= 1;
qed #2 : -1;
```

```
end scope;
qed dom;
```

The proof next simplifies the constraints and deletes the circuit, leaving only some constraints that use the \bar{s} variables to break the symmetry σ .

```
rup +1 t3 >= 1 : -1 22;
rup +1 t2 >= 1 : -1 20;
rup +1 t1 >= 1 : -1 18;
pol 11 x1 + s;
pol 13 x2 + s;
pol 15 x3 + s;
pol 11 ~x3 + s;
pol 13 ~x4 + s;
pol 15 ~x1 + s;
pol 16 136 + s;
pol 18 ~t1 3 * + 135 4 * + s;
pol 20 ~t2 3 * + 134 4 * + s;
pol 22 ~t3 3 * + 133 4 * + s;
del range 10 26;
del range 133 137;
```

By the end of this fragment, the net effect of symmetry breaking has been adding the following constraints to the derived set:

$$\begin{array}{ll} x_3 + s_1 \geq 1 & x_4 + \bar{s}_1 + s_2 \geq 1 \\ x_1 + \bar{s}_2 + s_3 \geq 1 & \bar{x}_1 + s_1 \geq 1 \\ \bar{x}_2 + \bar{s}_1 + s_2 \geq 1 & \bar{x}_3 + \bar{s}_2 + s_3 \geq 1 \\ \bar{x}_1 + x_3 \geq 1 & \bar{x}_2 + x_4 + \bar{s}_1 \geq 1 \\ x_1 + \bar{x}_3 + \bar{s}_2 \geq 1 & x_2 + \bar{x}_4 + \bar{s}_3 \geq 1 \end{array}$$

Since these constraints do not affect the core set, we can still break another symmetry. The symmetry τ is larger than σ , but the proof proceeds through similar strokes. First, a symmetry breaker circuit is introduced through redundancy-based strengthening.

```
red +1 ~s4 +1 x5 +1 ~x4 >= 1 : s4 -> 0;
red +2 s4 +1 ~x5 +1 x4 >= 2 : s4 -> 1;
red +3 ~s5 +2 s4 +1 x6 +1 ~x3 >= 3 : s5 -> 0;
red +2 s5 +2 ~s4 +1 ~x6 +1 x3 >= 2 : s5 -> 1;
red +3 ~s6 +2 s5 +1 x1 +1 ~x6 >= 3 : s6 -> 0;
red +2 s6 +2 ~s5 +1 ~x1 +1 x6 >= 2 : s6 -> 1;
red +3 ~s7 +2 s6 +1 x2 +1 ~x5 >= 3 : s7 -> 0;
red +2 s7 +2 ~s6 +1 ~x2 +1 x5 >= 2 : s7 -> 1;
red +3 ~s8 +2 s7 +1 x3 +1 ~x2 >= 3 : s8 -> 0;
red +2 s8 +2 ~s7 +1 ~x3 +1 x2 >= 2 : s8 -> 1;
red +1 ~t1 +1 ~x5 +1 x4 >= 1 : t1 -> 0;
red +2 t1 +1 x5 +1 ~x4 >= 2 : t1 -> 1;
red +4 ~t2 +3 t1 +1 ~s4 +1 ~x6 +1 x3 >= 4 : t2 -> 0;
red +3 t2 +3 ~t1 +1 s4 +1 x6 +1 ~x3 >= 3 : t2 -> 1;
red +4 ~t3 +3 t2 +1 ~s5 +1 ~x1 +1 x6 >= 4 : t3 -> 0;
red +3 t3 +3 ~t2 +1 s5 +1 x1 +1 ~x6 >= 3 : t3 -> 1;
red +4 ~t4 +3 t3 +1 ~s6 +1 ~x2 +1 x5 >= 4 : t4 -> 0;
red +3 t4 +3 ~t3 +1 s6 +1 x2 +1 ~x5 >= 3 : t4 -> 1;
red +4 ~t5 +3 t4 +1 ~s7 +1 ~x3 +1 x2 >= 4 : t5 -> 0;
red +3 t5 +3 ~t4 +1 s7 +1 x3 +1 ~x2 >= 3 : t5 -> 1;
red +4 ~t6 +3 t5 +1 ~s8 +1 ~x4 +1 x1 >= 4 : t6 -> 0;
red +3 t6 +3 ~t5 +1 s8 +1 x4 +1 ~x1 >= 3 : t6 -> 1;
```

Then, a symmetry breaking constraint is introduced through dominance-based strengthening.

```
dom +1 t6 >= 1 : x5 x4 x6 x3 x1 x6 x2 x5 x3 x2 x4 x1 :
subproof
```

Then the first dominance proof goal is resolved.

```
scope leq
proofgoal #1
rup +1 ~$d6 >= 1;
rup >= 0;
pol 170;
pol 171;
rup +1 ~$a1 +1 $a1 >= 1;
rup +1 $a1 +1 ~$a1 >= 1;
pol 172 -2 2 * +;
pol 173 -2 2 * +;
rup +1 ~$a2 +1 $a2 >= 1;
rup +1 $a2 +1 ~$a2 >= 1;
pol 174 -2 2 * +;
pol 175 -2 2 * +;
rup +1 ~$a3 +1 $a3 >= 1;
rup +1 $a3 +1 ~$a3 >= 1;
pol 176 -2 2 * +;
pol 177 -2 2 * +;
rup +1 ~$a4 +1 $a4 >= 1;
rup +1 $a4 +1 ~$a4 >= 1;
pol 178 -2 2 * +;
pol 179 -2 2 * +;
rup +1 ~$a5 +1 $a5 >= 1;
rup +1 $a5 +1 ~$a5 >= 1;
rup >= 0;
pol 180;
pol 181;
rup +1 ~$d1 +1 $d1 >= 1;
rup +1 $d1 +1 ~$d1 >= 1;
pol 182 -2 3 * + -22 +;
pol 183 -2 3 * + -24 +;
rup +1 ~$d2 +1 $d2 >= 1;
rup +1 $d2 +1 ~$d2 >= 1;
pol 184 -2 3 * + -22 +;
pol 185 -2 3 * + -24 +;
rup +1 ~$d3 +1 $d3 >= 1;
rup +1 $d3 +1 ~$d3 >= 1;
pol 186 -2 3 * + -22 +;
pol 187 -2 3 * + -24 +;
rup +1 ~$d4 +1 $d4 >= 1;
rup +1 $d4 +1 ~$d4 >= 1;
pol 188 -2 3 * + -22 +;
pol 189 -2 3 * + -24 +;
rup +1 ~$d5 +1 $d5 >= 1;
rup +1 $d5 +1 ~$d5 >= 1;
pol 190 -2 3 * + -22 +;
pol 191 -2 3 * + -24 +;
rup +1 $d1 +1 ~s4 >= 1;
rup +1 $d2 +1 ~s5 >= 1;
rup +1 $d3 +1 ~s6 >= 1;
rup +1 $d4 +1 ~s7 >= 1;
rup +1 $d5 +1 ~s8 >= 1;
rup +1 t1 +1 ~$a1 >= 1;
rup +1 t2 +1 ~$a2 >= 1;
rup +1 t3 +1 ~$a3 >= 1;
rup +1 t4 +1 ~$a4 >= 1;
rup +1 t5 +1 ~$a5 >= 1;
rup +1 t2 +1 ~t1 +1 $d1 >= 1;
rup +1 t3 +1 ~t2 +1 $d2 >= 1;
rup +1 t4 +1 ~t3 +1 $d3 >= 1;
rup +1 t5 +1 ~t4 +1 $d4 >= 1;
rup +1 t6 +1 ~t5 +1 $d5 >= 1;
rup +1 $d2 +1 ~$d1 +1 t1 >= 1;
```

```

rup +1 $d3 +1 ~$d2 +1 t2 >= 1;
rup +1 $d4 +1 ~$d3 +1 t3 >= 1;
rup +1 $d5 +1 ~$d4 +1 t4 >= 1;
rup +1 $d6 +1 ~$d5 +1 t5 >= 1;
rup +1 $d1 +1 t1 >= 1;
rup +1 $d1 +1 t2 >= 1;
rup +1 $d2 +1 t1 >= 1;
rup +1 $d2 +1 t2 >= 1;
rup +1 $d2 +1 t3 >= 1;
rup +1 $d3 +1 t2 >= 1;
rup +1 $d3 +1 t3 >= 1;
rup +1 $d3 +1 t4 >= 1;
rup +1 $d4 +1 t3 >= 1;
rup +1 $d4 +1 t4 >= 1;
rup +1 $d4 +1 t5 >= 1;
rup +1 $d5 +1 t4 >= 1;
rup +1 $d5 +1 t5 >= 1;
rup +1 $d5 +1 t6 >= 1;
rup +1 $d6 +1 t5 >= 1;
rup +1 $d6 +1 t6 >= 1;
rup >= 1;
qed #1 : -1;
end scope;

```

The second proof goal follows, and finishes the dominance proof:

```

scope geq
proofgoal #2
rup +1 $d5 >= 1;
rup +1 $d4 >= 1;
rup +1 $d3 >= 1;
rup +1 $d2 >= 1;
rup +1 $d1 >= 1;
rup +1 ~s4 +1 $a1 >= 1;
rup +1 ~s5 +1 $a2 >= 1;
rup +1 ~s6 +1 $a3 >= 1;
rup +1 ~s7 +1 $a4 >= 1;
rup +1 ~s8 +1 $a5 >= 1;
rup +1 t1 >= 1;
rup +1 t2 >= 1;
rup +1 t3 >= 1;
rup +1 t4 >= 1;
rup +1 t5 >= 1;
rup >= 1;
qed #2 : -1;
end scope;
qed dom;

```

Finally, some constraints are simplified.

```

rup +1 t5 >= 1 : -1 167;
rup +1 t4 >= 1 : -1 165;
rup +1 t3 >= 1 : -1 163;
rup +1 t2 >= 1 : -1 161;
rup +1 t1 >= 1 : -1 159;
pol 148 x5 + s;
pol 150 x6 + s;
pol 152 x1 + s;
pol 154 x2 + s;
pol 156 x3 + s;
pol 148 ~x4 + s;
pol 150 ~x3 + s;
pol 152 ~x6 + s;
pol 154 ~x5 + s;
pol 156 ~x2 + s;
pol 157 319 + s;

```

```

pol 159 ~t1 3 * + 318 4 * + s;
pol 161 ~t2 3 * + 317 4 * + s;
pol 163 ~t3 3 * + 316 4 * + s;
pol 165 ~t4 3 * + 315 4 * + s;
pol 167 ~t5 3 * + 314 4 * + s;
del range 147 171;
del range 314 320;

```

Through the second symmetry breaking proof, the following constraints are ultimately introduced:

$$\begin{array}{ll}
x_4 + s_4 \geq 1 & x_3 + \overline{s_4} + s_5 \geq 1 \\
x_6 + \overline{s_5} + s_6 \geq 1 & x_5 + \overline{s_6} + s_7 \geq 1 \\
x_2 + \overline{s_7} + s_8 \geq 1 & \overline{x_5} + s_4 \geq 1 \\
\overline{x_6} + \overline{s_4} + s_5 \geq 1 & \overline{x_1} + \overline{s_5} + s_6 \geq 1 \\
\overline{x_2} + \overline{s_6} + s_7 \geq 1 & \overline{x_3} + \overline{s_7} + s_8 \geq 1 \\
x_4 + \overline{x_5} \geq 1 & x_3 + \overline{x_6} + \overline{s_4} \geq 1 \\
\overline{x_1} + x_6 + \overline{s_5} \geq 1 & \overline{x_2} + x_5 + \overline{s_6} \geq 1 \\
x_2 + \overline{x_3} + \overline{s_7} \geq 1 & x_1 + \overline{x_4} + \overline{s_8} \geq 1
\end{array}$$

E Details on Crafted Benchmarks

In this appendix, we briefly describe the five crafted benchmark families that we used in our experimental evaluation. We use the standard notation $[n] = \{1, \dots, n\}$.

Pigeonhole principle (PHP). The classical *pigeonhole principle* formula (Haken 1985) encodes the claim that n pigeons can fly into $n - 1$ holes such that no two pigeons fly into the same hole. Clearly, this formula is UNSAT. The encoding has $n(n - 1)$ variables x_{ij} for $i \in [n]$ and $j \in [n - 1]$, where x_{ij} encodes that pigeon i flies into hole j . There are $(n - 1) \binom{n}{2}$ binary clauses of the form $\overline{x_{ik}} \vee \overline{x_{jk}}$, for all $1 \leq i < j \leq n$ and $k \in [n - 1]$, claiming that it is not the case that both pigeon i and pigeon j fly into hole k . Together, these clauses encode that no two pigeons fly into the same hole. In addition, there are n clauses of the form $\bigvee_{1 \leq j \leq n-1} x_{ij}$ for $i \in [n]$, encoding that pigeon i must fly into some hole. The symmetries of PHP are permuting the holes and permuting the pigeons. A set of generators for the corresponding symmetry group are $n - 2$ symmetries that swap two holes and $n - 1$ symmetries that swap two pigeons. Each such symmetry affects $O(n)$ variables.

Relativized pigeonhole principle (RPHP). The *relativized pigeonhole principle* formula (Atserias, Müller, and Oliva 2013; Atserias, Lauria, and Nordström 2016) encodes the claim that n pigeons can fly into m resting places and then onwards to $n - 1$ holes such that no two pigeons fly into the same resting place or the same hole. Formally, the RPHP formula claims that there exist maps $p: [n] \rightarrow [m]$ and $q: [m] \rightarrow [n - 1]$ such that p is injective and q is injective on the range of p . Similarly to PHP, this formula is UNSAT. Here we choose $m = 2n$. Then the formula contains $4n^2$ variables and $O(n^3)$ constraints. The symmetries of RPHP are permuting the holes, permuting the pigeons, and permuting the resting places.

Clique-coloring formulas (ClqCl). The *clique-coloring* formula (Pudlák 1997) encodes the claim that there exists

instance	variables	constraints	generators	support size
PHP(n)	$O(n^2)$	$O(n^3)$	$O(n)$	$O(n)$
RPHP(n)	$O(n^2)$	$O(n^3)$	$O(n)$	$O(n)$
ClqCl($n, 6, 5$)	$O(n^2)$	$O(n^2)$	$O(n)$	$O(n)$
TseitinGrid(n)	$O(n^2)$	$O(n^2)$	$O(n^2)$	varies
Count($n, 3$)	$O(n^3)$	$O(n^5)$	$O(n)$	varies

Table 1: Some data on the families of crafted benchmarks that we used: The number of variables and constraints of the instance, the number of generators that SATSUMA finds, and the support size of the symmetries that SATSUMA finds.

a graph on n vertices that contains a clique on k vertices and admits a coloring on c vertices. This formula is UNSAT for $k > c$. The formula contains $\binom{n}{2}$ variables to encode the edges of the graph, kn variables to encode a function from $[k]$ to $[n]$ representing the clique, and nc variables to encode a function from $[n]$ to $[c]$ representing the coloring. Here we choose $k = 6$ and $c = 5$. Then the formula has $O(n^2)$ constraints. The symmetries of the clique-coloring formula are the symmetries permuting the vertices of the graph.

Counting formulas (Count). The *counting* formula (Paris and Wilkie 1985) encodes the claim that we can partition n elements into sets of size k each. The formula has $\binom{n}{k}$ variables, where each variable encodes that a particular set is chosen. There are $O(n^{2k-1})$ constraints, encoding that any two chosen sets must be disjoint. In our experiments, we choose $k = 3$. The symmetries of the counting formula are permuting the elements.

Tseitin formulas on a grid (Tseitin Grid). The *Tseitin formulas* (Tseitin 1968) consider a graph $G = (V, E)$ and a charge function $f: V \rightarrow \{0, 1\}$. The formula has a variable for each edge, and the formula encodes that the XOR of the variables corresponding to the edges adjacent to some vertex v equals the charge $f(v)$. Here we consider the Tseitin formula on an $n \times n$ grid with even charge, i.e., $f(v) = 0$ for all $v \in V$. For each cycle in the graph G the Tseitin formula has a negation symmetry that flips all variables corresponding to edges in that cycle.

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