Proof Complexity as a Computational Lens: Lecture 22

Size-Space Trade-offs for Cutting Planes

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Recap: Configuration-Style Proofs

- Proof system operates with formulas of some syntactic form
- Proof/refutation is "presented on blackboard"
- Derivation steps:
  - Write down axiom clauses of CNF formula being refuted (as encoded by proof system)
  - Infer new lines by deductive rules of proof system
  - Erase lines not currently needed (to save space on blackboard)
- Refutation ends when (explicit) contradiction is derived

Cutting Planes (CP)

Clauses interpreted as linear inequalities
E.g., $x \lor y \lor \overline{z} \;\rightsquigarrow\; x + y + (1 - z) \geq 1 \;\rightsquigarrow\; x + y - z \geq 0$

Proof system also works for any system of 0–1 linear inequalities with integer coefficients

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Variable axioms $\dfrac{}{0 \leq x \leq 1}$

Multiplication $\dfrac{\sum a_i x_i \geq A}{\sum c a_i x_i \geq cA} \quad c \in \mathbb{N}^+$

Addition $\dfrac{\sum a_i x_i \geq A \quad \sum b_i x_i \geq B}{\sum (a_i + b_i) x_i \geq A + B}$

Division $\dfrac{\sum c a_i x_i \geq A}{\sum a_i x_i \geq \lceil A/c \rceil} \quad c \in \mathbb{N}^+$

Goal: Derive $0 \geq 1 \Leftrightarrow$ formula/system of inequalities unsatisfiable

Example: Cutting planes Refutation of Pigeonhole Principle

1. $x_{1,1} \lor x_{1,2}$
2. $x_{2,1} \lor x_{2,2}$
3. $x_{3,1} \lor x_{3,2}$
4. $\overline{x}_{1,1} \lor \overline{x}_{2,1}$
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Pigeonhole principle (PHP)

"$n+1$ pigeons don't fit into n holes"

Variables $x_{i,j}$ = "pigeon i goes into hole j"

$x_{i,1} \lor x_{i,2} \lor \cdots \lor x_{i,n}$	every pigeon i gets a hole
$\overline{x}_{i,j} \lor \overline{x}_{i',j}$	no hole j gets two pigeons $i \neq i'$

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History of derivation steps

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3. $x_{3,1} + x_{3,2} \geq 1$
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5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Write down axiom 4: $-x_{1,1} - x_{2,1} \geq -1$
Write down axiom 5: $-x_{1,1} - x_{3,1} \geq -1$
Add to get $-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2$
Erase the line $-x_{1,1} - x_{3,1} \geq -1$
Erase the line $-x_{1,1} - x_{2,1} \geq -1$
Write down axiom 6: $-x_{2,1} - x_{3,1} \geq -1$
Add to get $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Erase the line $-x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2$
Divide to get $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$

$$-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$$
$$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$$

Example: Cutting planes Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Write down axiom 5: $-x_{1,1} - x_{3,1} \geq -1$
Add to get $-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2$
Erase the line $-x_{1,1} - x_{3,1} \geq -1$
Erase the line $-x_{1,1} - x_{2,1} \geq -1$
Write down axiom 6: $-x_{2,1} - x_{3,1} \geq -1$
Add to get $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Erase the line $-x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2$
Divide to get $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$

$$-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$$
$$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$$

Example: Cutting planes Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Write down axiom 5: $-x_{1,1} - x_{3,1} \geq -1$
Add to get $-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2$
Erase the line $-x_{1,1} - x_{3,1} \geq -1$
Erase the line $-x_{1,1} - x_{2,1} \geq -1$
Write down axiom 6: $-x_{2,1} - x_{3,1} \geq -1$
Add to get $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Erase the line $-x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2$
Divide to get $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$

$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$

Example: Cutting planes Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Add to get $-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2$
Erase the line $-x_{1,1} - x_{3,1} \geq -1$
Erase the line $-x_{1,1} - x_{2,1} \geq -1$
Write down axiom 6: $-x_{2,1} - x_{3,1} \geq -1$
Add to get $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Erase the line $-x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2$
Divide to get $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Write down axiom 7: $-x_{1,2} - x_{2,2} \geq -1$

$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
$-x_{1,2} - x_{2,2} \geq -1$

Example: Cutting planes Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Erase the line $-x_{1,1} - x_{3,1} \geq -1$
Erase the line $-x_{1,1} - x_{2,1} \geq -1$
Write down axiom 6: $-x_{2,1} - x_{3,1} \geq -1$
Add to get $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Erase the line $-x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2$
Divide to get $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Write down axiom 7: $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 8: $-x_{1,2} - x_{3,2} \geq -1$

$$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$$
$$-x_{1,2} - x_{2,2} \geq -1$$
$$-x_{1,2} - x_{3,2} \geq -1$$

Example: Cutting planes Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Erase the line $-x_{1,1} - x_{2,1} \geq -1$
Write down axiom 6: $-x_{2,1} - x_{3,1} \geq -1$
Add to get $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Erase the line $-x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2$
Divide to get $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Write down axiom 7: $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 8: $-x_{1,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$

$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
$-x_{1,2} - x_{2,2} \geq -1$
$-x_{1,2} - x_{3,2} \geq -1$

Example: Cutting planes Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Erase the line $-x_{1,1} - x_{2,1} \geq -1$
Write down axiom 6: $-x_{2,1} - x_{3,1} \geq -1$
Add to get $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Erase the line $-x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2$
Divide to get $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Write down axiom 7: $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 8: $-x_{1,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$

$$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$$
$$-x_{1,2} - x_{2,2} \geq -1$$
$$-x_{1,2} - x_{3,2} \geq -1$$
$$-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$$

Example: Cutting planes Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Write down axiom 6: $-x_{2,1} - x_{3,1} \geq -1$
Add to get $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Erase the line $-x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2$
Divide to get $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Write down axiom 7: $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 8: $-x_{1,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$

Erase the line $-x_{1,2} - x_{3,2} \geq -1$

$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
$-x_{1,2} - x_{2,2} \geq -1$
$-x_{1,2} - x_{3,2} \geq -1$
$-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$

Example: Cutting planes Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Write down axiom 6: $-x_{2,1} - x_{3,1} \geq -1$
Add to get $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Erase the line $-x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2$
Divide to get $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Write down axiom 7: $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 8: $-x_{1,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{3,2} \geq -1$

$$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$$
$$-x_{1,2} - x_{2,2} \geq -1$$
$$-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$$

Example: Cutting planes Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Add to get $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Erase the line $-x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2$
Divide to get $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Write down axiom 7: $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 8: $-x_{1,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,2} - x_{2,2} \geq -1$

$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
$-x_{1,2} - x_{2,2} \geq -1$
$-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$

Example: Cutting planes Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Add to get $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Erase the line $-x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2$
Divide to get $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Write down axiom 7: $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 8: $-x_{1,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,2} - x_{2,2} \geq -1$

$$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$$
$$-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$$

Example: Cutting planes Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Erase the line $-x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2$
Divide to get $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Write down axiom 7: $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 8: $-x_{1,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,2} - x_{2,2} \geq -1$

Write down axiom 9: $-x_{2,2} - x_{3,2} \geq -1$

$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
$-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
$-x_{2,2} - x_{3,2} \geq -1$

Example: Cutting planes Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Erase the line $-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2$
Divide to get $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Write down axiom 7: $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 8: $-x_{1,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 9: $-x_{2,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$

$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
$-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
$-x_{2,2} - x_{3,2} \geq -1$

Example: Cutting planes Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Erase the line $-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2$
Divide to get $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Write down axiom 7: $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 8: $-x_{1,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 9: $-x_{2,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$

$$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$$
$$-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$$
$$-x_{2,2} - x_{3,2} \geq -1$$
$$-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$$

Example: Cutting planes Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Divide to get $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Write down axiom 7: $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 8: $-x_{1,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 9: $-x_{2,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$

Erase the line $-x_{2,2} - x_{3,2} \geq -1$

$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
$-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
$-x_{2,2} - x_{3,2} \geq -1$
$-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$

Example: Cutting planes Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Divide to get $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Write down axiom 7: $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 8: $-x_{1,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 9: $-x_{2,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$

Erase the line $-x_{2,2} - x_{3,2} \geq -1$

$$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$$
$$-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$$
$$-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$$

Example: Cutting planes Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
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8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Erase the line $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Write down axiom 7: $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 8: $-x_{1,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{3,2} \geq -1$
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Write down axiom 9: $-x_{2,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Erase the line $-x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$

$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
$-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
$-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$

Example: Cutting planes Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
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5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Erase the line $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Write down axiom 7: $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 8: $-x_{1,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{3,2} \geq -1$
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Write down axiom 9: $-x_{2,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Erase the line $-x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$

$$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$$
$$-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$$

Example: Cutting planes Refutation of Pigeonhole Principle

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6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
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History of derivation steps

Write down axiom 7: $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 8: $-x_{1,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
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Write down axiom 9: $-x_{2,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Erase the line $-x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Divide to get $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$

$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
$-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$

Example: Cutting planes Refutation of Pigeonhole Principle

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5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Write down axiom 7: $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 8: $-x_{1,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{3,2} \geq -1$
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Add to get $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Erase the line $-x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$

Divide to get $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$

$$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$$
$$-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$$
$$-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$$

Example: Cutting planes Refutation of Pigeonhole Principle

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2. $x_{2,1} + x_{2,2} \geq 1$
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4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Write down axiom 8: $-x_{1,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 9: $-x_{2,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Erase the line $-x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Divide to get $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$

$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
$-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
$-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$

Example: Cutting planes Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Write down axiom 8: $-x_{1,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 9: $-x_{2,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Erase the line $-x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Divide to get $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$

$$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$$
$$-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$$

Example: Cutting planes Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
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4. $-x_{1,1} - x_{2,1} \geq -1$
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6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Add to get $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 9: $-x_{2,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Erase the line $-x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Divide to get $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Add to get $-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$

$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
$-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$

Example: Cutting planes Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
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9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Add to get $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 9: $-x_{2,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Erase the line $-x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Divide to get $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Add to get $-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$

$$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$$
$$-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$$
$$-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$$

Example: Cutting planes Refutation of Pigeonhole Principle

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2. $x_{2,1} + x_{2,2} \geq 1$
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4. $-x_{1,1} - x_{2,1} \geq -1$
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9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Erase the line $-x_{1,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 9: $-x_{2,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Erase the line $-x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Divide to get $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Add to get $-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$

$$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$$
$$-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$$
$$-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$$

Example: Cutting planes Refutation of Pigeonhole Principle

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9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Erase the line $-x_{1,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 9: $-x_{2,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Erase the line $-x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Divide to get $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Add to get $-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$

$$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$$
$$-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$$

Example: Cutting planes Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
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4. $-x_{1,1} - x_{2,1} \geq -1$
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7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Erase the line $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 9: $-x_{2,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Erase the line $-x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Divide to get $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Add to get $-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$

$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
$-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$

Example: Cutting planes Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
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4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Erase the line $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 9: $-x_{2,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Erase the line $-x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Divide to get $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Add to get $-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$

$$-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$$

Example: Cutting planes Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Write down axiom 9: $-x_{2,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Erase the line $-x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Divide to get $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Add to get $-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Write down axiom 1: $x_{1,1} + x_{1,2} \geq 1$

$-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
$x_{1,1} + x_{1,2} \geq 1$

Example: Cutting planes Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Add to get $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Erase the line $-x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Divide to get $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Add to get $-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Write down axiom 1: $x_{1,1} + x_{1,2} \geq 1$
Write down axiom 2: $x_{2,1} + x_{2,2} \geq 1$

$-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
$x_{1,1} + x_{1,2} \geq 1$
$x_{2,1} + x_{2,2} \geq 1$

Example: Cutting planes Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Erase the line $-x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Divide to get $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Add to get $-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Write down axiom 1: $x_{1,1} + x_{1,2} \geq 1$
Write down axiom 2: $x_{2,1} + x_{2,2} \geq 1$
Add to get $x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} \geq 2$

$-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
$x_{1,1} + x_{1,2} \geq 1$
$x_{2,1} + x_{2,2} \geq 1$

Example: Cutting planes Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Erase the line $-x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Divide to get $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Add to get $-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Write down axiom 1: $x_{1,1} + x_{1,2} \geq 1$
Write down axiom 2: $x_{2,1} + x_{2,2} \geq 1$
Add to get $x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} \geq 2$

$-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
$x_{1,1} + x_{1,2} \geq 1$
$x_{2,1} + x_{2,2} \geq 1$
$x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} \geq 2$

Example: Cutting planes Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Erase the line $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Divide to get $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Add to get $-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Write down axiom 1: $x_{1,1} + x_{1,2} \geq 1$
Write down axiom 2: $x_{2,1} + x_{2,2} \geq 1$
Add to get $x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} \geq 2$
Erase the line $x_{2,1} + x_{2,2} \geq 1$

$$-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$$
$$x_{1,1} + x_{1,2} \geq 1$$
$$x_{2,1} + x_{2,2} \geq 1$$
$$x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} \geq 2$$

Example: Cutting planes Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Erase the line $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Divide to get $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Add to get $-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Write down axiom 1: $x_{1,1} + x_{1,2} \geq 1$
Write down axiom 2: $x_{2,1} + x_{2,2} \geq 1$
Add to get $x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} \geq 2$
Erase the line $x_{2,1} + x_{2,2} \geq 1$

$-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
$x_{1,1} + x_{1,2} \geq 1$
$x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} \geq 2$

Example: Cutting planes Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Divide to get $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Add to get $-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Write down axiom 1: $x_{1,1} + x_{1,2} \geq 1$
Write down axiom 2: $x_{2,1} + x_{2,2} \geq 1$
Add to get $x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} \geq 2$
Erase the line $x_{2,1} + x_{2,2} \geq 1$
Erase the line $x_{1,1} + x_{1,2} \geq 1$

$-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
$x_{1,1} + x_{1,2} \geq 1$
$x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} \geq 2$

Example: Cutting planes Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Divide to get $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Add to get $-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Write down axiom 1: $x_{1,1} + x_{1,2} \geq 1$
Write down axiom 2: $x_{2,1} + x_{2,2} \geq 1$
Add to get $x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} \geq 2$
Erase the line $x_{2,1} + x_{2,2} \geq 1$
Erase the line $x_{1,1} + x_{1,2} \geq 1$

$-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
$x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} \geq 2$

Example: Cutting planes Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Erase the line $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Add to get $-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Write down axiom 1: $x_{1,1} + x_{1,2} \geq 1$
Write down axiom 2: $x_{2,1} + x_{2,2} \geq 1$
Add to get $x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} \geq 2$
Erase the line $x_{2,1} + x_{2,2} \geq 1$
Erase the line $x_{1,1} + x_{1,2} \geq 1$
Write down axiom 3: $x_{3,1} + x_{3,2} \geq 1$

$-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
$x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} \geq 2$
$x_{3,1} + x_{3,2} \geq 1$

Example: Cutting planes Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Add to get $-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Write down axiom 1: $x_{1,1} + x_{1,2} \geq 1$
Write down axiom 2: $x_{2,1} + x_{2,2} \geq 1$
Add to get $x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} \geq 2$
Erase the line $x_{2,1} + x_{2,2} \geq 1$
Erase the line $x_{1,1} + x_{1,2} \geq 1$
Write down axiom 3: $x_{3,1} + x_{3,2} \geq 1$
Add to get $x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} + x_{3,1} + x_{3,2} \geq 3$

$-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
$x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} \geq 2$
$x_{3,1} + x_{3,2} \geq 1$

Example: Cutting planes Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Add to get $-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Write down axiom 1: $x_{1,1} + x_{1,2} \geq 1$
Write down axiom 2: $x_{2,1} + x_{2,2} \geq 1$
Add to get $x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} \geq 2$
Erase the line $x_{2,1} + x_{2,2} \geq 1$
Erase the line $x_{1,1} + x_{1,2} \geq 1$
Write down axiom 3: $x_{3,1} + x_{3,2} \geq 1$
Add to get $x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} + x_{3,1} + x_{3,2} \geq 3$

$-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
$x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} \geq 2$
$x_{3,1} + x_{3,2} \geq 1$
$x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} + x_{3,1} + x_{3,2} \geq 3$

Example: Cutting planes Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Erase the line $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Write down axiom 1: $x_{1,1} + x_{1,2} \geq 1$
Write down axiom 2: $x_{2,1} + x_{2,2} \geq 1$
Add to get $x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} \geq 2$
Erase the line $x_{2,1} + x_{2,2} \geq 1$
Erase the line $x_{1,1} + x_{1,2} \geq 1$
Write down axiom 3: $x_{3,1} + x_{3,2} \geq 1$
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Complexity Measures for Cutting Planes

Length = total # lines/inequalities in refutation

Size = sum also size of coefficients

Line space = max # lines in memory during refutation

Total space = max # bits in memory (sum also size of coefficients)

Size Lower Bounds for Cutting Planes

Clique-colouring formulas

"A graph with an m-clique is not $(m-1)$-colourable"

Exponential lower bound via interpolation and circuit complexity [Pud97]

Technique very specifically tied to structure of formula

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Random $\mathcal{O}(\log n)$-CNF formulas
"Large number of randomly sampled clauses can be satisfied"
Exponential lower bound via bottleneck counting argument [Sok24]
Very intriguing new technique! (Or circuit lower bound in disguise?)

What About Line Space in Cutting Planes?

Pebbling formulas

"Possible to get from sources to sink in connected directed acyclic graph"

Short cutting planes refutations of (lifted) pebbling formulas on certain DAGs must have large line space [HN12, GP18]

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Tseitin formulas

"Sum of degrees of vertices in graph is even"

Short refutations of (lifted) Tseitin formulas on expanders must have large line space [GP18]

Not clear whether such short refutations exist...

Size-Space Trade-offs for Cutting Planes?

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What about "true" trade-offs?

Are there trade-offs where the space-efficient cutting planes refutations have small coefficients? (Say, of polynomial or even constant size)

Focus of This Lecture (and Next Lecture)

Theorem (Informal sample)

There are families of 6*-CNF formulas* $\{F_N\}_{N=1}^{\infty}$ *of size* $\Theta(N)$ *such that:*

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- Upper bounds for # bits; lower bounds for # lines/inequalities
- Hold uniformly for resolution, polynomial calculus, and cutting planes
- Even for semantic proofs where anything implied by blackboard inferred in single step

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Outline of Proof

Proof is by carefully constructed chain of delicate reductions
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1. Short, space-efficient proof $\Rightarrow$ efficient communication protocol for falsified clause search problem [HN12]
2. Crucial twists:
   - Study real communication model [Kra98, BEGJ00]
   - Consider round efficiency of protocols
3. Protocol for composed search problem $\Rightarrow$ parallel decision tree [Val75] via simulation theorem à la [RM99, GPW15]
4. Parallel decision tree for pebbling formulas Peb_G [BW01] $\Rightarrow$ pebbling strategy for Dymond–Tompa game on graph G [DT85]
5. Construct graphs G with strong round-cost trade-offs for Dymond–Tompa pebbling inspired by [CS82, LT82, BN11, Nor12]

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- Two players:
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- When they reach any leaf, both Alice and Bob should have learned $f(x, y)$
- Function f solved by r-round deterministic communication in cost c
  if $\exists$ protocol tree such that along any path from root
  - $\#$ rounds $\leq r$
  - total $\#$ bits sent $\leq c$

Real Communication [Kra98]

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- Strictly stronger than standard deterministic communication ($\textsc{Equality}$ solved with real communication in 1 round with cost 2)

Falsified Clause Search Problem

Falsified clause search problem $\mathit{Search}(F)$

Set-up: Fixed (and unsatisfiable) CNF formula F

Input: Assignment α to $\mathit{Vars}(F)$

Output: Clause $C \in F$ such that α falsifies C

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For any standard proof system, refutation $\pi : F \vdash \perp$ (viewed as DAG) can be used to solve $\mathit{Search}(F)$:

- Start at sink (labelled by $\perp$)
- Walk backwards along nodes falsified by α
- Axiom clause $C \in F$ labelling source node is valid answer

Falsified Clause Search Problem (Communication Version)

Falsified clause search problem $\mathit{Search}(F)$ for Alice and Bob

Set-up: Fixed (and unsatisfiable) CNF formula F
And (devious) partition of $\mathit{Vars}(F)$ between Alice and Bob

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Actually, communication protocol should compute not function but relation — will mostly ignore this distinction

Succinct Refutations Yield Efficient Protocols

Evaluate blackboard configurations of a refutation of F under α

$\emptyset$																$\perp$

Succinct Refutations Yield Efficient Protocols

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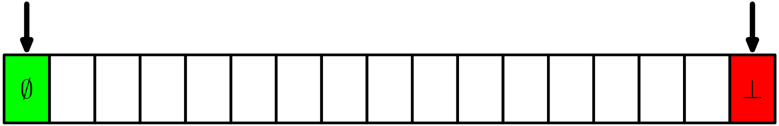

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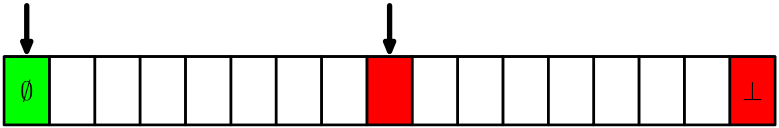

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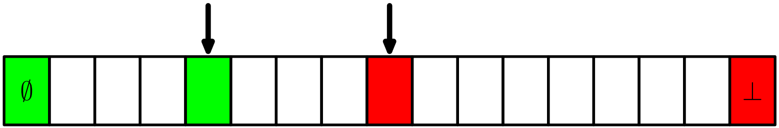

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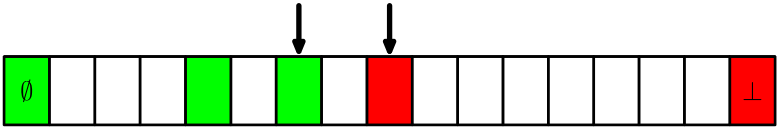

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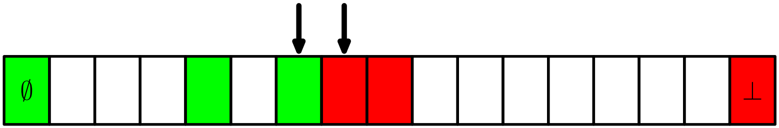

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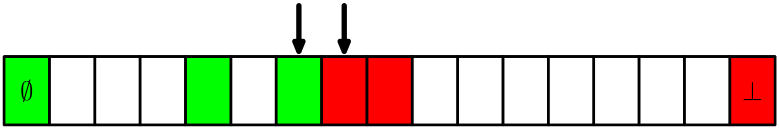

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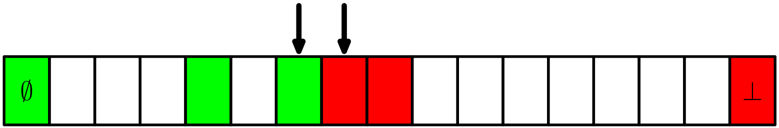

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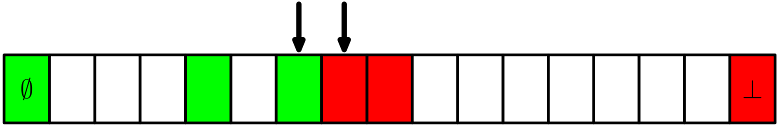

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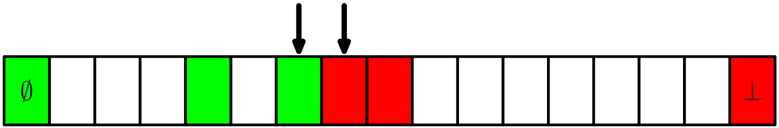

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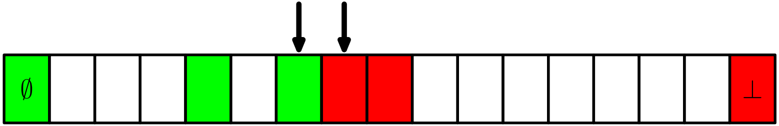

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(Alice and Bob simply evaluate their parts of each inequality and ask referee to compare)

Lifting of Functions

Construct hard communication problems by "hardness amplification" using lifting or composition

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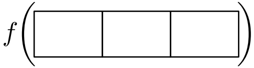
Start with function $f : \{0,1\}^m \to \{0,1\}$



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Construct new function on inputs $x \in [\ell]^m$ and $y \in \{0,1\}^{\ell m}$

$$\boxed{x_1}\boxed{x_2}\boxed{x_3}$$

$$\boxed{y_{1,1}}\boxed{y_{1,2}}\boxed{y_{2,1}}\boxed{y_{2,2}}\boxed{y_{3,1}}\boxed{y_{3,2}}$$

$$f\left(\boxed{}\boxed{}\boxed{}\right)$$

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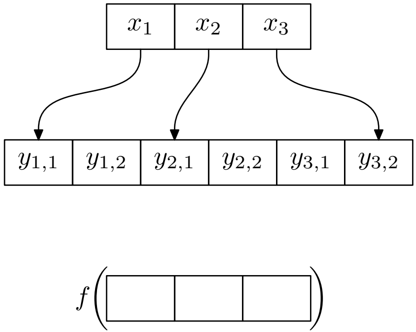

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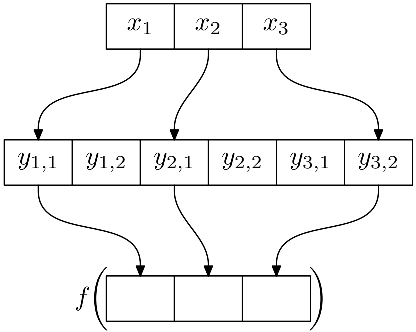

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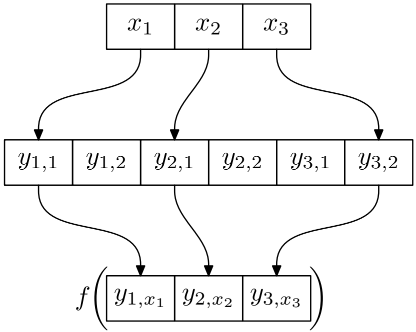
Length-ℓ lifting of f defined as
$Lift_\ell(f)(x,y) := f(y_{1,x_1}, \ldots, y_{m,x_m})$



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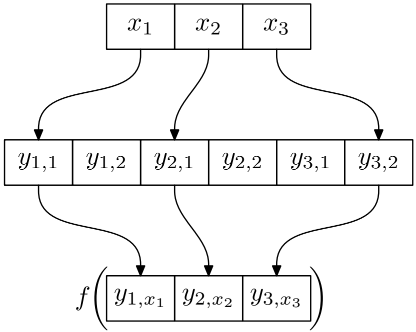
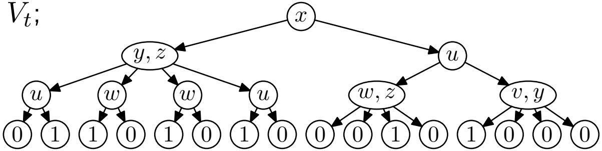
Building on ideas from e.g. [She08, BHP10]



Simulation of Protocols by Parallel Decision Trees [Val75]

Each node t in tree labelled by variables V_t; has $2^{|V_t|}$ outgoing edges



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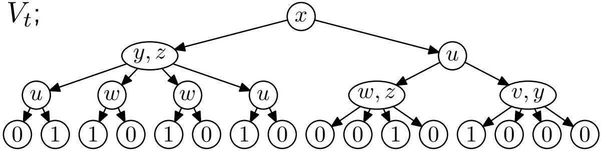

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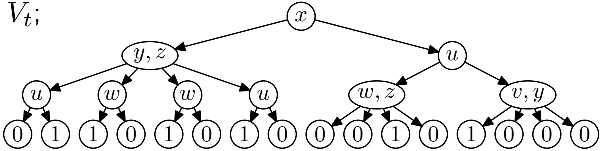

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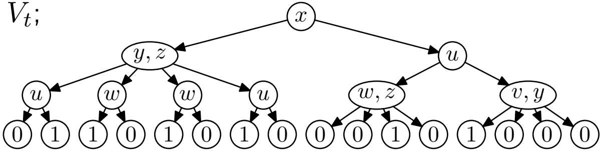

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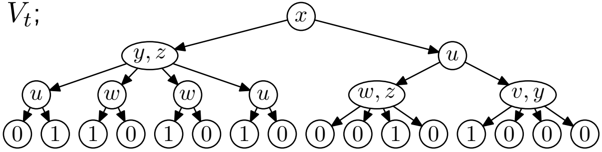

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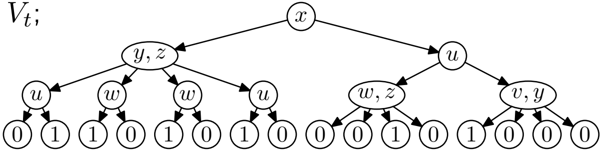

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Simulation theorem of protocol by decision tree (hard direction)

Let S search problem with domain $\{0,1\}^m$ and let $\ell = m^{3+\epsilon}$, $\epsilon > 0$. Then:
$\exists$ r-round real communication protocol in cost c solving $Lift_\ell(S)$
$\Rightarrow$ $\exists$ depth-r parallel decision tree solving S with $\mathcal{O}(c/\log \ell)$ queries.

Where to Get Formulas with Trade-off Properties?

Questions about time-space trade-offs fundamental in theoretical computer science

Well-studied (and well-understood) for pebble games modelling calculations described by DAGs

In particular, for black-white pebble game investigated by [CS76] and many others

Pebbling Contradictions

CNF formulas encoding black-white pebble game played on DAG G

1. u_1
2. u_2
3. u_3
4. $\overline{u}_1 \lor \overline{u}_2 \lor u_4$
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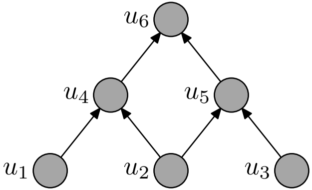

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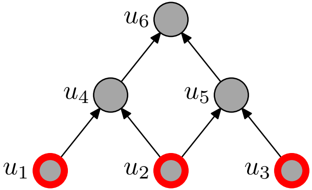

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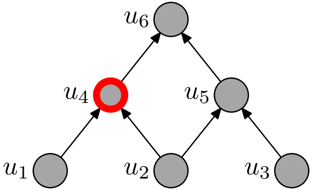

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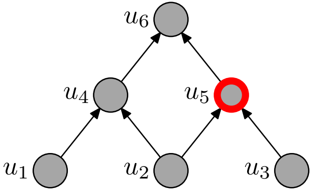

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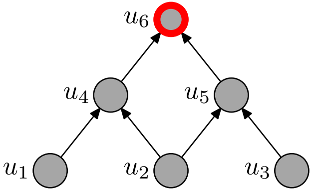

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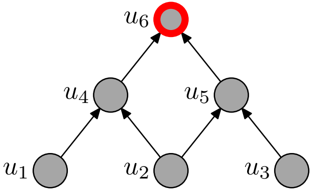

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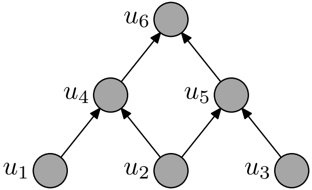

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Appeared in various contexts in e.g. [RM99, BEGJ00, BW01]

Used in [Nor06, NH08, BN08, BN11, BNT13] to study space and size-space trade-offs in resolution and polynomial calculus

Formulas inherit some DAG properties, but not enough — make them harder by lifting!

Lifted CNF Formulas

Given

- CNF formula F over variables $u_1, \ldots, u_n$
- lift length $\ell \in \mathbb{N}^+$

the lifted formula $Lift_\ell(F)$ has

- selector variables $\{x_{i,j}\}_{i\in[n],j\in[\ell]}$
- main variables $\{y_{i,j}\}_{i\in[n],j\in[\ell]}$
- for every $i \in [n]$ an auxiliary clause

$$x_{i,1} \lor x_{i,2} \lor \cdots \lor x_{i,\ell}$$

- for every $C = u_{i_1} \lor \cdots \lor u_{i_s} \lor \overline{u}_{i_{s+1}} \lor \cdots \lor \overline{u}_{i_t}$ in F and $(j_1, \ldots, j_t) \in [\ell]^t$ a main clause

$$\overline{x}_{i_1,j_1} \lor y_{i_1,j_1} \lor \cdots \lor \overline{x}_{i_s,j_s} \lor y_{i_s,j_s} \lor \overline{x}_{i_{s+1},j_{s+1}} \lor \overline{y}_{i_{s+1},j_{s+1}} \lor \cdots \lor \overline{x}_{i_t,j_t} \lor \overline{y}_{i_t,j_t}$$

Toy Example Lifted Pebbling Contradiction (Lift Length 2)

$$\begin{aligned}
& (x_{1,1} \lor x_{1,2}) \\
\land\ & (x_{2,1} \lor x_{2,2}) \\
\land\ & (x_{3,1} \lor x_{3,2}) \\
\land\ & (x_{4,1} \lor x_{4,2}) \\
\land\ & (x_{5,1} \lor x_{5,2}) \\
\land\ & (x_{6,1} \lor x_{6,2}) \\
\land\ & (\overline{x}_{1,1} \lor y_{1,1}) \\
\land\ & (\overline{x}_{1,2} \lor y_{1,2}) \\
\land\ & (\overline{x}_{2,1} \lor y_{2,1}) \\
\land\ & (\overline{x}_{2,2} \lor y_{2,2}) \\
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\land\ & (\overline{x}_{1,1} \lor \overline{y}_{1,1} \lor \overline{x}_{2,1} \lor \overline{y}_{2,1} \lor \overline{x}_{4,2} \lor y_{4,2}) \\
\land\ & (\overline{x}_{1,1} \lor \overline{y}_{1,1} \lor \overline{x}_{2,2} \lor \overline{y}_{2,2} \lor \overline{x}_{4,1} \lor y_{4,1}) \\
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\end{aligned}$$

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Lifted Pebbling Contradictions and the Simulation Theorem

Plug in the simulation theorem:

- From r-round real communication protocol in cost c solving $\mathit{Search}\big(\mathit{Lift}_\ell(\mathit{Peb}_G)\big)$
- Get depth-r parallel decision tree solving $\mathit{Search}\big(\mathit{Peb}_G\big)$ with $\mathcal{O}(c/\log \ell)$ queries

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Study pebble game on graph G, but other game than black-white pebbling

From Parallel Decision Trees to Dymond–Tompa Games

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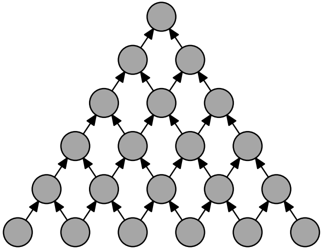

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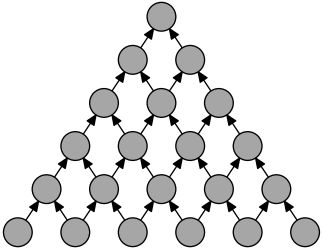

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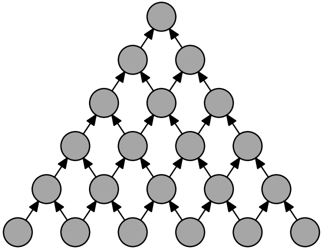

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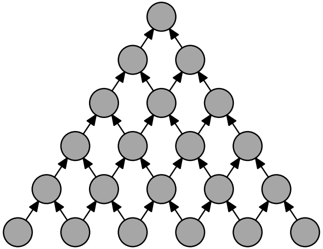

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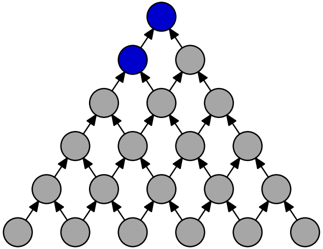

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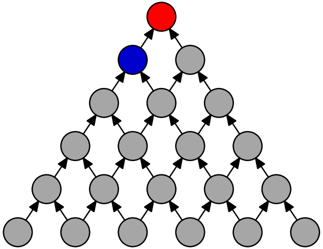

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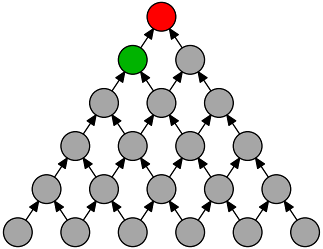

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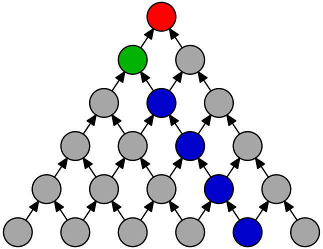

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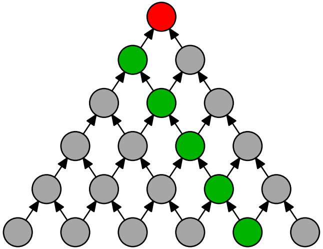

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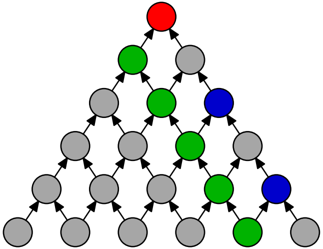

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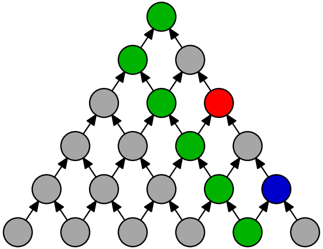

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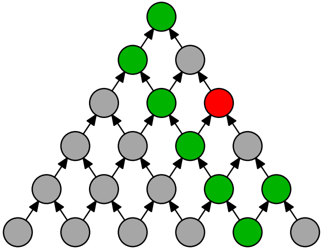

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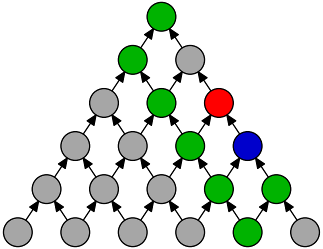

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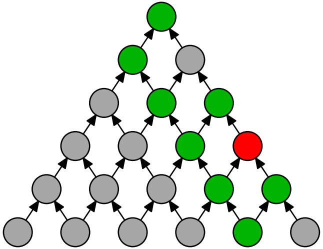

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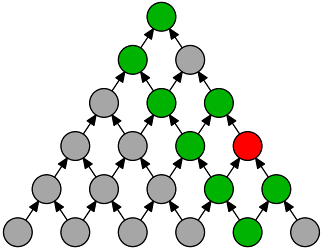

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Lemma

$\exists$ depth-r parallel decision tree for $Search(Peb_G)$ with $\leq c$ queries
$\Rightarrow$ Pebbler wins r-round Dymond–Tompa game on G in cost $\leq c+1$

Putting the Pieces Together

Prove round-cost trade-offs for Dymond–Tompa games on graphs G
(hacking graph constructions from [CS82, LT82, Nor12])

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Same communication lower bounds hold for $\mathit{Search}(\mathit{Lift}(\mathit{Peb}_G))$,
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$\Downarrow$

Cutting planes length-space trade-off for $\mathit{Lift}(\mathit{Peb}_G)$

Some Interesting Questions

Communication complexity

- Smaller length of lift?
- Simulation theorems for stronger communication models (in particular, for so-called DAG-like communication)

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- Reduction to black-white pebbling instead of Dymond–Tompa?
- Supercritical size-space trade-offs for Tseitin formulas à la [BNT13, BBI16]?
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And more to come in future lectures. . .

- But now it is time to switch to the board and do some proper proofs!

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