Pseudo-Boolean Proof Logging for Optimal Classical Planning

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Abstract

We introduce lower-bound certificates for classical planning tasks, which can be used to prove the unsolvability of a task or the optimality of a plan in a way that can be verified by an independent third party. We describe a general framework for generating lower-bound certificates based on pseudo-Boolean constraints, which is agnostic to the planning algorithm used. As a case study, we show how to modify the A* algorithm to produce proofs of optimality with modest overhead, using pattern database heuristics and $h^{\max}$ as concrete examples. The same proof logging approach works for any heuristic whose inferences can be efficiently expressed as reasoning over pseudo-Boolean constraints.

Introduction

Optimal classical planning algorithms make three promises: that the plans they produce are correct, that no cheaper plans achieving the goal exist, and that any task reported as unsolvable actually is. As McConnell et al. (2011) argue in their seminal work on *certifying algorithms*, there are many good reasons not to accept such promises blindly. Instead, a *certifying planning algorithm* outputs some kind of proof (a *certificate*) that an independent third party can use to verify the truthfulness of the planner's claims.

For the correctness of plans, such a verification is performed routinely: the generated plans themselves serve as certificates for this, and plan validation tools such as VAL (Howey and Long 2003) or INVAL (Haslum 2017) can be used to check that the produced plans are correct. For unsolvability, Eriksson et al. (Eriksson, Röger, and Helmert 2017, 2018; Eriksson and Helmert 2020) recently introduced two forms of unsolvability certificates that cover very diverse planning algorithms.

Certifying the *optimality* of plans, in contrast, is still in its infancy. The only existing work in this direction is a paper by Mugdan, Christen, and Eriksson (2023) which describes two approaches: one based on a compilation to unsatisfiability, and one based on an extension of the unsatisfiability proof system of Eriksson, Röger, and Helmert (2018). The first approach is in general not computationally feasible, as it requires a task reformulation that increases the number of state variables and actions exponentially in the size of the planning task. The second approach does not share this weakness, but is not sufficiently general to encompass a wide range of planning approaches. Mugdan, Christen, and Eriksson identify five essential properties for optimality certificates: *soundness* (if an optimality certificate is accepted by the verifier, then optimality holds), *completeness* (if a solution is optimal, then a certificate of its optimality exists), *efficient generation* (a non-certifying algorithm can be changed to produce certificates with reasonable, at most polynomial overhead), *efficient verification* (the verifier runtime is polynomial in the task and certificate size), and *generality* (certificates can be efficiently produced by a wide variety of different planning algorithms rather than being algorithm-specific).

Mugdan, Christen, and Eriksson show that their approach is sound and complete, but critically comment that "the other three properties for practical usability do not have clear-cut answers". Compared to the unsolvability proof system of Eriksson et al., their approach appears much more specific to heuristic forward search using heuristics that are themselves based on some kind of forward search, such as the $h^{\max}$ heuristic (Bonet and Geffner 2001). For example, it is not at all clear how to certify heuristics computed in a backward direction, such as pattern databases (Edelkamp 2001) or other abstraction heuristics without redoing the abstract state space exploration for every heuristic evaluation.

While in the planning community the interest in certifying algorithms is quite recent, they are standard in the SAT community, where unsatisfiability certificates based on proof systems like DRAT (Heule, Hunt, and Wetzler 2013) and LRAT (Cruz-Filipe et al. 2017) are required for participating in SAT competitions and supported by formally verified checkers (Tan, Heule, and Myreen 2023; Lammich 2020). A recently proposed alternative are pseudo-Boolean proofs based on cutting planes, which are supported by the formally verified checker VeriPB (Bogaerts et al. 2022). Unlike the other proof systems mentioned, cutting planes are able to directly incorporate linear arithmetic, making them very appealing for optimization problems such as optimal classical planning. Indeed, VeriPB has been applied to a wide range of optimization problems such as MaxSAT (Berg et al. 2023), ILP presolving (Hoen et al. 2024), constraint pro-

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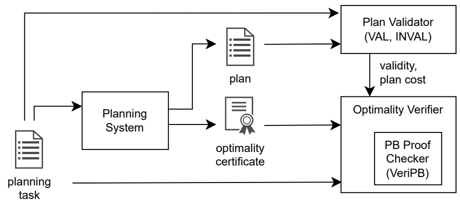

Figure 1: Interaction of Components

gramming (Gocht, McCreesh, and Nordström 2022; McIlree and McCreesh 2023; McIlree, McCreesh, and Nordström 2024), and dynamic programming (Demirović et al. 2024).

In this work, we propose to certify plan optimality by means of such pseudo-Boolean proofs. We claim that our optimality certificates have all five desirable properties discussed above: they are sound, complete, efficiently generatable, efficiently verifiable, and general. (For space reasons, the generality of the proof system must remain a conjecture for the time being, but at least we show that we can cover the techniques covered by Mugdan et al. as well as pattern database heuristics that their approach struggles with.)

The overall concept is shown in Figure 1. In addition to a plan, the certifying planning system produces an optimality certificate that proves that the input task has no cheaper solution. A plan validator such as VAL verifies that the plan solves the task and determines its cost. The optimality verifier has access to the original task, the plan cost determined by the validator, and the certificate from the planning system. On this basis it verifies that the plan is indeed optimal.

The certificate is based on cutting planes proofs with reification (Bogaerts et al. 2023), in which the atomic pieces of knowledge are pseudo-Boolean constraints. Such constraints allow us to reason conveniently about costs of actions and bounds on costs to reach a state. Roughly speaking, the certificate describes an overapproximation of which states can be reached at which cost and shows that no goal state can be reached at a cost below the claimed optimal plan cost. The heart of the proof is a standard pseudo-Boolean proof that can be verified by an off-the-shelf proof checker such as the formally verified VeriPB system.

Background

We first introduce the STRIPS planning formalism (Fikes and Nilsson 1971), followed by pseudo-Boolean constraints and the cutting planes with reification proof system.

STRIPS Planning Tasks

A STRIPS planning task $\Pi = \langle \mathcal{V}, \mathcal{A}, I, G \rangle$ consists of a finite set $\mathcal{V}$ of propositional *state variables*, a finite set $\mathcal{A}$ of *actions*, the *initial state* $I \subseteq \mathcal{V}$ and the *goal* $G \subseteq \mathcal{V}$.

A *state* $s \subseteq \mathcal{V}$ of Π induces a variable assignment ρ_s that maps every $v \in s$ to 1 and every $v \notin s$ to 0. Each action $a \in \mathcal{A}$ is a tuple $\langle \mathit{pre}(a), \mathit{add}(a), \mathit{del}(a), \mathit{cost}(a) \rangle$, where $\mathit{pre}(a) \subseteq \mathcal{V}$ is the set of *preconditions*, $\mathit{add}(a) \subseteq \mathcal{V}$ is the set of *add effects*, $\mathit{del}(a) \subseteq \mathcal{V} \setminus \mathit{add}(a)$ is the set of *delete effects*, and $\mathit{cost}(a) \in \mathbb{N}_0$ is the *cost* of a. We write $\mathit{evars}(a)$ for the set $\mathit{add}(a) \cup \mathit{del}(a)$ of affected variables. Action a is *applicable* in state s if $\mathit{pre}(a) \subseteq s$. The *successor state* is $s[\![a]\!] = (s \setminus \mathit{del}(a)) \cup \mathit{add}(a)$. For a sequence $\pi = a_1, \dots, a_n$ of actions that are successively applicable in state s, we write $s[\![\pi]\!]$ for the resulting state $s[\![a_1]\!] \dots [\![a_n]\!]$. If this state is a *goal state*, i.e. $G \subseteq s[\![\pi]\!]$, then π is a *plan* for s. A plan for the initial state I is a plan for task Π. The cost of π is $\mathit{cost}(\pi) = \sum_{i=1}^{n} \mathit{cost}(a_i)$. A plan is *optimal* if there is no plan of lower cost. Task Π is solvable if there is a plan for Π, otherwise it is *unsolvable*.

Pseudo-Boolean Formulas

A *Boolean* variable has domain $\{0, 1\}$. A literal is a Boolean variable x or its negation $\bar{x}$. A *pseudo-Boolean (PB) constraint* (in normalized form) over a finite set $X = \{x_1, \dots, x_n\}$ of Boolean variables is an inequality

$$\sum\nolimits_i a_i \ell_i \geq A,$$

where all literals ℓ_i are over distinct variables from X, $A \in \mathbb{N}_0$ and all coefficients a_i are from $\mathbb{N}_0$. We will also write linear constraints more flexibly, but they can always be transformed to normalized form by simple algebraic transformations. As syntactic sugar, we will also write $(\ell_1 \wedge \dots \wedge \ell_n) \rightarrow \ell$ as abbreviation for $\overline{\ell_1} + \dots + \overline{\ell_n} + \ell \geq 1$.

A *solution* of the constraint is an assignment $\rho : X \rightarrow \{0, 1\}$ such that the inequality is satisfied by replacing every variable x_i with $\rho(x_i)$ and every negated variable $\bar{x}_i$ with $1 - \rho(x_i)$. We also use assignments for all literals, implicitly requiring that $\rho(\bar{x}) = 1 - \rho(x)$ for all variables x.

For constraint $C \doteq \sum_i a_i \ell_i \geq A$, the negation $\neg C$ is the normalized form of $\sum_i a_i \ell_i \leq A - 1$.

A *pseudo-Boolean (PB) formula* (Buss and Nordström 2021) is a finite set $\mathcal{C}$ of pseudo-Boolean constraints over a set X of variables. An assignment $\rho : X \rightarrow \{0, 1\}$ is a *model* of the formula if it is a solution of all constraints. If $\mathcal{C}$ has no model, it is *unsatisfiable*. We say that a constraint C is *implied* by a PB formula $\mathcal{C}$ (written $\mathcal{C} \models C$) if every model of $\mathcal{C}$ is a solution of C and that PB formula $\mathcal{D}$ is implied by $\mathcal{C}$ (written $\mathcal{C} \models \mathcal{D}$) if $\mathcal{C} \models D$ for all $D \in \mathcal{D}$.

For a partial variable assignment $\rho : X \nrightarrow \{0, 1\}$ and constraint C, we write $C\!\upharpoonright_\rho$ for the constraint obtained from C by replacing each variable v in the domain of ρ by $\rho(v)$ and normalizing.

Cutting Planes with Reification Proof System

The VeriPB proof system (Bogaerts et al. 2023) is an extension of the cutting planes proof system. We use a subset of the VeriPB proof system which we call *cutting planes with reification* (CPR).

Proofs in the cutting planes proof system (Cook, Coullard, and Turán 1987) are built from one axiom and three derivation rules. For any literal ℓ, the *literal axiom* allows us to derive $\ell \geq 0$ without prerequisites. If we already have $\sum_i a_i \ell_i \geq A$ and $\sum_i b_i \ell_i \geq B$, we can derive

- $\sum_i (c_A a_i + c_B b_i) \ell_i \geq c_A A + c_B B$
  for every $c_A, c_B \in \mathbb{N}_0$ (*linear combination*),
- $\sum_i \lceil a_i / c \rceil \ell_i \geq \lceil A / c \rceil$
  for every $c \in \mathbb{N}^+$ (*division*), and

- $\sum_i \min\{a_i, A\}\ell_i \geq A$ (*saturation*).

All constraints that can be derived from a PB formula $\mathcal{C}$ by these rules are implied by $\mathcal{C}$.

Reverse Unit Propagation A constraint $C \in \mathcal{C}$ *unit propagates* literal ℓ under partial assignment ρ if all models of $C\!\upharpoonright_\rho$ assign ℓ to 1. When this happens, we can extend ρ with $\ell \mapsto 1$ and unit-propagate further literals under the extended assignment. If this process derives a conflict (assigning 0 and 1 to the same variable) starting from the empty variable assignment, then $\mathcal{C}$ is unsatisfiable.

Formula $\mathcal{C}$ implies constraint C by *reverse unit propagation* (RUP) if $\mathcal{C} \cup \{\neg C\}$ unit propagates to a conflict from the empty assignment. Any constraint implied by RUP can be derived by a cutting plane proof. As in VeriPB, our proof system allows adding any constraint implied by RUP directly in a single step. This is a useful shortcut that drastically compresses many cutting plane proofs.

Reification In addition to cutting planes reasoning, our proof system also allows *reification*, i.e., introducing a new variable that represents the truth value of a constraint. If C is a constraint and r is a new variable, we write $r \Leftrightarrow C$ to express that variable r must be 1 if C is true under the assignment and 0 otherwise. If C is $\sum_i a_i\ell_i \geq A$, this is a shorthand notation for the two constraints

$$A\bar{r} + \sum\nolimits_i a_i\ell_i \geq A, \text{ and}$$
$$(M - A + 1)r + \sum\nolimits_i a_i\bar{\ell}_i \geq M - A + 1$$

where $M = \sum_i a_i$. We use the notation $r \Rightarrow C$ for the first and $r \Leftarrow C$ for the second constraint.

To summarize, a *CPR proof* consists of a sequence of derivation steps from the cutting plane proof system (literal axiom, linear combination, division, and saturation), constraints derived by RUP, and reifications. In addition, we allow the *redundance-based strengthening* (RED) rule from VeriPB, which can be understood as a form of proof by contradiction. We only use it in the supplemental material and therefore describe it there.

Lower-bound Certificates

We propose certificates that prove that there is no plan of lower cost than a given bound B. A typical, but not the only, application of such lower-bound certificates is to certify that a plan of cost B is optimal. Before we define the full framework, we first introduce how we encode planning tasks by means of PB formulas.

Encoding Planning Tasks

The PB encoding of task $\Pi = \langle \mathcal{V}, \mathcal{A}, I, G\rangle$ uses the propositional variables from $\mathcal{V}$ as Boolean variables, and variables $\mathcal{V}_c = \{c_0, \dots, c_{\lceil \log_2(B)\rceil}\}$ as a binary representation of a number in the range $0, \dots, B$. These variables allow us to represent pairs $\langle s, c\rangle$ consisting of a state s and a number $c \leq B$, with the intuition that whenever such a pair is represented by a model in our encoding, we consider it possible that state s is reachable from the initial state at cost c while staying below the bound B. In addition, we introduce a number of reification variables.

Reification variable r_I is true in a model iff the state variables encode the initial state:

$$r_I \Leftrightarrow \sum_{v \in I} v + \sum_{v \in \mathcal{V}\setminus I} \bar{v} \geq |\mathcal{V}| \tag{1}$$

For the goal, we introduce a reification variable r_G, which is true in a model iff the state variables encode a goal state:

$$r_G \Leftrightarrow \sum_{v \in G} v \geq |G| \tag{2}$$

For the actions, we encode transitions from a state s to successor state $s[\![a]\!]$ in a similar way to symbolic search (e.g., Edelkamp and Kissmann 2009) or planning as satisfiability (e.g., Rintanen, Heljanko, and Niemelä 2006), encoding the successor state by means of additional variables v' for each state variable v. The variables c_i encode a cost by which state s can be reached, and analogously we use a variable c'_i for each c_i to encode the cost to reach $s[\![a]\!]$ via this transition. For this purpose, we need constraints that ensure that the difference between the two values corresponds to the cost of the action. We do this by means of additional reification variables $\Delta c^{=k}$ that express that the difference between the two numbers is k:

$$\Delta c^{=k} \Leftrightarrow \sum\nolimits_{i=0}^{\lceil \log_2 B\rceil} 2^i c'_i - \sum\nolimits_{i=0}^{\lceil \log_2 B\rceil} 2^i c_i = k \tag{3}$$

To express that the variables c_i or c'_i encode a value that is at least k for some $k \in \{0, \dots, B\}$, we use reification variables $\mathit{cost}_{\geq k}$ and $\mathit{cost}'_{\geq k}$:

$$\mathit{cost}_{\geq k} \Leftrightarrow \sum\nolimits_{i=0}^{\lceil \log_2 B\rceil} 2^i c_i \geq k \tag{4}$$

$$\mathit{cost}'_{\geq k} \Leftrightarrow \sum\nolimits_{i=0}^{\lceil \log_2 B\rceil} 2^i c'_i \geq k \tag{5}$$

Note that we do not introduce these reification variables for all values of k up to B, which would require an exponential number of variables in the encoding size of B. Rather, we lazily introduce only the variables used by the proof.

For handling the state variables v that are not affected by an action, we introduce reification variables $eq_{v,v'}$ that are true in a model iff it assigns v and v' the same value:

$$\begin{aligned} eq_{v,v'} &\Leftrightarrow leq_{v,v'} + geq_{v,v'} \geq 2 \\ geq_{v,v'} &\Leftrightarrow v + \overline{v'} \geq 1 \\ leq_{v,v'} &\Leftrightarrow \bar{v} + v' \geq 1 \end{aligned} \tag{6}$$

For each action $a \in \mathcal{A}$ we introduce a variable r_a expressing that whenever the action is applied, the cost is increased by $\mathit{cost}(a)$, the action precondition is satisfied, the primed variables truthfully represent the successor state, and the successor cost is within the cost bound:

$$\begin{aligned} r_a \Rightarrow \Delta c^{=\mathit{cost}(a)} &+ \sum_{v \in \mathit{pre}(a)} v + \sum_{v \in \mathit{add}(a)} v' + \sum_{v \in \mathit{del}(a)} \overline{v'} \\ &+ \sum_{v \in \mathcal{V}\setminus \mathit{evars}(a)} eq_{v,v'} + \overline{\mathit{cost}'_{\geq B}} \geq 2 + |\mathit{pre}(a)| + |\mathcal{V}| \end{aligned} \tag{7}$$

Observe that this constraint represents a conjunction: all literals must be true to meet the bound $2 + |pre(a)| + |\mathcal{V}|$.

Finally, reification variable r_T encodes that a state transition happens, i.e., some action variable is selected:

$$r_T \Leftrightarrow \sum_{a \in \mathcal{A}} r_a \geq 1 \tag{8}$$

This representation allows selecting several actions at the same time, but only if they all lead to the same state change under the same cost.

Definition 1. *For planning task* $\Pi = \langle \mathcal{V}, \mathcal{A}, I, G \rangle$ *and cost bound* $B \in \mathbb{N}_0$*, a* PB task encoding *is a tuple* $\mathcal{E}_\Pi = \langle \mathcal{C}_\Pi, r_I, r_G, r_T \rangle$*, where* $\mathcal{C}_\Pi = \langle \mathcal{C}_{\text{init}}, \mathcal{C}_{\text{goal}}, \mathcal{C}_{\text{trans}}, \mathcal{C}_{\geq} \rangle$ *such that* $\mathcal{C}_{\text{init}}$*,* $\mathcal{C}_{\text{goal}}$*,* $\mathcal{C}_{\text{trans}}$ *and* $\mathcal{C}_{\geq}$ *are sets of reifications from equations* (1)–(8)*,* $\mathcal{C}_{\text{init}}$ *from* (1) *and* (4)*,* $\mathcal{C}_{\text{goal}}$ *from* (2) *and* (4)*,* $\mathcal{C}_{\text{trans}}$ *from* (3)–(8)*,* $\mathcal{C}_{\geq}$ *from* (3)–(5) *and* r_I*,* r_G*, and* r_T *are the reification variables introduced in* (1)*,* (2)*, and* (8)*.*

Certifying Unsolvability under Cost Bound

A certificate shows that the task is unsolvable under a cost bound B, i.e., there is no plan π with $cost(\pi) < B$. Intuitively, it encodes a set of pairs $\langle s, c \rangle$ of states and costs such that (1) it covers the initial state with cost 0, (2) it covers *no* goal state with a cost strictly lower than B, and (3) whenever $\langle s, c \rangle$ is covered and action a is applicable in s, then $s[\![a]\!]$ is covered with cost $c + cost(a)$ if this is below B. The certificate must prove these three properties by means of three separate CPR proofs. We call property (1) the *initial state lemma*, property (2) the *goal lemma* and property (3) the *inductivity lemma*.

The set of pairs $\langle s, c \rangle$ is defined by a sequence of reifications. Alternatively, we can think of it as a circuit where each gate evaluates a PB constraint. The state variables and cost bits c_i are the inputs to the circuit, and the output of the circuit determines whether the input state-cost pair is contained in the set or not. We now formalize this notion.

Definition 2. *A* PB circuit *with input variables* V *is a pair* $\mathcal{R} = \langle R, r \rangle$*, where*

- R *is a sequence* $\langle r_1 \Leftrightarrow \varphi_1, \dots, r_n \Leftrightarrow \varphi_n \rangle$ *of PB reifications such that each PB constraint* φ_i *only has non-zero coefficients for variables from* $V \cup \{r_j \mid j < i\}$*, and*
- $r \in \{r_1, \dots, r_n\}$ *is the* output variable.

Slightly abusing notation, we also interpret a sequence of PB constraints as the PB formula consisting of its components (i.e., treat the sequence as a set). For a PB circuit $\langle R, r \rangle$ with input variables $\mathcal{V} \cup \mathcal{V}_c$, we write $M(R, r)$ for the represented set of pairs $\langle s, c \rangle$. In other words, for each model ρ of $R \cup \{r = 1\}$, the set $M(R, r)$ contains the pair $\langle s, c \rangle$ where $s = \{v \in \mathcal{V} \mid \rho(v) = 1\}$ and $c = \sum_i 2^i \rho(c_i)$.

Definition 3. *Let* $\Pi = \langle \mathcal{V}, \mathcal{A}, I, G \rangle$ *be a planning task and* $B \in \mathbb{N}_0$ *a cost bound. Let* $\mathcal{E}_\Pi = \langle \mathcal{C}_\Pi, r_I, r_G, r_T \rangle$ *be a PB task encoding for* Π *and* B*.*

A lower-bound certificate for Π with bound B *is a tuple* $\langle \langle \mathcal{C}_\varphi, r_\varphi \rangle, \mathcal{P}_{\text{init}}, \mathcal{P}_{\text{ind}}, \mathcal{P}_{\text{goal}} \rangle$*, where*

- $\langle \mathcal{C}_\varphi, r_\varphi \rangle$ *is a PB circuit with input variables* $\mathcal{V} \cup \mathcal{V}_c$ *not mentioning a primed variable.*
- ***initial state lemma:*** $\mathcal{P}_{\text{init}}$ *is a CPR proof for* $\mathcal{C}_{\text{init}} \cup \mathcal{C}_\varphi \models (r_I \wedge \overline{cost_{\geq 1}}) \rightarrow r_\varphi$
- ***goal lemma:*** $\mathcal{P}_{\text{goal}}$ *is a CPR proof for* $\mathcal{C}_{\text{goal}} \cup \mathcal{C}_\varphi \models (r_G \wedge \overline{cost_{\geq B}}) \rightarrow \overline{r_\varphi}$
- ***inductivity lemma:*** $\mathcal{P}_{\text{ind}}$ *is a CPR proof for* $\mathcal{C}_{\text{trans}} \cup \mathcal{C}_\varphi \cup \mathcal{C}'_\varphi \models (r_\varphi \wedge r_T) \rightarrow r'_\varphi$*, where* $\mathcal{C}'_\varphi$ *is a copy of* $\mathcal{C}_\varphi$ *where all PB variables are replaced with their primed version.*

Lower-bound certificates are sound:

Theorem 1. *If there is a lower-bound certificate for planning task* Π *with bound* B*, then the task has no plan* π *with* $cost(\pi) < B$*.*

Proof. Any solution for the task starts from the initial state with cost 0, which is covered by the certificate according to the initial state lemma. The PB encoding of the task is such that exactly the action applications that do not exceed the cost bound are captured by r_T. The inductivity lemma establishes that the resulting state-cost pair will always be covered by the certificate, so the certificate covers all states that are reachable below the cost bound. The goal lemma establishes that there is no goal state among these states, so there is no solution of cost strictly below B. □

The verifier component (see Fig. 1) receives the planning task, the lower-bound certificate, and the validated plan cost. Inside, it confirms that the reifications in $\mathcal{E}_\Pi$ are a system of pseudo-Boolean constraints that encode the original planning task. Additionally, it generates $\langle \mathcal{C}'_\varphi, r'_\varphi \rangle$ based on $\langle \mathcal{C}_\varphi, r_\varphi \rangle$ by creating a copy of each PB reification in $\mathcal{C}_\varphi$ where all variables are replaced by their primed counterparts. It then verifies that the proofs $\mathcal{P}_{\text{init}}$, $\mathcal{P}_{\text{goal}}$ and $\mathcal{P}_{\text{ind}}$ actually derive the statements in Definition 3. Only if all these steps are successful, the verifier confirms that the provided plan cost is optimal.

This concludes our formalization of lower-bound certificates and their verification. Lower-bound certificates are efficiently verifiable: the verifier runs in polynomial time because the underlying PB proof checker does and all steps other than the proof checking are easy to perform in polynomial time. In the rest of the paper we describe a case study of how certain optimal classical planning algorithms can be augmented to efficiently generate lower-bound certificates. Because the algorithms we discuss are complete, this also shows that lower-bound certificates are complete.

Proof-Logging Heuristic Search

We now examine how a planner can be transformed into a proof-logging system, in our case a planner that emits a lower-bound certificate for the cost of the found plan. The idea is to already log most relevant information about its internal reasoning during its normal operation, keeping the overhead for generating the proof low.

A* search (Hart, Nilsson, and Raphael 1968) with an admissible heuristic h, i.e. a distance estimator that provides a lower bound on the goal distance of a state, is the most common approach for optimal planning as heuristic search.

It maintains a priority queue *Open* of states s, ordered by $f = g + h(s)$, where g is the best known cost to reach s from the initial state. The open list is initialized with the initial state. A* removes a state s from the list and expands it until it removes a goal state. A state expansion generates all successor states and adds them to the open list if the implicit path via s improves its g-value.

One challenge when transforming the approach into a proof-logging system is that the bound B that should be certified is only known once a plan has been found. To handle this, we will use "placeholder" PB variables $K_{\geq l}$ and $K'_{\geq l}$ with integers l. Once B is known, the proof will be augmented by reifications that give them the intended meaning. The idea is that they represent $cost_{\geq l}$ and $cost'_{\geq l}$, but l will not necessarily be in the range $\{0, \dots, B\}$. For $l < 0$, we will instead use 0, and for $l > B$, we will use B. So the reifications for these variables will be

$$K_{\geq l} \Leftrightarrow cost_{\geq \min\{B, \max\{0,l\}\}} \text{ and} \tag{9}$$

$$K'_{\geq l} \Leftrightarrow cost'_{\geq \min\{B, \max\{0,l\}\}} \tag{10}$$

By $\mathcal{K}$, we denote the (infinite) set of all such variables. Upon termination A* adds a set $\mathcal{C}_K$ of reifications for the finitely many such variables that actually occur in the proof. In the proofs in this paper, we will silently switch from $K_{\geq \dots}$ to the corresponding $cost_{\geq \dots}$ expression, implicitly using (9).

The following two lemmas establish that these clipped costs behave as expected.[1] Intuitively, the first lemma says that if the costs represented by the variables in $\mathcal{V}_c$ exceed $j + k$, they also exceed the smaller value j:

Lemma 1. *Let $j \in \mathbb{Z}$ and $k \in \mathbb{N}_0$. It is possible to derive $cost_{\geq \min\{B, \max\{0,j+k\}\}} \rightarrow cost_{\geq \min\{B, \max\{0,j\}\}}$ from $\mathcal{C}_{\geq}$.*

The second lemma informally states that if the costs already exceed l and we spend cost m, the successor cost exceeds $l + m$ (clipping all costs into $\{0, \dots, B\}$).

Lemma 2. *For $l \in \mathbb{Z}$ and $m \in \mathbb{N}_0$ it is possible to derive $(cost_{\geq \min\{B, \max\{0,l\}\}} \wedge \Delta c^{=m}) \rightarrow cost'_{\geq \min\{B, \max\{0,l+m\}\}}$ from $\mathcal{C}_{\geq}$.*

Proof-Logging Heuristics

Parts of the overall proof must be contributed by the heuristic. Throughout its execution (a potential initialization phase and evaluations for a number of states), the heuristic maintains its own PB circuit $\langle H, r^h \rangle$ with input variables $\mathcal{V}$ and $\mathcal{K}$. We assume that the heuristic uses its own namespace, so it does not introduce reification variables that are also introduced by the search or some other heuristic. Whenever the search uses the heuristic to evaluate a state s, the heuristic not only returns the estimate but also a PB variable r^h_s with the following requirements:

- $M(H, r^h_s)$ contains all pairs $\langle s, \hat{c} \rangle$ with $\hat{c} + h(s) \geq B$.
- If $M(H, r^h_s)$ contains pair $\langle \hat{s}, \hat{c} \rangle$, it contains all pairs $\langle \tilde{s}, \tilde{c} \rangle$ such that $\tilde{s}$ is reachable from $\hat{s}$ with overall cost $\tilde{c} < B$ (i.e., there is an action sequence π with $\hat{s}[\![\pi]\!] = \tilde{s}$ and $\tilde{c} = \hat{c} + cost(\pi) < B$).
- $M(H, r^h_s)$ contains no pair $\langle \hat{s}, \hat{c} \rangle$ where $\hat{s}$ is a goal state and $\hat{c} < B$.

This motivation leads to the following definition:

Definition 4. *Let $\Pi = \langle \mathcal{V}, \mathcal{A}, I, G \rangle$ be a planning task and $B \in \mathbb{N}_0$ a cost bound. Let $\langle \mathcal{C}_\Pi, r_I, r_G, r_T \rangle$ be the PB task encoding for Π and B and s be a state over $\mathcal{V}$.*

A heuristic certificate for state s of Π with bound B is a tuple $\langle \langle H, r^h_s \rangle, \mathcal{P}^h_s, \mathcal{P}^h_{s,\text{ind}}, \mathcal{P}^h_{s,\text{goal}} \rangle$, where

- *$\langle H, r^h_s \rangle$ is a PB circuit with input variables $\mathcal{V} \cup \mathcal{K}$ not mentioning a primed variable.*
- ***state lemma:** $\mathcal{P}^h_s$ is a CPR proof for $\mathcal{C}_{\text{init}} \cup H \cup \mathcal{C}_K \cup \mathcal{C}_s \models (r_s \wedge cost_{\geq \max\{0, B-h(s)\}}) \rightarrow r^h_s$, where $\mathcal{C}_s = \{r_s \Leftrightarrow \sum_{v \in s} v + \sum_{v \in \mathcal{V} \setminus s} \bar{v} \geq |\mathcal{V}|\}$.*
- ***goal lemma:** $\mathcal{P}^h_{s,\text{goal}}$ is a CPR proof for $\mathcal{C}_{\text{goal}} \cup H \cup \mathcal{C}_K \models (r_G \wedge \overline{cost_{\geq B}}) \rightarrow \overline{r^h_s}$.*
- ***inductivity lemma:** $\mathcal{P}^h_{s,\text{ind}}$ is a CPR proof for $\mathcal{C}_{\text{trans}} \cup H \cup H' \cup \mathcal{C}_K \cup \models (r^h_s \wedge r_T) \rightarrow {r'}^h_s$, where H' is a copy of H where all PB variables are replaced with their primed version.*

It is not a coincidence that these heuristic certificates resemble lower-bound certificates under cost bound B: they can be seen as such certificates for state s, where some cost of at least $B - h(s)$ has already been spent to reach s.

We will later showcase for pattern database heuristics and $h^{\max}$ how such heuristic certificates can be generated. But first we show how they contribute to the overall lower-bound certificate generated by A*.

Proof-Logging A*

Proof-logging A* maintains a sequence A of reifications and a proof log L of derivations. During its initialization, the heuristic already writes some information to A and L.

Whenever A* removes a state s with g-value g from the open list, it adds a reification

$$r_{s,\text{g}\geq g} \Leftrightarrow \sum_{v \in s} v + \sum_{v \in \mathcal{V} \setminus s} \bar{v} + cost_{\geq g} \geq |\mathcal{V}| + 1 \tag{11}$$

to A, characterizing all pairs $\langle s, \tilde{g} \rangle$ with $\tilde{g} \geq g$. In addition, it adds $\langle s, g \rangle$ to the initially empty collection *Closed*. It also logs some derivations in L that we describe later. If a successor s' with g-value g' is not added to open because A*'s duplicate detection is aware of an earlier encounter with $g'' < g'$, it uses Lemma 1 and logs

$$K'_{\geq g'} \rightarrow K'_{\geq g''}. \tag{12}$$

Whenever the search uses the heuristic to evaluate a state, the heuristic can extend A with further reifications, adds derivations for the corresponding state lemma, inductivity lemma and goal lemma to L and returns the corresponding PB variable r^h_s to the search. In addition, we require it to log a proof for the state lemma in terms of the primed variables.

When the search terminates, it adds to A a reification

$$r_{\text{A}^*} \Leftrightarrow \sum_{\langle s,g \rangle \in Closed} r_{s,\text{g}\geq g} + \sum_{\langle s,g,h \rangle \in Open} r^h_s \geq 1 \tag{13}$$

[1]Throughout the paper, we omit proofs of technical lemmas that do not provide further insight. These proofs are included in the supplemental material.

and prepends A with the necessary reifications (9) and (10), where B is the cost of the found plan.

In the following we explain how we can generate the three proofs $\mathcal{P}_{\text{init}}, \mathcal{P}_{\text{ind}}$ and $\mathcal{P}_{\text{goal}}$ for the PB circuit $\langle A, r_{\text{A}^*}\rangle$. We start by showing how L can be extended to $\mathcal{P}_{\text{init}}$.

Lemma 3. *RUP can derive the initial state lemma* $(r_I \wedge \overline{\mathit{cost}_{\geq 1}}) \rightarrow r_{\text{A}^*}$ *from* $\mathcal{C}_{\text{init}} \cup A$.

Proof. Assume (a) $r_I \geq 1$, (b) $\overline{\mathit{cost}_{\geq 1}} \geq 1$, and (c) $\overline{r_{\text{A}^*}} \geq 1$. From (a) and (1), we receive (d) $v \geq 1$ for each $v \in I$ and $\overline{v} \geq 1$ for each $v \in \mathcal{V} \setminus I$. Reification (4) and (b) express $\sum_{i=0}^{\lceil \log_2 B \rceil} 2^i c_i < 1$, from them we get for all $c_i \in \mathcal{V}_c$ that $\overline{c_i} \geq 1$. With (4), we receive $\mathit{cost}_{\geq 0}$, which with (d) and (11) gives $r_{I,g\geq 0} \geq 0$. Since $\langle I, 0\rangle$ is in *Closed*, this yields with (a) and (13) $r_{\text{A}^*} \geq 1$, contradicting (c). □

The proof $\mathcal{P}_{\text{goal}}$ builds on the goal lemmas logged by the heuristic:

Lemma 4. *It is possible to derive the goal lemma* $(r_G \wedge r_{\text{A}^*}) \rightarrow \mathit{cost}_{\geq B}$ *from* $\mathcal{C}_{\text{goal}} \cup A$.

Proof. The heuristic certificate can derive the goal lemma for all open states. The rest is then by RUP, assuming (a) $r_G \geq 1$, (b) $r_{\text{A}^*} \geq 1$ and (c) $\overline{\mathit{cost}_{\geq B}} \geq 1$.

From (2) and (a) we get that (d) $v \geq 1$ for all goal variables. For all closed pairs $\langle s, g\rangle$, s is not a goal state or it is the reached goal state $s_\star$ with $g = B$. If s is not a goal state, then some $v \in G$ is false in s and (11) yields with (d) that $\overline{r_{s,g\geq g}} \geq 1$. We also get $\overline{r_{s_\star,g\geq B}} \geq 1$ from (11), using (c).

For all variables r^h_s, the goal lemma provided by the heuristic implies with (a) and (c) that $\overline{r^h_s} \geq 1$, so overall we get with (13) that $\overline{r_{\text{A}^*}} \geq 1$, contradicting (b). □

To support the derivation of the inductivity lemma, A* already extends L upon every expansion for every action with the derivation described in the following lemma:

Lemma 5. *For every action a and state s closed with cost g, it is possible to derive* $(r_{s,g\geq g} \wedge r_a) \rightarrow r'_{\text{A}^*}$ *from* $\mathcal{C}_{\text{trans}} \cup A$.

Proof. We first establish by Lemma 2 that (a) $(\mathit{cost}_{\geq g} \wedge \Delta c^{=\mathit{cost}(a)}) \rightarrow \mathit{cost}'_{\geq \min\{B, g+\mathit{cost}(a)\}}$.

The desired constraint follows by RUP: Assume (b) $r_{s,g\geq g} \geq 1$, (c) $r_a \geq 1$, and (d) $\overline{r'_{\text{A}^*}} \geq 1$.

From (b), we can derive with (11) that (e) $v \geq 1$ for all $v \in s$, (f) $\overline{v} \geq 1$ for all $v \in \mathcal{V} \setminus s$, and (g) $\mathit{cost}_{\geq g} \geq 1$. We use (c) with (7) to derive (h) $\Delta c^{=\mathit{cost}(a)} \geq 1$. With (h), (g), and (a) we get (i) $\mathit{cost}'_{\geq \min\{B, g+\mathit{cost}(a)\}} \geq 1$. If s is a goal state then $g = B$ and we get from (i) and (7) that $\overline{r_a} \geq 1$, contradicting (c). Otherwise proceed as follows:

If a is not applicable in s, some precondition is violated and we use (f) and (7) to derive $\overline{r_a} \geq 1$, contradicting (c).

Otherwise, a is applicable. We define $\tilde{s} = s[\![a]\!]$ and use (7), (c), (e), (f), (g), and (6) to derive that (j) the primed state variables encode $\tilde{s}$ and that (k) $K'_{\geq g+\mathit{cost}(a)} \geq 1$.

If the successor was not considered because of a duplicate $\langle \tilde{s}, \hat{g}\rangle$ with $\hat{g} < g + \mathit{cost}(a)$, we use the derived constraint (12) to derive (k') $K'_{\geq \hat{g}} \geq 1$.

If $\langle \tilde{s}, g + \mathit{cost}(a)\rangle \in \mathit{Closed}$ or $\langle \tilde{s}, \hat{g}\rangle \in \mathit{Closed}$, respectively, we use the primed version of (11) with (j) and (k) or (k') to derive $r_{\tilde{s},g\geq g+\mathit{cost}(a)}{}' \geq 1$ or $r_{\tilde{s},g\geq \hat{g}}{}' \geq 1$. With the primed version of (13) we derive $r'_{\text{A}^*} \geq 1$, contradicting (d).

Otherwise, there is an entry for $\tilde{s}$ in *Open*. Since A* closes all states with $f < B$, we know that in this case $g + \mathit{cost}(a) + h(\tilde{s}) \geq B$, so $g + \mathit{cost}(a) \geq \max\{0, B - h(\tilde{s})\}$. We can thus use (j) and (k) with the primed version of the state lemma for $\tilde{s}$ from the heuristic and receive $r^{h\,\prime}_{\tilde{s}} \geq 1$. The primed version of (13) yields $r'_{\text{A}^*} \geq 1$, contradicting (d). □

Lemma 6. *For every state s closed with cost g, it is possible to derive* $(r_{s,g\geq g} \wedge r_T) \rightarrow r'_{\text{A}^*}$ *from* $\mathcal{C}_{\text{trans}} \cup A$.

Proof. We establish by Lemma 5 that (a) $(r_{s,g\geq g} \wedge r_a) \rightarrow r'_{\text{A}^*}$ for every action $a \in \mathcal{A}$.

Then RUP derives the constraint: Assume (b) $r_{s,g\geq g} \geq 1$, (c) $r_T \geq 1$, and (d) $\overline{r'_{\text{A}^*}} \geq 1$. With (a), (b) and (d) we derive $\overline{r_a} \geq 1$ for each action a. With (8), this yields $\overline{r_T} \geq 1$, contradicting (c). □

These derivations are already logged during the execution of the search. We extend the log with derivations from the following lemma to generate $\mathcal{P}_{\text{ind}}$:

Lemma 7. *It is possible to derive the inductivity lemma* $(r_{\text{A}^*} \wedge r_T) \rightarrow r'_{\text{A}^*}$ *from* $\mathcal{C}_{\text{trans}} \cup A$.

Proof. We first derive by RUP for every s with some $\langle s, g, h\rangle \in \mathit{Open}$ that (a) $(r^h_s \wedge r_T) \rightarrow r'_{\text{A}^*}$: From the assumption $\overline{r'_{\text{A}^*}} \geq 1$, we get $\overline{r'^h_s} \geq 1$. Using the inductivity lemma from the heuristic for s with the other assumptions $r^h_s \geq 1$ and $r_T \geq 1$, we get the contradiction.

For every state s expanded with cost g, we get (b) $(r_{s,g\geq g} \wedge r_T) \rightarrow r'_{\text{A}^*}$ as described in Lemma 6.

Afterwards RUP can derive the constraint, assuming (c) $r_{\text{A}^*} \geq 1$, (d) $r_T \geq 1$, and (e) $\overline{r'_{\text{A}^*}} \geq 1$.

For every open state s, we get from (a,d,e) that (f) $\overline{r^h_s} \geq 1$. For every state s closed with g-value g, we get from (b, d, e) that (g) $\overline{r_{s,g\geq g}} \geq 1$. With (13), we get from (f) and (g) that $\overline{r_{\text{A}^*}} \geq 1$, contradicting (c). □

To analyze the overhead of proof-logging A*, we assume that each proof from the heuristic is provided with only a constant-factor overhead to the heuristic computation time. Logging a single reification constraint (11) is an operation that requires time linear in $|\mathcal{V}|$. A constraint of this kind has to be logged for each closed state. This is still a constant-factor overhead because the closed state has to be generated first, which is an operation with time linear in $|\mathcal{V}|$, too. The reification of (13) is linear in the number of states generated. Both the goal lemma and the initial state lemma are logged by single RUP statements of constant size.

In the proof of the inductivity lemma we first use as many constant-size statements as there are open states. The second RUP subproof in the inductivity lemma requires the constraint from Lemma 6 specified on each state $s \in \mathit{Closed}$, which in turn requires the constraint from Lemma 5 specified on each applicable action in s. This amortizes with the generation of the successors of s when it moves from *Open* to *Closed*. We see that in total there is a constant-factor overhead to the proof-logging of A*.

Proof-Logging Pattern Database Heuristics

Abstraction heuristics such as *pattern database (PDB) heuristics* use goal distances in an induced abstract task for the heuristic estimates. Let $\Pi = \langle \mathcal{V}, \mathcal{A}, I, G \rangle$ be a STRIPS planning task. A PDB heuristic is defined in terms of a *pattern* $P \subseteq \mathcal{V}$ and its abstraction function α maps each state s over $\mathcal{V}$ to an abstract state $\alpha(s)$ over P as $\alpha(s) = s \cap P$. Each action $a \in \mathcal{A}$ induces the abstract action a^α with $pre(a^\alpha) = pre(a) \cap P$, $add(a^\alpha) = add(a) \cap P$, $del(a^\alpha) = del(a) \cap P$, and $cost(a^\alpha) = cost(a)$. The abstract goal G^α is $G \cap P$. The heuristic estimate of the PDB for state s is the cost of an optimal solution of the abstract task $\langle P, \{a^\alpha \mid a \in \mathcal{A}\}, \alpha(s), G^\alpha \rangle$ or ∞ if it is unsolvable.

Let S_α be the set of all abstract states. In practice, PDB heuristics do not build an abstract task for each heuristic evaluation but precompute the abstract goal distances $d(s_\alpha)$ of all $s_\alpha \in S_\alpha$, and store them in a so-called pattern database. When a concrete state s is evaluated, the PDB heuristic computes $\alpha(s)$ and returns the stored goal distance $d(\alpha(s))$ as the heuristic estimate $h(s)$.

For each abstract state s_α, we will introduce a PB variable r^{s_α} that is true iff the variables from $\mathcal{V}$ encode a state s with $\alpha(s) = s_\alpha$, and a variable $r^{s_\alpha}_{\geq B - d(s_\alpha)}$ that is true iff the variables from $\mathcal{V}$ and $\mathcal{V}_c$ encode a pair $\langle s, c \rangle$ such that $\alpha(s) = s_\alpha$ and $c \geq \max\{B - d(s_\alpha), 0\}$. The overall represented set contains all pairs $\langle s, c \rangle$ such that the abstract goal distance d of $\alpha(s)$ is already so high that $c + d \geq B$, and thus it is impossible to reach the goal from s with a strictly lower cost than B if reaching s already costs c.

Definition 5. *Let $\Pi = \langle \mathcal{V}, \mathcal{A}, I, G \rangle$ be a STRIPS planning task and $P \subseteq \mathcal{V}$. The* PB circuit for the PDB heuristic with pattern P *is the PB circuit $\langle H_{\text{PDB}}, r_{\text{PDB}} \rangle$ with input variables $\mathcal{V} \cup \mathcal{V}_c$, where H_{PDB} contains for each abstract state $s_\alpha \subseteq P$ the two constraints*

$$r^{s_\alpha} \Leftrightarrow \sum_{v \in s_\alpha} v + \sum_{v \in P \setminus s_\alpha} \bar{v} \geq |P| \quad (14)$$

$$r^{s_\alpha}_{\geq B - d(s_\alpha)} \Leftrightarrow r^{s_\alpha} + K_{\geq B - d(s_\alpha)} \geq 2, \quad (15)$$

and the final reification

$$r_{\text{PDB}} \Leftrightarrow \sum_{s_\alpha \in S_\alpha} r^{s_\alpha}_{\geq B - d(s_\alpha)} \geq 1. \quad (16)$$

On the evaluation of a state s, the heuristic always returns reification variable r_{PDB} to the search. In the following we discuss how the required proofs can be generated.

We begin with the state lemma, which requires a new proof for each evaluated state.

Lemma 8. *RUP can derive the state lemma* $(r_s \wedge cost_{\geq \max\{0, B - h(s)\}}) \rightarrow r_{\text{PDB}}$ *from* $\mathcal{C}_{\text{init}} \cup H_{\text{PDB}} \cup \mathcal{C}_s$, *where* $\mathcal{C}_s = \{r_s \Leftrightarrow \sum_{v \in s} v + \sum_{v \in \mathcal{V} \setminus s} \bar{v} \geq |\mathcal{V}|\}$.

Since the PB circuit for the heuristic is the same for all evaluated states, we only need to include the proof for the goal lemma and for the inductivity lemma once in the overall generated proof. For the goal lemma, we use the following:

Lemma 9. *RUP can derive the goal lemma* $(r_G \wedge \overline{cost_{\geq B}}) \rightarrow \overline{r_{\text{PDB}}}$ *from* $\mathcal{C}_{\text{goal}} \cup H_{\text{PDB}}$.

For the inductivity lemma, we develop the derivation by means of four lemmas. The first one derives that applying an induced abstract action a^α in abstract state s_α leads to the abstract successor state.

Lemma 10. *For each action $a \in \mathcal{A}$ and abstract state s_α such that a^α is applicable in s_α, RUP can derive* $(r^{s_\alpha} \wedge r_a) \rightarrow r^{s_\alpha[\![a^\alpha]\!]\prime}$ *from* $\mathcal{C}_{\text{trans}} \cup H_{\text{PDB}} \cup H'_{\text{PDB}}$.

The next lemma again considers individual actions and abstract states, but takes cost into account and derives that the successor situation is covered by the certificate.

Lemma 11. *For each action $a \in \mathcal{A}$ and abstract state s_α such that a^α is applicable in s_α, it is possible to derive* $(r^{s_\alpha}_{\geq B - d(s_\alpha)} \wedge r_a) \rightarrow r'_{\text{PDB}}$ *from* $\mathcal{C}_{\text{trans}} \cup H_{\text{PDB}} \cup H'_{\text{PDB}}$.

Proof. We start by deriving some constraints over costs that we will use later in a RUP proof.

First, we derive (a) $(K_{\geq B - d(s_\alpha)} \wedge \Delta c^{=cost(a)}) \rightarrow K'_{\geq B - d(s_\alpha) + cost(a)}$ as described in Lemma 2.

We know that for the abstract goal distance it holds that $d(s_\alpha[\![a^\alpha]\!]) + cost(a^\alpha) \geq d(s_\alpha)$. Together with $cost(a^\alpha) = cost(a)$, we get $B - d(s_\alpha[\![a^\alpha]\!]) \leq B - d(s_\alpha) + cost(a)$. We use this to derive (b) $K'_{\geq B - d(s_\alpha) + cost(a)} \rightarrow K'_{\geq B - d(s_\alpha[\![a^\alpha]\!])}$ as described in Lemma 1. In addition, we establish (c) $(r^{s_\alpha} \wedge r_a) \rightarrow r^{s_\alpha[\![a^\alpha]\!]\prime}$ by RUP (Lemma 10).

Now we can derive the constraint in the claim by RUP, assuming that (d) $r^{s_\alpha}_{\geq B - d(s_\alpha)} \geq 1$, (e) $r_a \geq 1$, and (f) $\overline{r'_{\text{PDB}}} \geq 1$. From (d), we get with (15) that (g) $r^{s_\alpha} \geq 1$ and (h) $K_{\geq B - d(s_\alpha)} \geq 1$. From (e),(g) and (c) we derive (i) $r^{s_\alpha[\![a^\alpha]\!]\prime} \geq 1$.

From (e) and (7), we derive (j) $\Delta c^{=cost(a)} \geq 1$. From (a) together with (h) and (j) we derive that (k) $K'_{\geq B - d(s_\alpha) + cost(a)} \geq 1$. From (b) together with (k) we derive that (l) $K'_{\geq B - d(s_\alpha[\![a^\alpha]\!])} \geq 1$. We get with this, (i) and the primed constraint (15) from H_{PDB} that ${r^{s_\alpha[\![a^\alpha]\!]}_{\geq B - d(s_\alpha[\![a^\alpha]\!])}}'$. With the primed version of (16) from H'_{PDB} this yields $r'_{\text{PDB}} \geq 1$, contradicting (f). ☐

The third lemma generalizes the previous lemma from individual actions to the entire transition relation.

Lemma 12. *For each abstract state s_α it is possible to derive* $(r^{s_\alpha}_{\geq B - d(s_\alpha)} \wedge r_T) \rightarrow r'_{\text{PDB}}$ *from* $\mathcal{C}_{\text{trans}} \cup H_{\text{PDB}} \cup H'_{\text{PDB}}$.

Proof sketch. For each action $a \in \mathcal{A}$ that is applicable in s_α, establish $(r^{s_\alpha}_{\geq B - d(s_\alpha)} \wedge r_a) \rightarrow r'_{\text{PDB}}$ with Lemma 11. Then the desired constraint can be derived by RUP. ☐

The final step for the inductivity lemma generalizes this from individual abstract states to the entire PB circuit of the heuristic.

Lemma 13. *It is possible to derive the inductivity lemma* $(r_{\text{PDB}} \wedge r_T) \rightarrow r'_{\text{PDB}}$ *from* $\mathcal{C}_{\text{trans}} \cup H_{\text{PDB}} \cup H'_{\text{PDB}}$.

Proof sketch. Establish $(r^{s_\alpha}_{\geq B-d(s_\alpha)} \wedge r_T) \rightarrow r'_{\text{PDB}}$ for each abstract state s_α by Lemma 12. The rest follows by RUP. □

We considered PDB heuristics as one example of abstractions because they have a particularly simply structured abstraction function. To adapt the approach to other abstraction heuristics, the overall line of argument still works but we would have to replace (14) with one or more reifications that allow to identify the concrete states that correspond to a given abstract state. In addition, the derivations from Lemmas 8, 9 and 10 need to be adapted accordingly.

The described proof logging only has a constant-factor overhead for a somewhat naive implementation of PDBs. While the state and goal lemma only require a single RUP statement, the inductivity lemma requires effort in $O(|S_\alpha||\mathcal{A}|)$, iterating once over all actions for each abstract state. A better PDB implementation can deal more efficiently with abstract states that are not reverse-reachable. In the supplementary material we describe a variant of the certificate that does not explicitly represent such states and which guarantees a constant-factor overhead also for such a more efficient computation of PDBs.

Proof-Logging $h^{\max}$

The maximum heuristic (Bonet and Geffner 2001) $h^{\max}(s)$ is defined as $h^{\max}(s, G)$, where $h^{\max}(s, V)$ is the pointwise greatest function with $h^{\max}(s, V) =$

$$\begin{cases} 0 & \text{if } V \subseteq s \\ \min\limits_{\substack{a \in \mathcal{A},\\ v \in \mathit{add}(a)}} (\mathit{cost}(a) + h^{\max}(s, \mathit{pre}(a))) & \text{if } |V| = 1 \text{ and } V \not\subseteq s \\ \max\limits_{v \in V} h^{\max}(s, \{v\}) & \text{otherwise} \end{cases}$$

For $v \in \mathcal{V}$, the *max value* is $V^{\max}(v) = h^{\max}(s, \{v\})$. A typical efficient implementation of the maximum heuristic determines this value for all v with $V^{\max}(v) < h^{\max}(s)$. We define $W^{\max}(v) = \min\{V^{\max}(v), h^{\max}(s)\}$.

For each $v \in \mathcal{V}$, the heuristic adds a reification:

$$r_v \Leftrightarrow \bar{v} + K_{\geq B - h^{\max}(s) + W^{\max}(v)} \geq 1 \tag{17}$$

$$r^{\max}_s \Leftrightarrow \sum_{v \in \mathcal{V}} r_v \geq |\mathcal{V}| \tag{18}$$

The PB circuit for state s then is $\langle H_{\max}, r^{\max}_s \rangle$, where $H_{\max}$ is the sequence of all reifications (17) and (18).

Lemma 14. *RUP can derive the state lemma* $(r_s \wedge \mathit{cost}_{\geq \max\{0, B - h^{\max}(s)\}}) \rightarrow r^{\max}_s$ *from* $\mathcal{C}_{\text{init}} \cup H_{\max} \cup \mathcal{C}_s$, *where* $\mathcal{C}_s = \{r_s \Leftrightarrow \sum_{v \in s} v + \sum_{v \in \mathcal{V} \setminus s} \bar{v} \geq |\mathcal{V}|\}$.

A single RUP statement is sufficient for the goal lemma:

Lemma 15. *RUP can derive the goal lemma* $(r_G \wedge \overline{\mathit{cost}_{\geq B}}) \rightarrow \overline{r^{\max}_s}$ *from* $\mathcal{C}_{\text{goal}} \cup H_{\max}$.

For the inductivity lemma, we first show an analogous statement for a single action.

Lemma 16. *For every action a, it is possible to derive* $(r^{\max}_s \wedge r_a) \rightarrow r^{\max\prime}_s$ *from* $\mathcal{C}_{\text{trans}} \cup H_{\max} \cup H'_{\max}$.

Proof sketch. If $\mathit{pre}(a) \neq \emptyset$, let $p \in \mathit{pre}(a)$ be a precondition with maximal $W^{\max}$ value.

If $W^{\max}(p) = h^{\max}(s)$, we exploit that the $h^{\max}$ cost of $\mathit{pre}(a) \geq B$ is so high that $K_{\geq B} \geq 1$, which transfers to the primed cost.

If $W^{\max}(p) < h^{\max}(s)$ we derive $(r^{\max}_s \wedge r_a) \rightarrow r'_v$ for all variables v by RUP. If $v \in \mathit{del}(a)$, $(\overline{v'} \geq 1)$, then $r'_v \geq 1$, independent of the cost. If $v \in \mathit{add}(a)$, we exploit that $W^{\max}(p) = V^{\max}(p)$ and consequently $h^{\max}(s, \mathit{pre}(a)) = V^{\max}(p)$ because p has maximum $V^{\max}$-value among a's preconditions. This implies that that $V^{\max}(v) \leq V^{\max}(p) + \mathit{cost}(a)$ and consequently $W^{\max}(v) \leq W^{\max}(p) + \mathit{cost}(a)$. We can derive that $K'_{\geq B - h^{\max}(s) + W^{\max}(v)} \geq 1$, which is sufficient to derive $r_v \geq 1$ also in this case. If $v \notin \mathit{evars}(a)$, we can derive from assumption $\overline{r'_v} \geq 1$ that $v' \geq 1$ and $\overline{K'_{\geq B - h^{\max}(s) + W^{\max}(v)}} \geq 1$. Both properties can be propagated to the unprimed variables within some steps, contradicting the assumption $r^{\max}_s \geq 1$. With the subresults $(r^{\max}_s \wedge r_a) \rightarrow r'_v$, RUP can derive that $(r^{\max}_s \wedge r_a) \rightarrow r^{\max\prime}_s$.

The proof for an empty precondition is analogous without the need to use p to establish a certain $K_{\geq}$. □

The full inductivity lemma follows quite directly.

Lemma 17. *It is possible to derive the inductivity lemma* $(r^{\max}_s \wedge r_T) \rightarrow r^{\max\prime}_s$ *from* $\mathcal{C}_{\text{trans}} \cup H_{\max} \cup H'_{\max}$.

Proof sketch. Establish with Lemma 16 for every $a \in \mathcal{A}$ that $(r^{\max}_s \wedge r_a) \rightarrow r^{\max\prime}_s$. Then derive the constraint by RUP. □

Extending $h^{\max}$ with proof logging only leads to the following overhead: adding reifications (17) for each $v \in \mathcal{V}$ and (18) takes additional time linear in $|\mathcal{V}|$ because $h(s)$ and the values necessary for $W^{\max}$ are already computed during the normal evaluation. In addition, it has to reinitialize some values for each variable before each new evaluation, so the overhead is within a constant factor.

The state and goal lemmas each only require a single RUP statement. For the inductivity lemma, we need to establish the constraint from Lemma 16 for every action. For each action where $h^{\max}(s, \mathit{pre}(a)) \geq h^{\max}(s)$, we require constant effort. For all other actions, the effort for the action is linear in the number of variables.

Conclusion

We introduced lower-bound certificates for classical planning, which can be used to verify the optimality of optimal planners. The certificates are sound, complete and efficiently verifiable. We showed them to be efficiently generatable in a case study on A* with $h^{\max}$ or pattern database heuristics.

We believe these certificates to be much more general than ones considered in previous research because they work on a more fundamental level of abstraction, build on an established proof systems that has proved its utility in many other areas of AI, and are able to directly incorporate numerical reasoning. In future work, this generality has to be confirmed by applying these certificates to other optimal planning approaches that current approaches cannot handle. In particular, we believe that pseudo-Boolean constraints can generically handle the concept of *cost partitioning* (Katz and Domshlak 2010), perhaps *the* most important optimal planning technique, because they are able to reason numerically at the level of individual state transitions in a way that previously proposed certificates cannot.

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Supplemental Material "Pseudo-Boolean Proof Logging for Optimal Classical Planning"

Proofs for Cost Lemmas

We will use an extended cutting planes proof technique that we did not introduce in the background section, namely redundance-based strengthening (RED): if for a PB constraint C there is a substitution ρ such that $\mathcal{C} \cup \{\neg C\} \models \mathcal{C} \cup \{\neg C\} \upharpoonright_\rho$ then we can infer C. We only use the simple form with an empty substitution. Since it cannot invalidate the core formula $\mathcal{C}$, we only need to derive that $\mathcal{C} \cup \{\neg C\} \models \neg C$ to establish C. We can think of this as a more flexible form of a proof by contradiction than simple RUP.

We now provide the proofs for the two cost-related lemmas from the paper.

Lemma 1. *Let $j \in \mathbb{Z}$ and $k \in \mathbb{N}_0$. It is possible to derive $cost_{\geq \min\{B,\max\{0,j+k\}\}} \to cost_{\geq \min\{B,\max\{0,j\}\}}$ from $\mathcal{C}_\geq$.*

Proof. By RED with an empty witness. Since the witness is empty, all constraints from $\mathcal{C}_\geq$ are trivially implied, so we only need to derive $C \doteq cost_{\geq \min\{B,\max\{0,j+k\}\}} \to cost_{\geq \min\{B,\max\{0,j\}\}}$.

We may assume the negation of this constraint, which gives us by weakening that (a) $cost_{\geq \min\{B,\max\{0,j+k\}\}} \geq 1$ and (b) $\overline{cost_{\geq \min\{B,\max\{0,j\}\}}} \geq 1$. Using (4), we get from (a) that (c) $\sum_{i=0}^{\lceil \log_2 B \rceil} 2^i c_i \geq \min\{B, \max\{0, j+k\}\}$.

Define $\Delta = \min\{B, \max\{0, j+k\}\} - \min\{B, \max\{0, j\}\}$ and observe that $\Delta \in \{0, \ldots, B\}$.

Let set M contain exactly the numbers i such that the ith bit in the binary representation of Δ is 1. With a full weakening[2] of (c) with M, we receive (d) $\sum_{i=\in\{0,\ldots,\lceil \log_2 B \rceil\} \setminus M} 2^i c_i \geq \min\{B, \max\{0, j\}\}$.

For every $i \in M$, we can multiply the axiom lemma $c_i \geq 0$ with 2^i and receive $2^i c_i \geq 0$. The sum over all these constraints and (d) is $\sum_{i=0}^{\lceil \log_2 B \rceil} 2^i c_i \geq \min\{B, \max\{0, j\}\}$. With (4), this yields $cost_{\geq \min\{B,\max\{0,j\}\}}$. □

Lemma 2. *For $l \in \mathbb{Z}$ and $m \in \mathbb{N}_0$ it is possible to derive $(cost_{\geq \min\{B,\max\{0,l\}\}} \wedge \Delta c^{=m}) \to cost'_{\geq \min\{B,\max\{0,l+m\}\}}$ from $\mathcal{C}_\geq$.*

Proof. By RED with an empty witness. Since the witness is empty, all constraints from $\mathcal{C}_\geq$ are trivially implied, so we only need to derive $C \doteq (cost_{\geq \min\{B,\max\{0,l\}\}} \wedge \Delta c^{=m}) \to cost'_{\geq \min\{B,\max\{0,l+m\}\}}$.

We may assume the negation of this constraint, which gives us by weakening that (a) $cost_{\geq \min\{B,\max\{0,l\}\}} \geq 1$, (b) $\Delta c^{=m} \geq 1$, and (c) $\overline{cost'_{\geq \min\{B,\max\{0,l+m\}\}}} \geq 1$.

From (b) and (3) we get that (d) $\sum_{i=0}^{\lceil \log_2 B \rceil} 2^i c'_i - \sum_{i=0}^{\lceil \log_2 B \rceil} 2^i c_i \geq m$. From (a) and (4), we derive (e) $\sum_{i=0}^{\lceil \log_2 B \rceil} 2^i c_i \geq \min\{B, \max\{0, l\}\}$. The sum of (d) and (e) is (f) $\sum_{i=0}^{\lceil \log_2 B \rceil} 2^i c'_i \geq \min\{B, \max\{0, l\}\} + m$. From (c), we derive (g) $\sum_{i=0}^{\lceil \log_2 B \rceil} 2^i c'_i < \min\{B, \max\{0, l+m\}\}$, which in normalized form is $\sum_{i=0}^{\lceil \log_2 B \rceil} 2^i \overline{c'_i} \geq \sum_{i=0}^{\lceil \log_2 B \rceil} 2^i - \min\{B, \max\{0, l+m\}\} + 1$. The sum of (f) and (g) is (h) $0 \geq \min\{B, \max\{0, l\}\} + m - \min\{B, \max\{0, l+m\}\} + 1$. Since $\min\{B, \max\{0, l\}\} + m \geq \min\{B, \max\{0, l+m\}\}$, the right-hand side of (h) is positive, so we have a contradiction, from which we can derive any constraint, thus also C. □

[2] Full weakening is syntactic sugar that corresponds to multiplying the literal axioms $\ell \geq 0$ with the coefficients of ℓ and adding this to the constraint.

Proof-logging PDB Heuristics: Proofs

Lemma 8. *RUP can derive the state lemma $(r_s \wedge cost_{\geq \max\{0,B-h(s)\}}) \to r_{\text{PDB}}$ from $\mathcal{C}_{\text{init}} \cup H_{\text{PDB}} \cup \mathcal{C}_s$, where $\mathcal{C}_s = \{r_s \Leftrightarrow \sum_{v \in s} v + \sum_{v \in \mathcal{V} \setminus s} \bar{v} \geq |\mathcal{V}|\}$.*

Proof. Assume (a) $r_s \geq 1$, (b) $K_{\geq B-h(s)} \geq 1$, and (c) $\overline{r_{\text{PDB}}} \geq 1$. From (a) and $\mathcal{C}_s$, we receive $v \geq 1$ for each $v \in s$ and $\bar{v} \geq 1$ for each $v \in \mathcal{V} \setminus s$, and in particular (d) $v \geq 1$ for each $v \in \alpha(s)$ and $\bar{v} \geq 1$ for each $v \in P \setminus \alpha(s)$. With constraints (14) and (d), we derive (e) $r^{\alpha(s)} \geq 1$. Since $h(s) = d(\alpha(s))$, we can use (b) and (e) with (15) to derive (f) $r^{\alpha(s)}_{\geq B-d(\alpha(s))}$. With (16), this gives us $r_{\text{PDB}} \geq 1$, a contradiction to (c). □

Lemma 9. *RUP can derive the goal lemma $(r_G \wedge \overline{cost_{\geq B}}) \to \overline{r_{\text{PDB}}}$ from $\mathcal{C}_{\text{goal}} \cup H_{\text{PDB}}$.*

Proof. Assume (a) $r_G \geq 1$, (b) $r_{\text{PDB}} \geq 1$ and (c) $\overline{cost_{\geq B}} \geq 1$. From (a) and (2) we can derive that (d) $v \geq 1$ for each goal variable $v \in G$.

We now iterate over all abstract states s_α. If s_α is *not* an abstract goal state, we use (d) to derive from (14) that $\overline{r^{s_\alpha}} \geq 1$ and consequently with (15) $\overline{r^{s_\alpha}_{\geq B-d(s_\alpha)}} \geq 1$. If s_α is an abstract goal state, we exploit that $d(s_\alpha) = 0$ and use (c) with (15) to derive $\overline{r^{s_\alpha}_{\geq B-d(s_\alpha)}} \geq 1$. So for all abstract states s_α we derived $\overline{r^{s_\alpha}_{\geq B-d(s_\alpha)}} \geq 1$. This gives with (16) that $\overline{r_{\text{PDB}}} \geq 1$, a contradiction to (b). □

Lemma 10. *For each action $a \in \mathcal{A}$ and abstract state s_α such that a^α is applicable in s_α, RUP can derive $(r^{s_\alpha} \wedge r_a) \to r^{s_\alpha[\![a^\alpha]\!]'}$ from $\mathcal{C}_{\text{trans}} \cup H_{\text{PDB}} \cup H'_{\text{PDB}}$.*

Proof. Assume (a) $r^{s_\alpha} \geq 1$, (b) $r_a \geq 1$, and (c) $\overline{r^{s_\alpha[\![a^\alpha]\!]'}} \geq 1$. From (a) and (14) we get (d) $v \geq 1$ for all $v \in s_\alpha$ and $\bar{v} \geq 1$ for all $v \in P \setminus s_\alpha$. From (b) and (7), we get (e) $v' \geq 1$ for all $v \in add(a)$, (f) $\overline{v'} \geq 1$ for all $v \in del(a)$, and (g) $eq_{v,v'} \geq 1$ for all $v \in \mathcal{V} \setminus evars(a)$. From (d) and (g) and (6), we derive for all $v \in P \setminus evars(a)$ that $v' \geq 1$ if $v \in s_\alpha$ and $\overline{v'} \geq 1$ if $v \notin s_\alpha$. Combining this with (e), (f) and (14) from H'_{PDB} for $s_\alpha[\![a^\alpha]\!]$, we derive $r^{s_\alpha[\![a^\alpha]\!]'} \geq 1$, contradicting (c). □

Lemma 12. *For each abstract state s_α it is possible to derive $(r^{s_\alpha}_{\geq B-d(s_\alpha)} \wedge r_T) \to r'_{\text{PDB}}$ from $\mathcal{C}_{\text{trans}} \cup H_{\text{PDB}} \cup H'_{\text{PDB}}$.*

Proof. For each action $a \in \mathcal{A}$ that is applicable in s_α, we establish (a) $(r^{s_\alpha}_{\geq B-d(s_\alpha)} \wedge r_a) \to r'_{\text{PDB}}$ with Lemma 11.

Afterwards, the desired constraint can be derived by RUP, assuming (b) $r^{s_\alpha}_{\geq B-d(s_\alpha)} \geq 1$, (c) $r_T \geq 1$ and (d) $\overline{r'_{\text{PDB}}} \geq 1$.

For each action $a \in \mathcal{A}$, we derive (e) $\overline{r_a} \geq 1$ as follows: If a^α is applicable in s_α, we use (a), (b) and (d). Otherwise, there is a $v \in \mathit{pre}(a) \cap P$ with $v \notin s_\alpha$. From (b) and (15), we derive $r^{s_\alpha} \geq 1$, which with (14) gives us (f) $\overline{v} \geq 1$. Since $v \in \mathit{pre}(a)$, we can use (f) with (7) to derive (e) also for this case. Since we have derived (e) for all actions $a \in \mathcal{A}$, we can use (8) to derive $\overline{r_T} \geq 1$, a contradiction to (c). □

Lemma 13. *It is possible to derive the inductivity lemma* $(r_{\text{PDB}} \wedge r_T) \rightarrow r'_{\text{PDB}}$ *from* $\mathcal{C}_{\text{trans}} \cup H_{\text{PDB}} \cup H'_{\text{PDB}}$.

Proof. We first use lemma 12 to establish for each abstract state s_α (a) $(r^{s_\alpha}_{\geq B-d(s_\alpha)} \wedge r_T) \rightarrow r'_{\text{PDB}}$.

The rest can be derived by RUP, assuming (b) $r_{\text{PDB}} \geq 1$, (c) $r_T \geq 1$, and (d) $\overline{r'_{\text{PDB}}} \geq 1$. Using (c) and (d) with (a), we can derive for each abstract state s_α that $\overline{r^{s_\alpha}_{\geq B-d(s_\alpha)}} \geq 1$. With (16), this gives $\overline{r_{\text{PDB}}} \geq 1$, a contradiction to (b). □

Proof-logging $h^{\max}$: Proofs

Lemma 14. *RUP can derive the state lemma* $(r_s \wedge \mathit{cost}_{\geq \max\{0, B-h^{\max}(s)\}}) \rightarrow r^{\max}_s$ *from* $\mathcal{C}_{\text{init}} \cup H_{\max} \cup \mathcal{C}_s$, *where* $\mathcal{C}_s = \{r_s \Leftrightarrow \sum_{v \in s} v + \sum_{v \in \mathcal{V} \setminus s} \bar{v} \geq |\mathcal{V}|\}$.

Proof. Assume (a) $r_s \geq 1$, (b) $\mathit{cost}_{\geq \max\{0, B-h^{\max}(s)\}} \geq 1$, and (c) $\overline{r^{\max}_s} \geq 1$. From (a), we get for all $v \in \mathcal{V} \setminus s$ that $\bar{v} \geq 1$, so with (17) we get for all such v that $r_v \geq 1$. For all $v \in s$, the heuristic determines $W^{\max}(v) = V^{\max}(v) = 0$, and thus $B - h^{\max}(s) + W^{\max}(v) \leq B$, so from (b) and (17) we get $r_v \geq 1$ also for these variables. Together, we derive with (18) that $r^{\max}_s \geq 1$, contradicting (c). □

Lemma 15. *RUP can derive the goal lemma* $(r_G \wedge \overline{\mathit{cost}_{\geq B}}) \rightarrow \overline{r^{\max}_s}$ *from* $\mathcal{C}_{\text{goal}} \cup H_{\max}$.

Proof. Assume (a) $r_G \geq 1$, (b) $\overline{\mathit{cost}_{\geq B}} \geq 1$ and (c) $r^{\max}_s \geq 1$. Consider a goal variable g with maximum max value. For this g, it holds that $V^{\max}(g) = h^{\max}(s)$. From (a) we derive that $g \geq 1$, which gives together with (b) and (17) $\overline{r_g} \geq 1$. Thus we get from (18) that $\overline{r^{\max}_s} \geq 1$, contradicting (c). □

Lemma 16. *For every action a, it is possible to derive* $(r^{\max}_s \wedge r_a) \rightarrow r^{\max\prime}_s$ *from* $\mathcal{C}_{\text{trans}} \cup H_{\max} \cup H'_{\max}$.

Proof. If $\mathit{pre}(a) \neq \emptyset$, let $p \in \mathit{pre}(a)$ be a precondition with maximal $W^{\max}$ value.

If $W^{\max}(p) = h^{\max}(s)$, we use Lemma 2 to establish (*) $(K_{\geq B} \wedge \Delta c^{=\mathit{cost}(a)}) \rightarrow K'_{\geq B}$ and proceed by RUP, assuming (a) $r^{\max}_s \geq 1$ and (b) $r_a \geq 1$. From (a) and (18) we get (c) $r_p \geq 1$. From (b) and (7) we get (d) $p \geq 1$, (e) $\overline{\mathit{cost}'_{\geq B}} \geq 1$ and (f) $\Delta c^{=\mathit{cost}(a)} \geq 1$. From (c) and (d), we get with (17) that (g) $K_{\geq B-h^{\max}(s)+W^{\max}(p)} \geq 1$. Then (f), (g) and (*) yield $K'_{\geq B} \geq 1$, a contradiction to (e).

If $W^{\max}(p) < h^{\max}(s)$, we use Lemma 2 to establish (**) $(K_{\geq B-h^{\max}(s)+W^{\max}(v)} \wedge \Delta c^{=\mathit{cost}(a)}) \rightarrow K'_{\geq B-h^{\max}(s)+W^{\max}(v)+\mathit{cost}(a)}$ for all $v \in \mathcal{V}$. We use Lemma 1 to establish (***) $K'_{\geq B-h^{\max}(s)+W^{\max}(v)+\mathit{cost}(a)} \rightarrow K'_{\geq B-h^{\max}(s)+W^{\max}(v)}$ for every $v \in \mathcal{V} \setminus \mathit{evars}(a)$. For every $v \in \mathit{add}(a)$, it holds by the definition of $h^{\max}$ that $V^{\max}(v) \leq V^{\max}(p) + \mathit{cost}(a)$ and consequently $W^{\max}(v) \leq W^{\max}(p) + \mathit{cost}(a)$. We thus can establish with Lemma 1 that (****) $K'_{\geq B-h^{\max}(s)+W^{\max}(p)+\mathit{cost}(a)} \rightarrow K'_{\geq B-h^{\max}(s)+W^{\max}(v)}$.

Then we derive $(r^{\max}_s \wedge r_a) \rightarrow r'_v$ for all variables v by RUP: Assume (h) $r^{\max}_s \geq 1$, (i) $r_a \geq 1$, and (j) $\overline{r'_v} \geq 1$. From (i) and (7) we get (k) $\Delta c^{=\mathit{cost}(a)} \geq 1$. From (h) with (17) and (18) we derive (l) $K_{\geq B-h^{\max}(s)+W^{\max}(p)} \geq 1$ as in case $W^{\max}(p) = h^{\max}(s)$.

If $v \in \mathit{del}(a)$, we have from (i) and (7) that $\overline{v'} \geq 1$. With (17) this yields $r'_v \geq 1$, contradicting (j).

If $v \in \mathit{add}(a)$, (**) with (l) and (k) to derive $K'_{\geq B-h^{\max}(s)+W^{\max}(p)+\mathit{cost}(a)}$, which together with (****) gives $K'_{\geq B-h^{\max}(s)+W^{\max}(v)}$. With (17) from $H'_{\max}$, we get $r'_v \geq 1$, contradicting (j).

If $v \in \mathcal{V} \setminus \mathit{evars}(a)$, (j) and (17) yield (m) $v' \geq 1$ and (l) $\overline{K'_{\geq B-h^{\max}(s)+W^{\max}(v)}} \geq 1$. From (i) and (7) we get $eq_{v,v'} \geq 1$, which with (m) yields (n) $v \geq 1$. We use (l) with (***) to get $\overline{K'_{\geq B-h^{\max}(s)+W^{\max}(v)+\mathit{cost}(a)}} \geq 1$, which gives with (**) and (k) that $\overline{K_{\geq B-h^{\max}(s)+W^{\max}(v)}} \geq 1$. With (17) and (n) this yields $\overline{r_v} \geq 1$, which with (18) gives $\overline{r^{\max}_s} \geq 1$, contradicting (h).

RUP can now derive that $(r^{\max}_s \wedge r_a) \rightarrow r^{\max\prime}_s$: Assume (o) $r^{\max}_s \geq 1$, (p) $r_a \geq 1$ and (q) $\overline{r^{\max\prime}_s} \geq 1$. For all $v \in \mathcal{V}$, we get with (o), (p) and the previously derived $(r^{\max}_s \wedge r_a) \rightarrow r'_v$, that $r'_v \geq 1$. With (18) from $H_{\max}$, this yields $r^{\max\prime}_s \geq 1$, contradicting (q).

If $\mathit{pre}(a) = \emptyset$ the proof is analogous but simpler: instead of using p to derive $\mathit{cost}_{\geq B-h^{\max}(s)+W^{\max}(p)} \geq 1$, we use $\mathit{cost}_{\geq B-h^{\max}(s)} \geq 1$. Then case $h^{\max}(s) = 0$ works like the previous case for $W^{\max}(p) = h^{\max}(s)$. Otherwise, we proceed as in the case $W^{\max}(p) < h^{\max}(s)$. If $v \in \mathit{add}(a)$, we used $W^{\max}(p)$ to refer to $h^{\max}(s, \mathit{pre}(a))$, which is 0 with an empty precondition. We can adapt the proof by replacing all occurrences of $W^{\max}(p)$ with 0. □

Lemma 17. *It is possible to derive the inductivity lemma* $(r^{\max}_s \wedge r_T) \rightarrow r^{\max\prime}_s$ *from* $\mathcal{C}_{\text{trans}} \cup H_{\max} \cup H'_{\max}$.

Proof. Establish with Lemma 16 for every $a \in \mathcal{A}$ that (a) $(r^{\max}_s \wedge r_a) \rightarrow r^{\max\prime}_s$. Afterwards, continue by RUP: Assume (b) $r^{\max}_s \geq 1$, (c) $r_T \geq 1$, and (d) $\overline{r^{\max\prime}_s} \geq 1$. With (a), (b) and (d), derive (e) $\overline{r_a} \geq 1$ for every $a \in \mathcal{A}$. With (8), this gives $\overline{r_T} \geq 1$, contradicting (c). □

State Set Extension Lemma

The proofs in the next section require to switch from a description of a set of states in form of a constraint that fixes the value of some state variables, to a description of the same set of states in form of an enumeration of tighter constraints describing subsets of the original set of states. The next lemma shows how to construct a proof for this statement.

In the following, we write x^b to denote the literal over x that evaluates to 1 exactly when $x \mapsto b$, i.e., $x^b = x$ if $b = 1$ and $x^b = \overline{x}$ if $b = 0$.

Lemma 18. *Let Y, Z be two sets of variables with $Y \subseteq Z$. Let α be a fixed assignment of the variables in Y and let $\mathcal{B}$ be the set of all assignments β of Z with $\beta \supseteq \alpha$ extending α. Suppose that there are reified constraints*

$$r_\alpha \Rightarrow \sum_{y \in Y} y^{\alpha(y)} \geq |Y| \tag{19}$$

and for each $\beta \in \mathcal{B}$ constraints

$$r_\beta \Leftarrow \sum_{z \in Z} z^{\beta(z)} \geq |Z| \tag{20}$$

defining variables for all these assignments. Then there is a CPR derivation from (19) *and* (20) *of*

$$\overline{r_\alpha} + \sum_{\beta \in \mathcal{B}} r_\beta \geq 1 \tag{21}$$

in $\mathcal{O}(|\mathcal{B}|)$ steps.

Proof. Let $Z \backslash Y = \{z_1, \dots, z_n\}$ and $Z_i = \{z_1, \dots, z_i\} \cup Y$. Additionally, let $\mathcal{B}_i$ be the set of all assignments of Z_i with $\beta_i \supseteq \alpha$ extending α (i.e. $\mathcal{B}_n = \mathcal{B}$ and $\mathcal{B}_0 = \{\alpha\}$).

Step 1: For each of the 2^{n-1} many assignments $\beta_{n-1} : Z_{n-1} \to \{0, 1\}$ in $\mathcal{B}_{n-1}$ we derive the constraint

$$\sum_{\beta \supseteq \beta_{n-1}} r_\beta + \sum_{z \in Z_{n-1}} z^{1-\beta_{n-1}(z)} \geq 1 \tag{a}$$

by RUP: Assuming the negation of (a), we get (b) $\overline{r_\beta} \geq 1$ for all $\beta \supseteq \beta_{n-1}$, and (c) $z^{\beta_{n-1}(z)} \geq 1$ for all $z \in Z_{n-1}$. Note that $\{\beta_{n-1} \cup \{z_n \mapsto 0\}, \beta_{n-1} \cup \{z_n \mapsto 1\}\} \subseteq \mathcal{B}_n$. Thus, by using (b) and (c) with (20) for $\beta_{n-1} \cup \{z_n \mapsto 0\}$, we derive (d) $z_n \geq 1$, and similarly by using (b) and (c) with (20) for $\beta_{n-1} \cup \{z_n \mapsto 1\}$ we derive $\overline{z_n} \geq 1$, a contradiction to (d).

Step k: As induction hypothesis we have

$$\sum_{\beta \supseteq \beta_{n-k+1}} r_\beta + \sum_{z \in Z_{n-k+1}} z^{1-\beta_{n-k+1}(z)} \geq 1 \tag{e}$$

for each $\beta_{n-k+1} \in \mathcal{B}_{n-k+1}$. We derive the constraint

$$\sum_{\beta \supseteq \beta_{n-k}} r_\beta + \sum_{z \in Z_{n-k}} z^{1-\beta_{n-k}(z)} \geq 1 \tag{f}$$

by RUP: Assuming the negation of (f), we get (g) $\overline{r_\beta} \geq 1$ for all $\beta \supseteq \beta_{n-k}$, and (h) $z^{\beta_{n-k}(z)} \geq 1$ for all $z \in Z_{n-k}$. We use (g) and (h) with (e) for $\beta_{n-k} \cup \{z_{n-k+1} \mapsto 0\}$ obtained in step $k-1$, to derive (i) $z_{n-k+1} \geq 1$, and similarly we use (g) and (h) with (e) for $\beta_{n-k} \cup \{z_{n-k+1} \mapsto 1\}$ to derive $\overline{z_{n-k+1}} \geq 1$, a contradiction to (i).

The constraint derived in step n is

$$\sum_{\beta \in \mathcal{B}} r_\beta + \sum_{y \in Y} y^{1-\alpha(y)} \geq 1 \tag{j}$$

as $\beta_0 = \alpha$ and $Z_0 = Y$.

We can now derive the desired constraint (21) by RUP: Assuming that $r_\alpha + \sum_{\beta \in \mathcal{B}} \overline{r_\beta} \geq 1 + |\mathcal{B}|$ we obtain (k) $r_\alpha \geq 1$ and (l) $\overline{r_\beta} \geq 1$ for all $\beta \in \mathcal{B}$. From (k) and (19) we obtain that (m) $y^{\alpha(y)} \geq 1$ for all $y \in Y$, and thus (j) with (l) and (m) leads to a contradiction. □

Efficiently Proof-Logging Pattern Database Heuristics

In the paper, we described PDB certificates where the overhead is not within a constant factor of an efficient implementation of a PDB heuristic, which only traverses the part of the abstract state space that is backwards-reachable from the goal. Here we describe an alternative approach that is constant-factor efficient for such an implementation.

Definition 6. *Let $\Pi = \langle \mathcal{V}, \mathcal{A}, I, G \rangle$ be a STRIPS planning task and $P \subseteq \mathcal{V}$. The* PB circuit for the PDB heuristic with pattern P *is the PB circuit $\langle H_{\text{PDB}}, r_{\text{PDB}} \rangle$ with input variables $\mathcal{V} \cup \mathcal{V}_c$, where H_{PDB} contains for each abstract state $s_\alpha \subseteq P$ with $d(s_\alpha) < \infty$ two constraints*

$$r^{s_\alpha} \Leftrightarrow \sum\nolimits_{v \in s_\alpha} v + \sum\nolimits_{v \in P \setminus s_\alpha} \bar{v} \geq |P|, \text{ and} \tag{22}$$

$$r^{s_\alpha}_{\geq B - d(s_\alpha)} \Leftrightarrow r^{s_\alpha} + K_{\geq B - d(s_\alpha)} \geq 2, \tag{23}$$

and with $S = \{s_\alpha \subseteq P \mid d(s_\alpha) < \infty\}$ the constraint

$$r^\infty \Leftrightarrow \sum\nolimits_{s_\alpha \in S} \overline{r^{s_\alpha}} \geq |S| \tag{24}$$

as well as the final reification

$$r_{\text{PDB}} \Leftrightarrow r^\infty + \sum\nolimits_{s_\alpha \in S} r^{s_\alpha}_{\geq B - d(s_\alpha)} \geq 1. \tag{25}$$

On the evaluation of a state s, the heuristic always returns reification variable r_{PDB} to the search. In the following we discuss how the required proofs can be generated.

We begin with the state lemma, which requires a new proof for each evaluated state.

Lemma 19. *RUP can derive the state lemma* $(r_s \wedge \mathit{cost}_{\geq \max\{0, B-h(s)\}}) \to r_{\text{PDB}}$ *from $\mathcal{C}_{\text{init}} \cup H_{\text{PDB}} \cup \mathcal{C}_s$, where $\mathcal{C}_s = \{r_s \Leftrightarrow \sum_{v \in s} v + \sum_{v \in \mathcal{V} \setminus s} \bar{v} \geq |\mathcal{V}|\}$.*

Proof. Assume (a) $r_s \geq 1$, (b) $\mathit{cost}_{\geq \max\{0, B-h(s)\}} \geq 1$, and (c) $\overline{r_{\text{PDB}}} \geq 1$. From (a) and $\mathcal{C}_s$, we receive $v \geq 1$ for each $v \in s$ and $\bar{v} \geq 1$ for each $v \in \mathcal{V} \setminus s$, and in particular (d) $v \geq 1$ for each $v \in \alpha(s)$ and $\bar{v} \geq 1$ for each $v \in P \setminus \alpha(s)$.

If $d(\alpha(s)) < \infty$, we derive with constraints (22) and (d), that (e) $r^{\alpha(s)} \geq 1$. Since $h(s) = d(\alpha(s))$, we can use (b) and (e) with (23) to derive (f) $r^{\alpha(s)}_{\geq B - d(\alpha(s))}$. With (25), this gives us r_{PDB}, a contradiction to (c).

If $d(\alpha(s)) = \infty$, we get from constraints (22) and (d) for every s_α with $d(s_\alpha) < \infty$, that (g) $\overline{r^{s_\alpha}} \geq 1$, which gives with (24) that $r^\infty \geq 1$. With (25), this gives us $r_{\text{PDB}} \geq 1$, a contradiction to (c). □

Since the PB circuit for the heuristic is the same for all evaluated states, we only need to include the proof for the goal lemma and for the inductivity lemma once in the overall generated proof. For the goal lemma, we use the following:

Lemma 20. *It is possible to derive the goal lemma* $(r_G \wedge \overline{\mathit{cost}_{\geq B}}) \to \overline{r_{\text{PDB}}}$ *from $\mathcal{C}_{\text{goal}} \cup H_{\text{PDB}}$.*

Proof. Define a reification of the abstract goal as

$$r_{G^\alpha} \Leftrightarrow \sum_{v \in G \cap P} v \geq |G \cap P|.$$

We first establish (a) $\overline{r_{G^\alpha}} + \sum_{s_\alpha \subseteq P, G\cap P \subseteq s_\alpha} r^{s_\alpha} \geq 1$ using lemma 18: Use $\alpha = \{v \mapsto 1 \mid v \in G \cap P\}$ and for each abstract state s_α with $G \cap P \subseteq s_\alpha$, $\mathcal{B}$ contains the assignment $\beta : P \to \{0, 1\}$ with $\beta(v) = 1$ iff $v \in s_\alpha$.

Next, we use RUP to obtain (b) $r_G \to \overline{r^\infty}$, assuming (c) $r_G \geq 1$ and (d) $r^\infty \geq 1$. From (24) and (d) we obtain (e) $\overline{r^{s_\alpha}} \geq 1$ for each $r^{s_\alpha} \in S$ and in particular for each r^{s_α} where s_α is an abstract goal state which have $d(s_\alpha) = 0 < \infty$ From (c), we get $v \geq 1$ for all $v \in G$, an in particular for $v \in G \cap P$. From these we obtain (f) $r_{G^\alpha} \geq 1$ From (a) and (e) we obtain $\overline{r_{G^\alpha}}$ contradicting (f).

We now can establish the desired constraint via RUP: Assume (g) $r_G \geq 1$, (h) $r_{\text{PDB}} \geq 1$ and (i) $\overline{\mathit{cost}_{\geq B}} \geq 1$. From (g) and (2) we can derive that (j) $v \geq 1$ for each goal variable $v \in G$.

We now iterate over all abstract states s_α with $d(s_\alpha) < \infty$. If s_α is *not* an abstract goal state, we use (j) to derive from (22) that $\overline{r^{s_\alpha}} \geq 1$ and consequently with (23) $\overline{r^{s_\alpha}_{\geq B-d(s_\alpha)}} \geq 1$. If s_α is an abstract goal state, we exploit that $d(s_\alpha) = 0$ and use (i) with (23) to derive $\overline{r^{s_\alpha}_{\geq B-d(s_\alpha)}} \geq 1$. So for all abstract states s_α with $d(s_\alpha) < \infty$ we derived (k) $\overline{r^{s_\alpha}_{\geq B-d(s_\alpha)}} \geq 1$. With (h), (k) and (25), we derive $r^\infty \geq 1$, and from (g) and (b) that $\overline{r^\infty} \geq 1$, a contradiction. □

For the inductivity lemma, we develop the derivation by means of several lemmas. The first one considers the case where applying an induced abstract action a^α in abstract state s_α leads to an abstract successor state that is backwards reachable from the goal.

Lemma 21. *For each action $a \in \mathcal{A}$ and abstract state s_α such that a^α is applicable in s_α and $d(s_\alpha[\![a^\alpha]\!]) < \infty$, RUP can derive*
$(r^{s_\alpha} \wedge r_a) \to r^{s_\alpha[\![a^\alpha]\!]\prime}$ *from* $\mathcal{C}_{\text{trans}} \cup H_{\text{PDB}} \cup H'_{\text{PDB}}$.

Proof. By RUP assume (a) $r^{s_\alpha} \geq 1$, (b) $r_a \geq 1$, and (c) $\overline{r^{s_\alpha[\![a^\alpha]\!]\prime}} \geq 1$. From (a) and (22) we get (d) $v \geq 1$ for all $v \in s_\alpha$ and $\bar{v} \geq 1$ for all $v \in P \setminus s_\alpha$. From (b) and (7), we get (e) $v' \geq 1$ for all $v \in \mathit{add}(a)$, (f) $\overline{v'} \geq 1$ for all $v \in \mathit{del}(a)$, and (g) $\mathit{eq}_{v,v'} \geq 1$ for all $v \in \mathcal{V} \setminus \mathit{evars}(a)$. From (d) and (g) and (6), we derive for all $v \in P \setminus \mathit{evars}(a)$ that $v' \geq 1$ if $v \in s_\alpha$ and $\overline{v'} \geq 1$ if $v \notin s_\alpha$. Combining this with (e), (f) and the primed (22) from H'_{PDB} for $s_\alpha[\![a^\alpha]\!]$, we derive $r^{s_\alpha[\![a^\alpha]\!]\prime} \geq 1$, contradicting (c). □

The next lemma again considers individual actions and abstract states, but takes cost into account and allows to derive that the successor situation is covered by the certificate.

Lemma 22. *For each action $a \in \mathcal{A}$ and abstract state s_α such that a^α is applicable in s_α and $d(s_\alpha[\![a^\alpha]\!]) < \infty$, it is possible to derive*
$(r^{s_\alpha}_{\geq B-d(s_\alpha)} \wedge r_a) \to r'_{\text{PDB}}$ *from* $\mathcal{C}_{\text{trans}} \cup H_{\text{PDB}} \cup H'_{\text{PDB}}$.

Proof. We start by deriving some constraints over costs that we will use later in a RUP proof.

First, we derive (a) $(\mathit{cost}_{\geq \max\{0, B-d(s_\alpha)\}} \wedge \Delta c^{=\mathit{cost}(a)}) \to \mathit{cost}'_{\geq \min\{B, \max\{0, B-d(s_\alpha)+\mathit{cost}(a)\}\}}$ as described in lemma 2.

We know that for the abstract goal distance it holds that $d(s_\alpha[\![a^\alpha]\!]) + \mathit{cost}(a^\alpha) \geq d(s_\alpha)$. Together with $\mathit{cost}(a^\alpha) = \mathit{cost}(a)$, it follows that $B - d(s_\alpha[\![a^\alpha]\!]) \leq B - d(s_\alpha) + \mathit{cost}(a)$. We use this to derive (b) $K'_{\geq B-d(s_\alpha)+\mathit{cost}(a)} \to K'_{\geq B-d(s_\alpha[\![a^\alpha]\!])}$ as described in lemma 1.

In addition, we establish (c) $(r^{s_\alpha} \wedge r_a) \to r^{s_\alpha[\![a^\alpha]\!]\prime}$ by RUP (lemma 21).

Now we can derive the constraint in the claim by RUP, assuming that (d) $r^{s_\alpha}_{\geq B-d(s_\alpha)} \geq 1$, (e) $r_a \geq 1$, and (f) $\overline{r'_{\text{PDB}}} \geq 1$. From (d), we get with (23) that (g) $r^{s_\alpha} \geq 1$ and (h) $K_{\geq B-d(s_\alpha)} \geq 1$. From (e), (g) and (c) we derive (i) $r^{s_\alpha[\![a^\alpha]\!]\prime} \geq 1$.

From (e) and (7), we derive (j) $\Delta c^{=\mathit{cost}(a)} \geq 1$. From (a) together with (h) and (j) we derive that (k) $K'_{\geq B-d(s_\alpha)+\mathit{cost}(a)} \geq 1$. From (b) together with (k) we derive that (l) $K'_{\geq B-d(s_\alpha[\![a^\alpha]\!])} \geq 1$. From (l), (i) and the primed constraint (23) from H'_{PDB} we get $r^{s_\alpha[\![a^\alpha]\!]}_{\geq B-d(s_\alpha[\![a^\alpha]\!])}{}'$. With the primed version of (25) from H'_{PDB} this yields $r'_{\text{PDB}} \geq 1$, contradicting (e). □

The next lemma considers the case that the successor state of some backwards-reachable state is not backwards reachable, and allows to derive that this situation is covered by the certificate.

Lemma 23. *For each action $a \in \mathcal{A}$ and abstract state s_α such that a^α is applicable in s_α and $d(s_\alpha) < \infty$ and $d(s_\alpha[\![a^\alpha]\!]) = \infty$, it is possible to derive*
$(r^{s_\alpha}_{\geq B-d(s_\alpha)} \wedge r_a) \to r'_{\text{PDB}}$ *from* $\mathcal{C}_{\text{trans}} \cup H_{\text{PDB}} \cup H'_{\text{PDB}}$.

Proof. By RUP assume (a) $r^{s_\alpha}_{\geq B-d(s_\alpha)} \geq 1$ and (b) $r_a \geq 1$ and (c) $\overline{r'_{\text{PDB}}} \geq 1$. From (a) and (23) we derive (d) $r^{s_\alpha} \geq 1$.

From (b) and (7), we get (e) $v' \geq 1$ for all $v \in \mathit{add}(a)$, (f) $\overline{v'} \geq 1$ for all $v \in \mathit{del}(a)$, and (g) $\mathit{eq}_{v,v'} \geq 1$ for all $v \in \mathcal{V} \setminus \mathit{evars}(a)$. From (d) and (22) we obtain that (h) $v \geq 1$ for all $v \in s_\alpha$ and (i) $\overline{v} \geq 1$ for all $v \in P \setminus s_\alpha$. Using (6) with (g), (h) and (i), we derive for all $v \in P \setminus \mathit{evars}(a)$ that (j) $v' \geq 1$ if $v \in s_\alpha$ and (k) $\overline{v'} \geq 1$ if $v \notin s_\alpha$. As $d(s_\alpha[\![a^\alpha]\!]) = \infty$, we can derive from the primed versions of (22) from H'_{PDB} and (e), (f), (j) and (k) that (l) $\overline{\widehat{r^{s_\alpha}}{}'} \geq 1$ for all $\widehat{r^{s_\alpha}} \in S$. Therefore, using the primed version of (24) from H'_{PDB} we obtain (m) $r^{\infty\prime} \geq 1$, which with (25) yields $r'_{\text{PDB}} \geq 1$ contradicting (b). □

The next lemma generalizes the previous lemmas from individual actions to the entire transition relation:

Lemma 24. *For each abstract state s_α with $d(s_\alpha) < \infty$, it is possible to derive*
$(r^{s_\alpha}_{\geq B-d(s_\alpha)} \wedge r_T) \to r'_{\text{PDB}}$ *from* $\mathcal{C}_{\text{trans}} \cup H_{\text{PDB}} \cup H'_{\text{PDB}}$.

Proof. For each action $a \in \mathcal{A}$ that is applicable in s_α, we establish (a) $(r^{s_\alpha}_{\geq B-d(s_\alpha)} \wedge r_a) \to r'_{\text{PDB}}$ by means of lemmas 22 and 23.

Afterwards, the desired constraint can be derived by RUP, assuming (b) $r^{s_\alpha}_{\geq B-d(s_\alpha)} \geq 1$, (c) $r_T \geq 1$ and (d) $\overline{r'_{\text{PDB}}} \geq 1$.

For each action $a \in \mathcal{A}$, we derive (e) $\overline{r_a} \geq 1$ as follows: If a^α is applicable in s_α, we use (a), (b) and (d). Otherwise, there is a $v \in \mathit{pre}(a) \cap P$ with $v \notin s_\alpha$. From (b) and (23), we derive $r^{s_\alpha} \geq 1$, which with (22) gives us (f) $\overline{v} \geq 1$. Since $v \in \mathit{pre}(a)$, we can use (f) with (7) to derive (e) also for this case. Since we have derived (e) for all actions $a \in \mathcal{A}$, we can use (8) to derive $\overline{r_T} \geq 1$, a contradiction to (c). □

The previous lemmas have covered the cases where the considered state before a transition is backwards-reachable. The next series of lemmas covers the cases where a transition starts in a region that is not backwards-reachable.

We say that action a is *consistent* with abstract state s_α, if there is an abstract state $\widehat{s_\alpha}$ such that the application of a^α in $\widehat{s_\alpha}$ leads to s_α. An action a is consistent with s_α iff

- all add effects of a from P are true in s_α $\mathit{add}(a) \cap P \subseteq s_\alpha$,
- all delete effects of a from P are false in s_α $\mathit{del}(a) \cap P \cap s_\alpha = \emptyset$, and
- all preconditions of a from P that are not affected by a are still true in s_α: $(\mathit{pre}(a) \cap P) \setminus \mathit{evars}(a) \subseteq s_\alpha$

If the action is not consistent with s_α, we say it is *inconsistent*.

The following lemma shows that we can derive that a backwards-reachable state cannot be reached with an inconsistent action from a region that is not backwards-reachable.

Lemma 25. *For each action $a \in \mathcal{A}$ and abstract state s_α with $d(s_\alpha) < \infty$ such that a is inconsistent with s_α, RUP can derive $(r^\infty \wedge r_a) \rightarrow \overline{r^{s_\alpha}}'$ from $\mathcal{C}_{\text{trans}} \cup H_{\text{PDB}} \cup H'_{\text{PDB}}$.*

Proof. Assume (a) $r^{s_\alpha}{}' \geq 1$, (b) $r_a \geq 1$, and (c) $r^\infty \geq 1$.

From (a) and (22) from H'_{PDB}, we get (d) $v' \geq 1$ for all $v \in s_\alpha$ and (e) $\overline{v'} \geq 1$ for all $v \in P \setminus s_\alpha$.

From (b) and (7), we get (f) $v' \geq 1$ for all $v \in \mathit{add}(a)$, (g) $\overline{v'} \geq 1$ for all $v \in \mathit{del}(a)$, (h) $\mathit{eq}_{v,v'} \geq 1$ for all $v \in \mathcal{V} \setminus \mathit{evars}(a)$ and (i) $v \geq 1$ for all $v \in \mathit{pre}(a)$.

Since a is inconsistent with s_α, (*) there is a $v \in P \cap \mathit{add}(a)$ with $\overline{v'} \geq 1$, or (**) there is a $v \in P \cap \mathit{del}(a)$ with $v' \geq 1$, or (***) there is a $v \in (\mathit{pre}(a) \cap P) \setminus \mathit{evars}(a)$ with $\overline{v'} \geq 1$. In cases (*) and (**), we can derive a contradiction with (f) and (e), or (g) and (e), respectively. In case (***), we can for this v derive from (h) and (6) that $\overline{v} \geq 1$, contradicting (i). □

The next lemma allows us to derive the same constraint as the previous lemma but for a consistent action.

Lemma 26. *For each action $a \in \mathcal{A}$ and abstract state s_α with $d(s_\alpha) < \infty$ such that a is consistent with s_α, it is possible to derive $(r^\infty \wedge r_a) \rightarrow \overline{r^{s_\alpha}}'$ from $\mathcal{C}_{\text{trans}} \cup H_{\text{PDB}} \cup H'_{\text{PDB}}$.*

Proof. Define $L = (\mathit{pre}(a) \cap P) \cup (s_\alpha \setminus \mathit{evars}(a)) \cup \{\overline{v} \mid v \in (P \setminus s_\alpha) \setminus \mathit{evars}(a)\}$ and the corresponding reification (a) $r_L \Leftrightarrow \sum_{\ell \in L} \ell \geq 1$. Define $L^+ = \{\ell \in L \mid \ell = v \text{ for some } v \in P\}$ and $L^- = L \setminus L^+$.

Define set $Q = \{\widehat{s_\alpha} \subseteq P \mid L^+ \subseteq \widehat{s_\alpha} \text{ and } L^- \subseteq P \setminus \widehat{s_\alpha}\}$. This set contains exactly the abstract states $\widehat{s_\alpha}$ such that $\widehat{s_\alpha}[\![a^\alpha]\!] = s_\alpha$. Since s_α has a finite abstract goal distance, this must also be true for all these $\widehat{s_\alpha}$. We can use lemma 18 to derive (b) $\overline{r_L} + \sum_{r^{\widehat{s_\alpha}} \in Q} r^{\widehat{s_\alpha}} \geq 1$.

We are now ready to establish the desired constraint with RUP: Assume (c) $r^{s_\alpha}{}' \geq 1$, (d) $r_a \geq 1$, and (e) $r^\infty \geq 1$.

From (c) and (22) from H'_{PDB}, we get (f) $v' \geq 1$ for all $v \in s_\alpha$ and (g) $\overline{v'} \geq 1$ for all $v \in P \setminus s_\alpha$.

From (d) and (7), we get (h) $v' \geq 1$ for all $v \in \mathit{add}(a)$, (i) $\overline{v'} \geq 1$ for all $v \in \mathit{del}(a)$, (j) $\mathit{eq}_{v,v'} \geq 1$ for all $v \in \mathcal{V} \setminus \mathit{evars}(a)$ and (k) $v \geq 1$ for all $v \in \mathit{pre}(a)$.

We derive from (6), (f) and (j) that (l) $v \geq 1$ for all $v \in s_\alpha \setminus \mathit{evars}(a)$ and analogously from (g) and (j) that (m) $\overline{v} \geq 1$ for all $v \in (P \setminus s_\alpha) \setminus \mathit{evars}(a)$.

From (a) with (k), (l) and (m) we derive (n) $r_L \geq 1$.

From (e) and (24) we get that (o) $\overline{r^{\widehat{s_\alpha}}} \geq 1$ for all $\widehat{s_\alpha} \in S$ and in particular for all $\widehat{s_\alpha} \in Q$.

From (b) and (o) we get $\overline{r_L} \geq 1$, contradicting (n). □

The next lemma generalizes the previous lemmas from individual actions to the entire transition relation:

Lemma 27. *For abstract state s_α with $d(s_\alpha) < \infty$, RUP can derive $\mathcal{C}_\Pi \cup H_{\text{PDB}} \cup H'_{\text{PDB}} \vdash (r^\infty \wedge r_T) \rightarrow \overline{r^{s_\alpha}{}'}$.*

Proof. For each action $a \in \mathcal{A}$, we establish (a) $(r^\infty \wedge r_a) \rightarrow \overline{r^{s_\alpha}{}'}$ by means of lemmas 25 and 26, respectively.

Afterwards, the desired constraint can be derived by RUP, assuming (b) $r^{s_\alpha}{}' \geq 1$, (c) $r_T \geq 1$ and (d) $r^\infty \geq 1$. For every $a \in \mathcal{A}$, we get from (b), (d) and (a) that $\overline{r_a} \geq 1$. With (8), this yields $\overline{r_T} \geq 1$, contradicting (c). □

The following lemma allows us to derive that any successor of any state in a region that is not backwards-reachable, is covered by the certificate:

Lemma 28. *It is possible to derive $(r^\infty \wedge r_T) \rightarrow r'_{\text{PDB}}$ from $\mathcal{C}_{\text{trans}} \cup H_{\text{PDB}} \cup H'_{\text{PDB}}$.*

Proof. We first establish (a) $(r^\infty \wedge r_T) \rightarrow \overline{r^{s_\alpha}{}'}$ for all abstract states $s_\alpha \in S$ using lemma 27.

Then we can derive the desired constraint by RUP, assuming (b) $r^\infty \geq 1$, (c) $r_T \geq 1$, and (d) $\overline{r'_{\text{PDB}}} \geq 1$. From (b) and (c) and (a) we obtain (e) $\overline{r^{s_\alpha}{}'} \geq 1$ for all abstract states $s_\alpha \in S$. Using the primed version of (24) from H'_{PDB} and (e), we obtain (f) $r^{\infty}{}' \geq 1$, which together with the primed version of (25) from H'_{PDB} yields $r'_{\text{PDB}} \geq 1$ in contradiction to (d). □

The final step for the inductivity lemma generalizes the previous results to the entire PB circuit of the heuristic.

Lemma 29. *It is possible to derive the inductivity lemma $(r_{\text{PDB}} \wedge r_T) \rightarrow r'_{\text{PDB}}$ from $\mathcal{C}_{\text{trans}} \cup H_{\text{PDB}} \cup H'_{\text{PDB}}$.*

Proof. We first establish with lemma 24 that (a) $(r^{s_\alpha}_{\geq B-d(s_\alpha)} \wedge r_T) \rightarrow r'_{\text{PDB}}$ for each abstract state s_α with $d(s_\alpha) < \infty$ and with lemma 28 that (b) $(r^\infty \wedge r_T) \rightarrow r'_{\text{PDB}}$ holds.

The desired constraint can now be derived by RUP, assuming (c) $r_{\text{PDB}} \geq 1$, (d) $r_T \geq 1$, and (e) $\overline{r'_{\text{PDB}}} \geq 1$. Using (d) and (e) with (a), we can derive for each abstract state a^α with $d(a^\alpha) < \infty$ that (f) $\overline{r^{s_\alpha}_{\geq B-d(s_\alpha)}} \geq 1$. Using (d) and (e) with (b) we get (g) $\overline{r^\infty} \geq 1$. From (25) with (f) and (g) we get $\overline{r_{\text{PDB}}} \geq 1$, contradicting (c). □