



Computability and Complexity: Problem Set 4

Due: Thursday April 9 at 23:59 AoE.

Submission: Please submit your solutions via *Absalon* as a PDF file. State your name and e-mail address at the top of the first page. Solutions should be written in L^AT_EX or some other math-aware typesetting system with reasonable margins on all sides (at least 2.5 cm). Please try to be precise and to the point in your solutions and refrain from vague statements. Never just state an answer, but make sure to also explain your reasoning. *Write so that a fellow student of yours can read, understand, and verify your solutions.* In addition to what is said below, the general rules for problem sets stated on the course webpage always apply.

Collaboration: Discussions of ideas in groups of two to three people are allowed—and indeed, encouraged—but you should always write up your solutions completely on your own, from start to finish, and you should understand all aspects of them fully. It is not allowed to compose draft solutions together and then continue editing individually, or to share any text, formulas, or pseudocode. Also, no such material may be downloaded from or generated via the internet to be used in draft or final solutions. Submitted solutions will be checked for plagiarism. You should also clearly acknowledge any collaboration. State close to the top of the first page of your problem set solutions if you have been collaborating with someone and if so with whom. *Note that collaboration is on a per problem set basis, so you should not discuss different problems on the same problem set with different people.*

Reference material: Some of the problems are “classic” and hence it might be possible to find solutions on the internet, in textbooks or in research papers. It is not allowed to use such material in any way unless explicitly stated otherwise. Anything said during the lectures or in the lecture notes, or any material found in Arora–Barak, should be fair game, though, unless you are specifically asked to show something that we claimed without proof in class. All definitions should be as given in class or in Arora–Barak and cannot be substituted by versions from other sources. It is hard to pin down 100% watertight, formal rules on what all of this means—when in doubt, ask the main instructor.

Grading: A total score of $\langle E \rangle$ points will be enough for grade 02, $\langle D \rangle$ points for grade 4, $\langle C \rangle$ points for grade 7, $\langle B \rangle$ points for grade 10, and $\langle A \rangle$ points for grade 12 on this problem set. Please note that problems are not necessarily presented in order of difficulty. Unless otherwise stated, every subproblem can be solved independently of other subproblems. Any revised versions of the problem set with clarifications and/or corrections will be posted on the course webpage jakobnordstrom.se/teaching/CoCo26/.

Questions: Please do not hesitate to ask the instructors or TA if any problem statement is unclear, but please make sure to send private messages when using Absalon—sometimes specific enough questions could give away the solution to your fellow students, and we want all of you to benefit from working on, and learning from, the problems. Good luck!

- 1 (10 p) Prove that any (non-constant) monotone Boolean function can be computed by a monotone Boolean circuit.

2 (40 p) In the monotone circuit lower bound for clique presented in the lectures, we used the concept of *sunflowers*, which are collections of sets X_1, \dots, X_p such that there is a set X with the property that $X_i \cap X_j = X$ for all $1 \leq i < j \leq p$. This is quite a strict requirement, and so one can ask whether the definition could be relaxed a bit without breaking the lower bound proof (and perhaps even yielding a slightly better bound).

2a Let us say that a collection of sets X_1, \dots, X_p is a *sub-sunflower* if there is a set X with the property that $X_i \cap X_j \subseteq X$ for all $1 \leq i < j \leq p$. Would the lower bound we did in class still go through if we did the “plucking” with sub-sunflowers? Please explain how to adapt the argument or point out where it fails.

2b Say that X_1, \dots, X_p form a *super-sunflower* if there is a set X such that $X_i \cap X_j \supseteq X$ for all $1 \leq i < j \leq p$. Would the lower bound we did in class still go through if we used super-sunflowers? Please explain how to adapt the argument or point out where it fails.

Remark: You should not expect to have to write pages of detailed arguments in the cases where you believe the proof still works or can be adapted to work. Also, when it seems that the proof cannot be made to work you do not have to prove beyond all doubt that no way of formalizing an argument along similar lines can possibly work in any universe—it is enough to point out, briefly but concretely, what technical difficulties arise, and why they seem hard to circumvent.

3 (30 p) Following the lecture notes, prove that for every matrix M of rank r and even $k > 0$ it holds that

$$\text{rank}(A) \leq \binom{r+k}{k},$$

where the entries of A are $A_{ij} = M_{ij}^k$.

4 (40 p) For an $n \times n$ real matrix M , define the norm

$$\|M\|_\nu = \inf \left\{ \sum_z |c_z| : M = \sum_z c_z M_z \right\},$$

where the matrices M_z are rank-1 matrices such that $\|M_z\|_\infty \leq 1$. Prove that for the $n \times n$ identity matrix I it holds that $\|I\|_\nu = 1$.

5 (40 p) Denote by Δ_n the set of $x \in [0, 1]^n$ such that $\sum_{i=1}^n x_i = 1$. Then *von Neumann's minimax theorem* says that for every $A \in \mathbb{R}^{n \times m}$ it holds that

$$\max_{x \in \Delta_n} \min_{y \in \Delta_m} x^\top A y = \min_{y \in \Delta_m} \max_{x \in \Delta_n} x^\top A y.$$

Following the lecture notes, recall that $D_\mu(f)$ denotes the distributional communication complexity f over a distribution μ , and that $R(f)$ denotes the (public-randomness) randomized communication complexity of f (both with error probability $\frac{1}{3}$).

Using the minimax theorem, prove that the equality

$$R(f) = \max_{\mu} D_\mu(f)$$

holds for every two-party function f .

- 6 (50 p) Let X, Y be finite sets of size $|X| = |Y| = n$. A rectangle R is a set of the form $A \times B$ where $A \subseteq X$ and $B \subseteq Y$. For a function $f : X \times Y \rightarrow \{\pm 1\}$, define

$$\text{disc}_R(f) = \frac{1}{|X| \cdot |Y|} \left| \sum_{x \in A, y \in B} f(x, y) \right|$$

and

$$\text{disc}(f) = \max_R \text{disc}_R(f) ,$$

where the maximum is over all rectangles R . Prove that for every function f as above it holds that $\text{disc}(f) \geq \frac{1}{10\sqrt{n}}$.