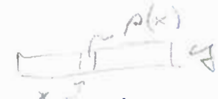
①

$R \subseteq [m]^n \times (\{0,1\}^m)^n$ is ρ-like iff $G(R) = C_\rho^{-1}(1)$

$\iff \forall z \in \{0,1\}^n$ consistent with ρ:
$\exists\, x \in [m]^n;\ y \in (\{0,1\}^m)^n$:
$G(x,y) = \mathrm{Ind}_m^n(x,y) = z.$

A random variable $X \in [m]^J$ is h-dense if for every $I \neq \emptyset \subseteq J$:
X_I has min-entropy $H_\infty(X_I) \geq h \cdot |I|.$
$\hookrightarrow \min_x \log\left(\frac{1}{\Pr[X=x]}\right)$

A rectangle $R = X \times Y$ is ρ-structured if



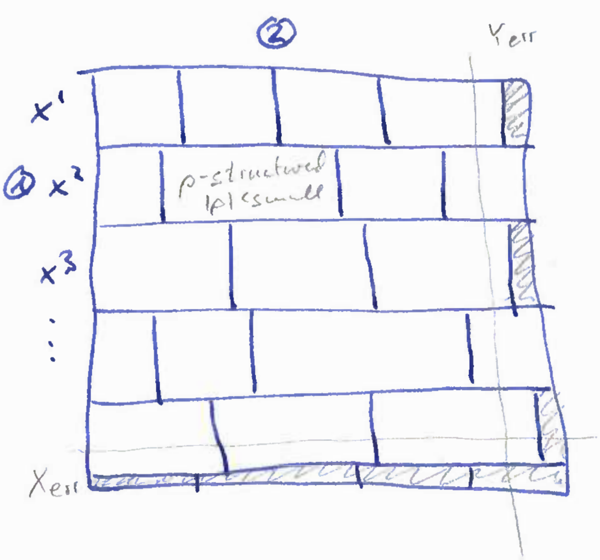
1) $X_{\mathrm{dom}(\rho)}$ is fixed, and every $z \in G(R)$: $z \in C_\rho^{-1}(1)$
$\iff y_x$ is chosen appropriately.

2) $X_{\mathrm{free}(\rho)}$ is $0.95 \log m$-dense
$\rho^{-1}(*)$

3) Y is large: $H_\infty(Y) \geq$ ~~...~~ $m \cdot |\rho^{-1}(*)| - n \cdot \log m.$

<u>Full range lemma:</u>
If $X \times Y$ is ρ-structured, then there is an $x \in X$ such that $\{x\} \times Y$ is ρ-like.

How to go from a rectangle $R := X \times Y \in \Pi$ to structured rectangles.



① Let $I_i \subseteq [n]$ be maximal such that X_{I_i} has min-entropy $\leq 0.95 \log_m |I_i|$.
Let $\alpha_i \in [m]^{I_i}$ witness this:
$\Pr[X_{I_i} = \alpha_i] > m^{-0.95|I_i|}.$

$X^i := \{x : x_{I_i} = \alpha_i\}$
$X = X \setminus X^i.$

② For each X^i; $\gamma \in \{0,1\}^{I_i}$:
$Y^{i,\gamma} := \{y : g^{I_i}(\alpha_i, y) = \gamma\}$

output $\{R^{i,\gamma} : X^i \times Y^{i,\gamma}\}$
$\neq \emptyset$

Rectangle Lemma

Let $R = X \times Y$ and $d < n$; let $R = \cup R^i_\ell$ be the rectangles from the above partition. Then, there are error sets $X_{err} \subseteq X$; $Y_{err} \subseteq Y$ with density $\leq 2^{-2d \log m}$ in $[m]^n$ and $(\{0,1\}^m)^n$ respectively such that either

- R^i is ρ^i structured for ρ^i of size $\leq O(d)$.
- R^i is covered by error rows/cols;
  $R^i \subseteq X_{err} \times (\{0,1\}^m)^n \cup Y_{err} \times [m]^n$.

Finally: for $x \in [m]^n \setminus X_{err}$ there is an $I_x \subseteq [n]$: $|I_x| \leq O(d)$ and every structured R^i intersecting row x has $\mathrm{dom}(\rho^i) \subseteq I_x$.

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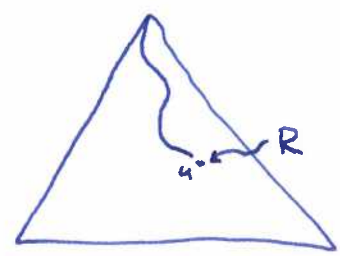
given a rectangle-dag Π solving $S \circ G$ of size $|\Pi| = m^d$, then $w(S) \leq O(d)$.

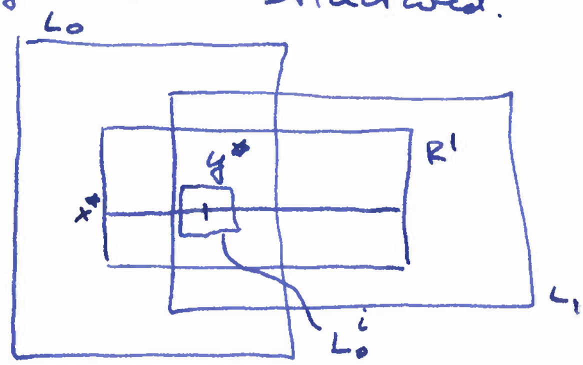
[ignoring error sets].

- maintain a ρ-structured $R' \subseteq R$. ($\leq O(d)$)



(1) Root: the rectangle is $*^n$-structured.

(2) Step:



$X' \times Y' =: R'$ is ρ-structured $\Rightarrow \exists x^* \in X'$: $x^* \times Y'$ is ρ-like

- Consider partition of L_0; L_1.
  $\leadsto$ the rectangles intersecting row x^*:
  $\exists I_0; I_1$: $\forall L^i_b$ intersecting x^* : $\mathrm{dom}(\rho^i_b) \subseteq I_b$.
  (L_b ρ^i_b-like)
  $\leadsto$ query $I_0 \cup I_1$ $\leadsto \rho^*$ (small; $O(d)$).
- $x^* \times Y'$ is ρ-like $\to \exists y^* \in Y'$: $g(x^*, y^*) = z$ is consistent with ρ^*.

$\to$ Consider L^i_b : $(x^*, y^*) \in L^i_b$. (ρ^i_b-structured) $\leadsto$ forget everything except $\mathrm{dom}(\rho^i_b)$.

(3) Leaf case: game state ρ; R': ρ-struct.

leaf labelled by $o \in O$:

$$R' \subseteq (S \circ G)^{-1}(o)$$

$$\Longleftrightarrow C_\rho^{-1}(1) = G(R') \subseteq S^{-1}(o)$$

Error: traverse π in topological order from leaves to root;

$R_1, \dots, R_{m^d}$.

$$X^*_{err}; \; Y^*_{err} = \emptyset.$$

Consider R_i:

- update $R_i \leftarrow R_i \setminus (X^*_{err} \times (\{0,1\}^n)^n \cup [m]^n \times Y^*_{err})$
- apply partition scheme; keep the structured rects.
- $X^*_{err} \leftarrow X^*_{err} \cup X_{err}$; $Y^*_{err} \leftarrow Y^*_{err} \cup Y_{err}$.

$\rightsquigarrow$ same proof as before on $(X \setminus X_{err}) \times (Y \setminus Y_{err})$.

(1) Root: the density of the error sets on X $< m^d \cdot m^{-2d} \ll 1\%$.

on Y less than m^{-d} fraction

$\rightarrow$ the remaining rectangle is $*^n$-structured.

(2) Step: Error sets shrink as we walk down the proof π.

$\rightarrow$ cover property is maintained.

Proof of the rectangle lemma:

- X_{err}: while there is $R^i = X \times Y$ such that $|I^i| > 40d$ [$= \mathrm{dom}(\rho^i)$]
  update $X_{err} \leftarrow X_{err} \cup X$.

- Y_{err}: while there is $R^i = X \times Y$ such that $|Y \setminus Y_{err}| < 2^{m \cdot |\bar{I}^i| - 5d\log m}$ [$-5d\log m$]
  update $Y_{err} \leftarrow Y_{err} \cup Y$.
  [$\setminus Y_{err}$: needed? don't think so?]

Claim 1: if R^i is not covered by X_{err}; Y_{err}, then R^i is ρ^i-struct ~~R^i is fixed on I;~~
with $|\mathrm{dom}(\rho^i)| \leq O(d)$.

- P1: obvious;
- P2: min-entropy holds by maximality.
- P3: by construction.

$\rightsquigarrow$ error set density?

$$|X_{err}| \leq m^n \cdot 2^{-2d\log m}$$

unless X_{err} is empty $\exists j$: (min)

x^j added to X_{err}.

$\rightarrow |I_j| > 40d$.

(1) $|X^j| \leq |X^{\geq j}| \cdot 2^{-0.95|I_j|\log m}$

$$|X^j| = |X^{\geq j}| \cdot \Pr_{x \sim X^{\geq j}}[x_{I_j} = \alpha_j] \leq |X^{\geq j}| \cdot 2^{-0.95|I_j|\log m}.$$

$\rightarrow H_\infty(X^j) \geq H_\infty(X^{\geq j}) - .95|I_j|\log m$

$$(n - |I_j|)\log m \geq H_\infty(X^j).$$

$$\rightsquigarrow H_\infty(X^{\geq j}) \leq (n - 0.05|I_j|)\log m.$$

$$|X_{err}| \leq |X^{\geq j}| < 2^{(n - 0.05 \cdot 40d)\log m}$$

$$\leq m^n \cdot 2^{-2d\log m}.$$

$\mathcal{Y}_{err}$: each $Y^{i,\gamma}$ is defined by

$$(I_i, \alpha_i, \gamma)$$

for $k \in [40d]$: # of such $Y^{i,\gamma} \leq \binom{n}{k} m^k 2^k < 2^{3k \log m}$.

$\rightsquigarrow$ by a union bound:

$$|\mathcal{Y}_{err}| \leq \sum_{k=1}^{40d} 2^{3k \log m} \cdot 2^{m(n-k) - 5d \log m}$$

$$\leq 40d \cdot 2^{m(n-1) - 2d \log m} \ll 2^{mn} 2^{-2d \log m}.$$

<u>Full range lemma</u>. $R := X \times Y$; ρ-~~like~~ structured

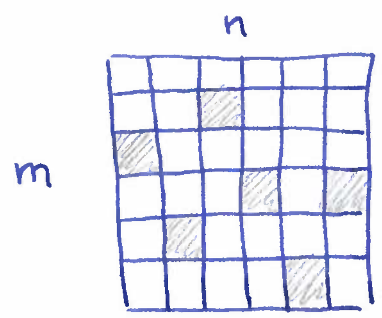
want to argue that there is a row x^* such that

$$\mathrm{Ind}_m^n(x^*, Y) = \underbrace{C_\rho^{-1}(1)}_{\text{all assignments "compatible" with } \rho}$$

By contradiction: For every row $x \in [m]^n$, there is a $z \in \{0,1\}^n$:

$$\forall y \in Y: \; (y_{1x_1}, \dots, y_{nx_n}) \neq z \quad (Y \subseteq \{0,1\}^{m \cdot n})$$

$$\Longleftrightarrow \quad z \notin \mathrm{Ind}(x, Y).$$



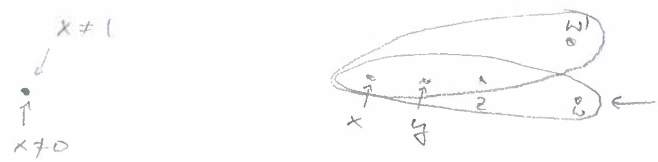
x picks a box per column.

$\rightarrow$ no matter what value $y \in Y$ you choose, you will never see the assignment z in the boxes.

We want to argue that since these constraints have high ~~min-entropy~~ h-density, there cannot be many y satisfying these constraints.

But first, let us think of what the "worst case" is w.r.t. the constraints; when do they rule out the fewest $y \in (\{0,1\}^m)^n$ (with respect to the choice of z).

__Claim:__ setting all $z = \vec{1}$ is the worst-case; max y will satisfy the constraints.



← all assignments with $x = y = z = w = 1$ are ruled out.

w.l.o.g. $w' = 1$

if $x = 1 = y = z \rightarrow$ max overlap of ruled out subcubes; assignments

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want to analyze the event that for $y \sim_{u.a.r.} \{0,1\}^{mn}$ the boxes chosen by x are all different from $\vec{1}$.

$\rightarrow$ Apply Janson's inequality:

$$\Pr_y\left[\ \forall x \in X:\ x \nsubseteq y\ \right] \leq e^{(-\mu^2/\Lambda)}$$

(set indicator)

$$\mu := \mathbb{E}[\#\text{of contained sets}] = |X| \cdot 2^{-n}$$

$$\Lambda := \sum_{\substack{(i,j):\\ X_i \cap X_j \neq \emptyset}} \mathbb{E}\left[\mathbb{1}_{\{X_i \cup X_j \subseteq y\}}\right]$$

Remains to bound Λ.

1) Fix the set $x \in X$

2) Fix the size of the intersection a.

Use denseness to argue that there are few sets that intersect in a given choice of a points;

$$|X| \cdot m^{-0.95 \cdot a}.$$

$$\Rightarrow \Lambda \leq |X| \cdot \sum_{a=1}^{n} \binom{n}{a} |X| \cdot m^{-0.95 \cdot a} \cdot 2^{-2n+a}$$

$$\leq \mu^2 \cdot \left(\left(1 + \frac{2}{m^{0.95}}\right)^n - 1 \right)$$

$$\leq \mu^2 \cdot \frac{4n}{m^{0.95}}$$

$$\Rightarrow \Pr_y [\forall x \in X: x \not\subseteq y] \leq \exp\left(-\frac{m^{0.95}}{4n}\right) \leq \exp(-n \cdot \log m)$$

$$\Rightarrow |Y| \leq 2^{nm - n \cdot \log m} \quad ; \quad \text{contradiction} \qquad \square.$$

If we want to optimize m, need to be more careful ~~with~~ with the used bounds; see [Rao20, Lemma 4].

$\rightsquigarrow$ get $m \sim n^{1+\varepsilon}$.