Cutting Planes:

operates on integer linear inequalities

$$\sum a_i x_i \geq A.$$

syntactic derivation rules

Variables:

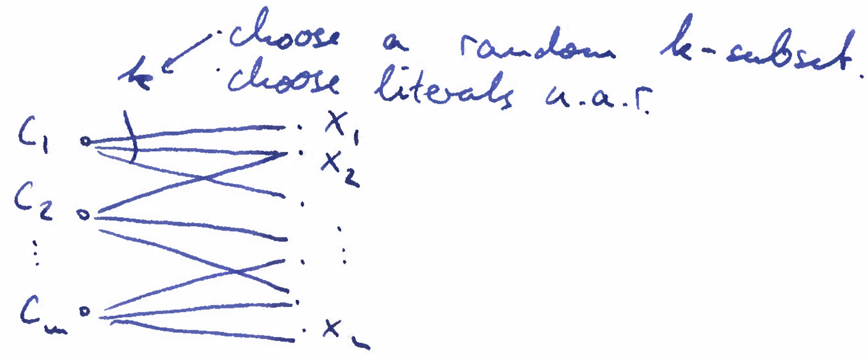
$$\frac{}{x_i \geq 0} \qquad \frac{}{-x_i \geq -1}$$

Addition:
$$\frac{\sum_i a_i x_i \geq A \qquad \sum b_i x_i \geq B}{\sum_i (a_i + b_i) x_i \geq A + B}$$

Multiplication:
$$\frac{\sum a_i x_i \geq A}{\sum c \cdot a_i x_i \geq c \cdot A}$$

Division:
$$\frac{\sum_i c a_i x_i \geq A}{\sum a_i x_i \geq \lceil A/c \rceil}$$

Random kCNF:



$$C_i = \bigvee_j x_j \qquad \mathcal{F}(m, n, k).$$

For enough clauses; $m \geq \ln 2 \cdot 2^k \cdot n$ this formula is unsat.

<u>Thm.</u> [Hrubeš, Pudlák and Fleming, Pankratov, Pitassi, Robere and Sokolov]

If $\varphi \sim \mathcal{F}(m, 2n, 2k)$ for $m = \Theta(n 2^{2k})$ and $k \geq c \cdot \log n$,

then every <u>semantic</u> CP refutation of φ is of size $2^{\Omega(n)}$.

derive any linear inequality that follows from $(\sum a_i x_i \geq A) \wedge (\sum b_i x_i \geq B)$ over $\{0,1\}$.
stronger than CP
Hrubeš, Filmus, Lauria

How can we prove CP lower bounds?

- interpolation for real monotone circuits
- no really useful width measure; PHP requires large width but is trivial to refute.
- restrictions?

$$\sum c_i x_i \geq A \restriction_{x_j = 1}$$

$$\sum_{i \neq j} c_i x_i \geq A - c$$

- well [HP, FPPR] figured out a way to use interpolation.
  ⇝ Sokolov merged the real monotone circuit lower bound (a bottleneck counting argument) with the CP l.b.

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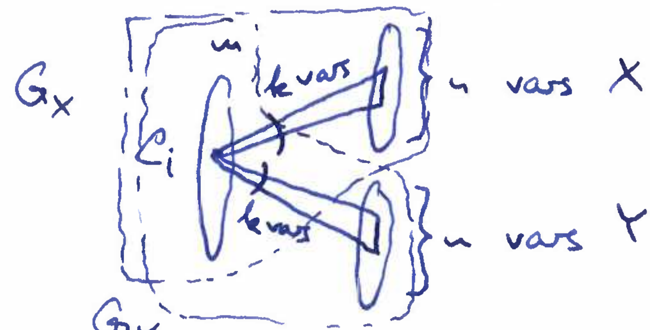
① We will prove the theorem not for the distribution $\mathcal{F}(m, n, k)$ but rather for a "bipartite" version $\mathcal{B}(m, n, k)$

(r, k, c)-exp:
- $\forall S : |S| \leq r$: $|N(S)| \geq |S| \cdot c$.
- left-degree is k.



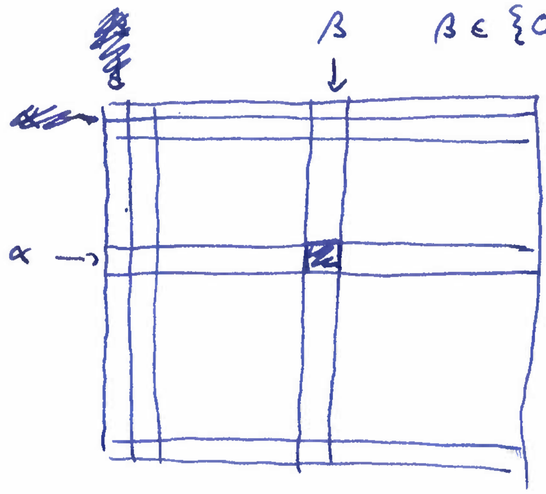
Sample each C_i by choosing a random k-subset from X as well as a k-subset from Y; choose literals u.a.r.

<u>Lemma:</u>

For $k = \Theta(\log n)$; $m \leq O(n 2^k)$ ~~any any constant ε > 0~~ there is a constant $c > 0$ such that for $G_X \cup G_Y \sim \mathcal{B}(m, n, k)$ it holds that G_X, G_Y are $(c \cdot n/k, k, \frac{k}{2})$-expanders.

Consider an inequality $\sum_i a_i x_i + \sum b_i y_i \geq A$. ③

$\beta \in \{0,1\}^Y$



$\alpha \in \{0,1\}^X$

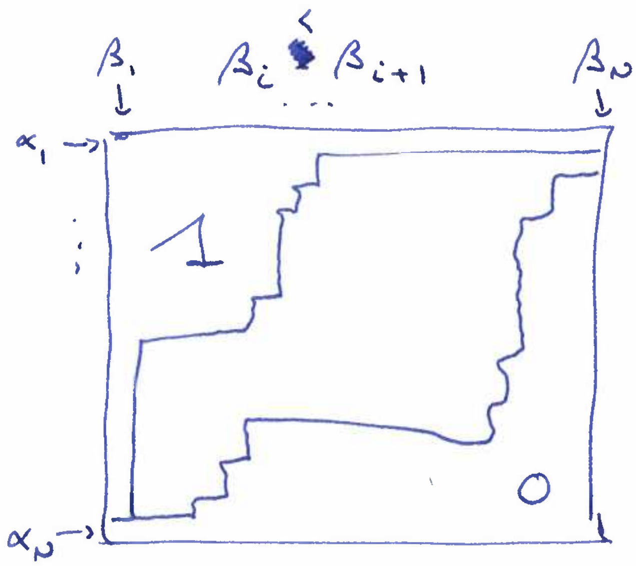
$2^n \times 2^n$ matrix M

$$\sum_i a_i x_i \restriction_\alpha = a(\alpha)$$

$$-A + \sum_i b_i y_i \restriction_\beta = b(\beta)$$

$\leadsto$ the cell $M(\alpha, \beta) := \begin{cases} 1 & \text{if } a(\alpha) + b(\beta) < 0 \\ 0 & \text{otherwise} \end{cases}$

$\Rightarrow$ can order the rows; columns so that ~~M(α, β)~~ M is ~~a~~ a "triangular" matrix



$\alpha_i < \alpha_{i+1}$

$\leadsto$ associate with each inequality such a triangle $T \subseteq 2^X \times 2^Y$.

The ~~unsatisfied~~ falsified clause search problem

$$\mathrm{Search}_\varphi \subseteq \{0,1\}^X \times \{0,1\}^Y \times \underbrace{[m]}_{\text{clauses}}$$

of φ is defined by

$$(\alpha, \beta, i) \in \mathrm{Search}_\varphi \text{ iff}$$

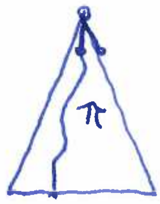
$$C_i \restriction_{\alpha \cup \beta} = \text{false}.$$

If φ is unsat, then Search_φ is total; for all $\alpha \in \{0,1\}^X$; $\beta \in \{0,1\}^Y$ there is an i s.t.

$$(\alpha, \beta, i) \in \mathrm{Search}_\varphi.$$

Problem: given (α, β) find i such that $(\alpha, \beta, i) \in \mathrm{Search}_\varphi$.

This problem can be solved by any refutation π of φ!

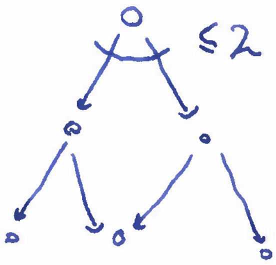


start at the root and follow the falsified line until we reach a falsified clause.

Let us give a characterization of semantic CP in terms of triangles.

A triangle ~~DA~~ DAG solving the Search_φ problem satisfies:

- every node u labelled with a triangle $T_u \subseteq 2^X \times 2^Y$.
- the root r is labelled with $T_r = 2^X \times 2^Y$.



- For a node u and out-children v, w:

$$T_u \subseteq T_v \cup T_w$$

- Every leaf node u is labelled with a clause C_i such that

$$\forall \quad (\alpha, \beta) \in T_u : \quad C_i \restriction_{\alpha \cup \beta} = \text{false}.$$

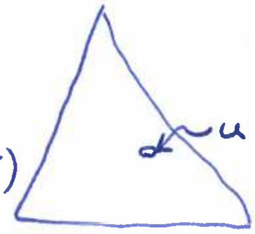
Thm.
There is a semantic CP refutation of φ iff there is a triangle DAG solving Search_φ.

Thm.
Any triangle-DAG solving Search_φ for $\varphi \sim \mathcal{B}(m,n,k)$ is of size $2^{\Omega(n)}$.

Proof idea:

· Given a small Δ-DAG π.

· $\mu: 2^X \cup 2^Y \longrightarrow V(\pi)$ such that
partial



· $|\mu^{-1}(u)| \leq 2^{n-\frac{w\cdot k}{4}}$ $\forall u \in V(\pi)$

· $|\mathrm{Dom}(\mu)| \geq 2^{n-1}$

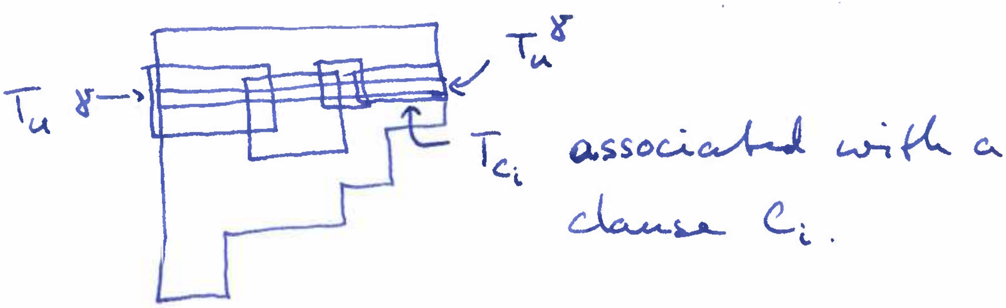
$$\Rightarrow \quad |\pi| \geq \frac{2^{n-1}}{2^{n-\frac{wk}{4}}} = 2^{\frac{wk}{4}}$$

for $w \leq \min\{2^{k/4-1}, c\cdot\frac{n}{k}\}$

· How do we define μ?

Consider an assignment $\gamma \in 2^X \cup 2^Y$.

$$w(u,\gamma) := \min|\{S \subseteq [m] : \bigcup_{i\in S} T_{C_i} \text{ covers } T_u^\gamma\}|$$



$w(u,\gamma)$:= the smallest collection of axioms that covers T_u^γ.

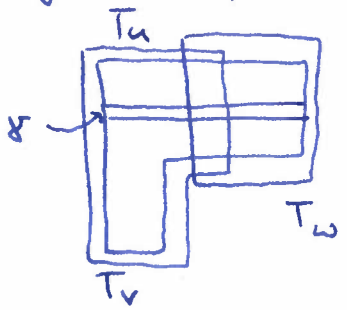
For intuition:

in case of resolution

$w(u)$:= min # of axioms that imply C_u.

<u>Claim:</u> 

For any γ: $\omega(u,\gamma) \leq \omega(v,\gamma) + \omega(w,\gamma)$.



the covers of T_v^γ; T_w^γ also cover T_u^γ

The measure μ is defined as follows.

Consider $u_1, \dots, u_{|\Pi|} \in \Pi$ sorted topologically.

for $\alpha \in 2^X$ do:

  if $\omega(u_i, \alpha) > w$, then

    · $\mu(\alpha) = u_i$

    · erase the α line from all T_u for $u \in \Pi$.

for $\beta \in 2^Y$ do:

  same.

<u>Claims:</u>

· It holds that $\omega(u_i, \gamma) \leq 2^w$ before running the above algo.

· After running the above algo: $\omega(u_i, \gamma) \leq w$.

<u>Lemma 1:</u> $|\mathrm{Dom}(\mu)| \geq \frac{1}{2} 2^n$.

Proof by contradiction; suppose that $|\mathrm{Dom}(\mu)| < \frac{1}{2} \cdot 2^n$.

· after defining μ we are left with

$T_r = A_r \times B_r$ where $|A_r|, |B_r| > 2^{n-1}$

For $\alpha \in A_r$ we have that $\omega(r, \alpha) \leq w$,

that is, there are at most w rectangles covering T_r^α.

In other words, any point $\beta \in B_r$ does not satisfy at least one clause in a set S of size $|S| \le w$.

On the other hand

$$\Pr_{\beta \sim 2^r}[\beta \in B_r] \le \Pr[\beta \text{ does not satisfy some clause } C \in S]$$
$$\le \sum_{C \in S} \Pr[\beta \text{ does not sat } C]$$
$$\le |S| \cdot \max_{C \in S} \Pr[\beta \text{ does not sat } C]$$
$$\le w \cdot 2^{-k}.$$

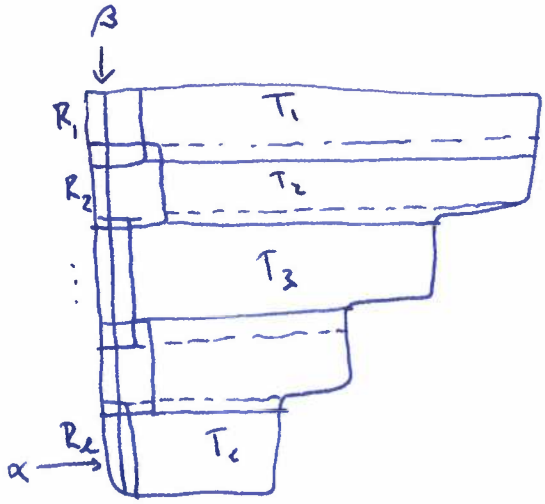
$2^k > \frac{w}{2} \iff w < 2^{k+1}$

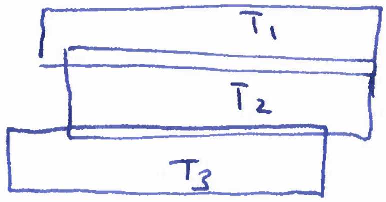
$\Rightarrow |B_r| \le w \cdot 2^{-k} \cdot 2^{|r|} < 2^{n-1}$ for $k > \log\left(\frac{w}{2}\right)$.

<u>Lemma 2:</u> For all $u \in V(\pi)$ it holds that

$$|\mu^{-1}(u)| \le 2^{n-\dots}$$

We want to argue that not too many assignments may be mapped to a single node u of the refutation π.
Fix $u \in V(\pi)$. Consider the situation <u>before</u> running the algo.
we know that for all β: $w(u, \beta) \le 2w$.
we want to use this fact to argue that there are few α such that $w(u, \alpha) > w$.





know: $\omega(u,\beta) \le 2w$.

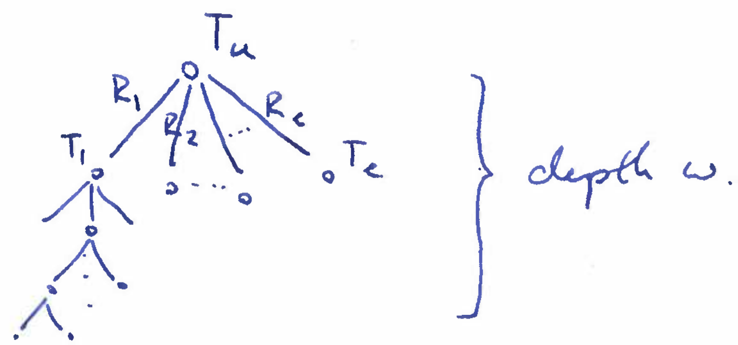
$\Leftrightarrow$ Can cover T_u^{β} with a set $S \subseteq [m]$ of axioms of size $|S| \le 2w$.

$\rightarrow$ every α is covered by this set S;

$\forall \alpha : \exists i \in S : C_i \restriction_\alpha \neq$ true.

$\rightsquigarrow$ recurse.

$\ell \le 2w$.



Every assignment α such that $w(u,\alpha) > w$ must end up in this tree at depth w.

$\Rightarrow$ ending in a lower leaf gives a "witness" covering of size $\le w$.

$\Rightarrow$ let us bound the number of $|\{\alpha \mid \omega(u,\alpha) > w\}|$ by bounding the maximum number of assignments in any such leaf.

we have at most $(2w)^w$ paths of length w.

Any assignment α at depth w ~~falsifies~~ does not satisfy w clauses S. The number of such α may thus be bounded by

$$2^{n-|N(S)|} \overset{\text{expansion; } \varepsilon = \frac{1}{2}.}{\leq} 2^{n - \frac{|S| \cdot k}{2}} = 2^{n - \frac{k \cdot w}{2}}$$

Hence the number of α s.t. $w(u, \alpha) > w$ is bounded by

$$2^{n - w\left(\frac{k}{2} - \log 2w\right)}$$

A similar argument for β shows that

$$|\mu^{-1}(u)| \leq 2 \cdot 2^{n - w\left(\frac{k}{2} - \log(2w)\right)}$$
$$\leq 2 \cdot 2^{n - \frac{k \cdot w}{4}}$$

$2w \leq 2^{k/4}$

<u>Open problems:</u>

1) Constant k?

2) Balanced predicates?
$\rightsquigarrow$ applications to $\mathcal{H}$...?