Proof Complexity as a Computational Lens

Final Lecture

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Outline

1. Proof Systems Covered in This Course
   - Resolution
   - Nullstellensatz and Polynomial Calculus
   - Cutting Planes
2. Proof Systems That We Didn't Manage to Cover
   - Stabbing Planes
   - Sherali–Adams and Sum-of-Squares
   - Resolution over Parities
3. More Proof Systems and Perspectives
   - Even Stronger Methods of Reasoning
   - Other Techniques
   - Applications of Proof Complexity in Other Areas

An Apology

- Slides prepared in great haste
- Pretty much all references missing
- See lecture notes for concrete lectures for more details
- Proof complexity chapter in *Handbook of Satisfiability* [BN21] should be good source
- Krajíček's book *Proof Complexity* [Kra19] better for advanced topics
- And semialgebraic proof systems covered in F&TTCS survey *Semialgebraic Proofs and Efficient Algorithm Design* [FKP19]

Resolution Length/Size Lower Bounds

In our lectures on resolution we covered some "classic" size lower bounds:

- Pigeonhole principle (PHP) formulas
- Tseitin formulas
- Random k-CNF formulas
- Clique-colouring formulas

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And some more recent results:

- Trade-offs between different complexity measures for resolution (length/size, width, space)
- Clique lower bound for regular resolution
- Non-automatability (efficient proof search for resolution is NP-hard)

Proof Techniques for Resolution

- Prosecutor-defendant game
- Random restrictions
- Size-width lower bounds
- Monotone feasible interpolation
- Decision tree reductions

Some Resolution Topics We Didn't Cover

- Pseudorandom generators (more about this later)
- Separations between different subsystems of resolution
- Polynomial simulation of resolution by conflict-driven clause learning (CDCL)

Resolution Width

Resolution width lower bounds for k-CNF formulas imply:

- length/size lower bounds (if width $\gg \sqrt{\#\text{ variables}}$)
- clause space lower bounds
- total space lower bounds (width squared)

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But width and clause space (almost) maximally separated

Open Problems for Resolution Space

- Does linear clause space lower bounds imply width/length lower bounds?
- Must a refutation in constant clause space also have polynomial length?
- Possible to exhibit supercritical trade-offs for
  - length/size vs. clause space with better parameters?
  - width vs. clause space for space larger than formula size?

More Open Problems for Resolution

- Tight bounds for weak PHP formulas
  - with n pigeonholes and $\gg n^2$ pigeons
  - also for graph PHP formulas
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  - worst-case
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- Understand resolution complexity of NP-complete problems? (Using good encodings)
- How hard is it to search for a shortest resolution refutation?

Nullstellensatz

- Only talked briefly about Nullstellensatz
- More interested in polynomial calculus
- Main focus has been on degree measure
- Degree lower bounds $\Leftrightarrow$ existence of designs

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Open problems:

- Size-degree trade-offs for Nullstellensatz with dual variables
- Also without dual variables, would be nice to have stronger trade-offs — related to reversible pebbling
- Size lower bounds for more concise representation of polynomials than linear combination of monomials — leads to superstrong ideal proof system!

Polynomial Calculus

- Models Gröbner basis computations
- Assumes polynomials represented as linear combinations of monomials
- Exponentially stronger than resolution (assuming use of dual variables)
- Again main focus on degree complexity measure
- Degree lower bounds from pseudo-reductions faking polynomial ideal reductions
- Superpolynomial size lower bounds for constant-degree input if degree $\gg \sqrt{\# \text{ variables}}$
- Less tools in toolbox than for resolution

Some Results for Polynomial Calculus

Some hard formulas for resolution are easy for polynomial calculus:

- Tseitin formulas on expander graphs if $\mathbb{F} = \mathrm{GF}(2)$ (do Gaussian elimination)
- Onto functional pigeonhole principle over any field (count modulo characteristic)

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But other formulas remain hard for polynomial calculus:

- Tseitin-like formulas for counting $\bmod\ p$ if $p \neq$ field characteristic
- "vanilla" PHP, onto PHP, and functional PHP formulas
- Random k-CNF formulas
- Colouring formulas (worst-case and average-case)

Some Questions Motivated by Algebraic Solving

- Gröbner basis algorithm works with respect to fixed order — obtain proof complexity separations between different orders?
- Efficient algorithms for polynomials with dual variables?
- Conflict-driven algebraic solving?

Polynomial Calculus: Additional Topics

Some topics we didn't talk about:

- Pseudorandom generators
- Lower bound techniques for concrete field characteristics
  - change to "Fourier basis"
  - immunity (axioms without low-degree implications)

Open Problems for Polynomial Calculus Size and Degree

- Combine immunity with generalized constraint-variable incidence graphs (CVIGs)?
- Improve techniques for degree lower bounds
  - dense linear ordering (DLO) formulas
  - homomorphism problems
  - dichotomy results for constraint satisfaction problems (CSPs)
- Lower bounds for pseudorandom generators
- Size lower bounds without using degree
  - weak PHP formulas
  - clique formulas

Open Problems for Polynomial Calculus Space

- Separate monomial space from resolution clause space(?)
- Optimal monomial space lower bounds for
  - Tseitin formulas
  - Functional PHP formulas
- Monomial space $\geq$ resolution width?
- Monomial space lower bounds for pebbling formulas
- Separations of degree and space independent of characteristic
- Supercritical size-space trade-offs independent of characteristic
- Total space lower bounds for polynomial-size formulas

(Easier to prove some space lower bounds without dual variables)

Polynomial Calculus over Roots of Unity

- Some recent, quite mysterious, results — can we gain better understanding?
- Prove implication degree lower bound $\Rightarrow$ size lower bound for single formula?
- Clean general result saying that if
  - constraint-variable incidence graph is expander and
  - constraints have property $\mathcal{P}$

  then size lower bound follows?
- Transformation between $\{0, 1\}$ and roots of unity can be viewed as extension variables — possible to deal with more general definitions?
- What about space lower bounds for polynomial calculus over roots of unity?

Cutting Planes

Recap of some basics

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Proof techniques:

- Monotone feasible interpolation
- Lifting theorems in "classic" communication complexity
- Lifting theorems in "DAG-like" communication complexity (more recent)
- Bottleneck counting (very recent)

Open Problems for Cutting Planes Size

- Better parameters for DAG-like lifting
- Proof techniques for non-lifted formulas
- Proof techniques for distinguishing syntactic derivation rules (e.g., different cuts)
- Lower bounds for random k-CNF formulas
- Is cutting planes with polynomially bounded coefficients weaker than general cutting planes?

Open Problems for Cutting Planes Space

- General cutting planes refutes any infeasible 0–1 ILP in line space 5
- Possible to prove line space lower bounds for cutting planes with polynomially bounded coefficients?
- True trade-offs for cutting planes with polynomially bounded coefficients that don't apply to general cutting planes?
- Related problems:
  - Round-efficient lifting theorems in other settings
  - "Algorithmic" parity decision tree lower bounds for pebbling formulas
- Size-space trade-offs for general cutting planes with (much) better parameters would also be nice

Algorithmic Challenges for Pseudo-Boolean Solving

Pseudo-Boolean (PB) solvers use cutting planes + SAT-inspired methods for 0–1 ILPs

Challenging to make competitive with conflict-driven clause learning (CDCL)

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   - How to keep coefficient sizes down to make integer arithmetic feasible?
   - How to compare and assess quality of constraints?

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2. Designing search and conflict analysis
   - Cutting planes much smarter method of reasoning than resolution
   - But this also makes it trickier to design smart search algorithms
   - Also harder to compare and assess quality of 0–1 linear inequalities

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3. Pseudo-Boolean solvers terrible for CNF input
   - Can try to rewrite CNF to more helpful 0-1 linear inequalities
   - Tricky to get this to work well in practice

Stabbing Planes

- Stabbing planes introduced in [BFI$^+$18] to model (more modern) 0–1 ILP solving
- Decision tree that
  - branches over 0–1 linear inequalities
  - gets LP solving for free (so terminate when residual LP infeasible over $\mathbb{R}$)
- Originally believed to be much stronger than cutting planes
- But stabbing planes with polynomially bounded coefficients simulated by general cutting planes with a quasi-polynomial blow-up [DT20, FGI$^+$21]
- And recently, lower bounds for stabbing planes shown via interpolation [GP24]

Sherali–Adams (SA) and Sum-of-Squares (SoS)

Refutation of $p_i \in \mathbb{R}[x_1, \ldots, x_n]$, $i \in [m]$, and $x_j^2 - x_j$, $j \in [n]$

Nullstellensatz

$$\sum_{i=1}^{m} q_i \cdot p_i + \sum_{j=1}^{n} r_j \cdot (x_j^2 - x_j) = 1$$

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$$\sum_{i=1}^{m} q_i \cdot p_i + \sum_{j=1}^{n} r_j \cdot (x_j^2 - x_j) + \sum_{k=1}^{t} \alpha_k \prod_{i \in \mathcal{P}_t} (1 - x_i) \cdot \prod_{j \in \mathcal{N}_t} x_j = -1$$

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Sum-of-squares (SoS) $(s_k \in \mathbb{R}[x_1, \ldots, x_n])$

$$\sum_{i=1}^{m} q_i \cdot p_i + \sum_{j=1}^{n} r_j \cdot (x_j^2 - x_j) + \sum_{k=1}^{s} s_k^2 = -1$$

Sherali–Adams, Sum-of-Squares, and Relations to Other Proof Systems

Sherali–Adams models linear programming (LP) hierarchies

Sum-of-squares models semidefinite programming (SDP) hierarchies
Strong connections to several best known approximation algorithms
(But Tseitin formulas are hard)

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Strict hierarchy (over $\mathbb{R}$):

- Nullstellensatz
- Sherali–Adams
- Sum of squares

Sum of squares is strictly stronger than polynomial calculus (over $\mathbb{R}$)
Sherali-Adams and polynomial calculus are incomparable [Ber18]

More Results and Open Problems for Sherali–Adams and Sum-of-Squares

- Separation between general Sherali–Adams and Sherali–Adams with polynomially bounded coefficients (unary Sherali–Adams or uSA) [GHJ$^+$24]
- What about different coefficient sizes in sum-of-squares?
- Average-case clique lower bounds for unary Sherali–Adams [dRPR23]
- Average-case colouring lower bounds for SoS [PX25], but (much) worse parameters than for polynomial calculus
- Size-degree lower bounds analogous to resolution [BW01] and polynomial calculus [IPS99] hold also for Sherali–Adams and SoS [AH19]
- What about size-degree trade-offs?
- Or non-automatability results?

Resolution over Parities

- Resolution, but clauses are disjunctions over parities
- First obstacle towards proving lower bounds for bounded-depth Frege with MOD connectives (more later)
- Currently very active area of research
- Size lower bounds, but only for bounded depth
- Current barrier at quadratic depth
- Better lifting theorems needed (ideally DAG-like)

Frege Proof Systems

- Standard natural deduction proof system taught in intro logics course
- Different flavours are polynomially equivalent
- Currently seems way beyond techniques for (unconditional) lower bounds
- Even lack of good candidates for hard formulas (except random k-CNF and other formulas that are too hard to prove lower bounds for)
- What about conditional lower bounds for assumptions weaker than NP $\neq$ coNP?

Bounded-Depth Frege Proof Systems

k-DNF resolution: clauses are k-DNF formulas (disjunctions of conjunctions)

- Random k-CNF formulas are hard
- Weak PHP formulas are not well understood
- Random restrictions turn into switching lemmas

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Bounded-depth Frege: formulas of arbitrary but constant depth

- Lower bounds for
  - PHP formulas
  - Tseitin formulas
- But weak PHP formulas are easy
- Major challenges to prove lower bounds for
  - random k-CNF formulas
  - random k-XOR formulas (not Tseitin formulas)

Bounded-Depth Frege and Circuit Complexity

Known results in circuit complexity:

- Depth hierarchy for bounded-depth circuits
- Strong lower bounds for bounded-depth circuits with MOD gates

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Analogous problems remain open in proof complexity:

- Depth hierarchy for bounded-depth Frege (for CNF formulas)
- Lower bounds for bounded-depth Frege with MOD connectives (this is why resolution over parities is so interesting)
- Switching lemmas for bounded-depth Frege are very complex
- Need other tools

Extended and Substitution Frege Proof Systems

- Extended Frege: Introduce new variable to be equivalent to subformula
- Substitution Frege: Recycle any subderivation in single step
- Believed to be exponentially stronger than Frege
- Known to be polynomially equivalent
- Open problem: Does this hold also if we define extension and substitution for weaker proof systems?

Ideal Proof System

- **Very** rough explanation of ideal proof system:
  Nullstellensatz, but represent polynomials as you like
- For instance, with arithmetic circuits
- Yields very strong proof system!
- Conditional results establishing relations with extended Frege and other proof systems
- Unconditional results for restricted arithmetic circuits

Bounded Arithmetic

- Bridge between logic and computational complexity theory
- Weak formal theories of arithmetic
- Peano Arithmetic, but
  - restricted power of induction hypotheses
  - restricted quantifiers
- Designed to capture feasible reasoning
- Correspondence between bounded arithmetic theories and proof systems
- Bounded arithmetic proof can be translated to family of propositional logic proofs

Some Interesting Proof Techniques Worth Closer Study

- Duality
- Reductions
  - via low-depth decision trees
  - via low degree polynomials
- Switching lemmas
- Pseudo-width
- Top-down analysis

Proof Complexity Applications in Computational Complexity Theory

Total NP search problems (TFNP)

- Tight correspondence between TFNP problems and proof systems
- Breakthrough results from proof complexity separations
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- But proof complexity gets stuck earlier
- Intriguing interplay between proof complexity, circuit complexity, and communication complexity

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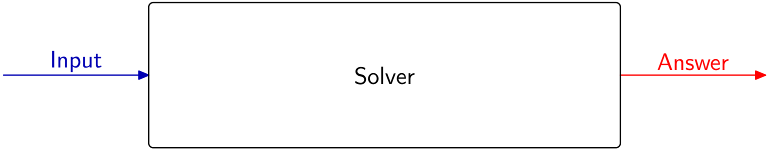
Extension complexity

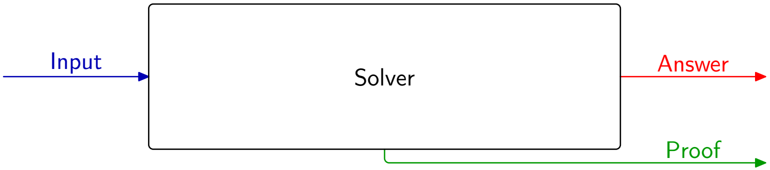
- Impossibility results for LP and SDP formulations
- Lower bounds for Sherali–Adams and sum-of-squares

Proof Complexity for Certified Combinatorial Solving



1. Run combinatorial solving algorithm on problem input

Proof Complexity for Certified Combinatorial Solving



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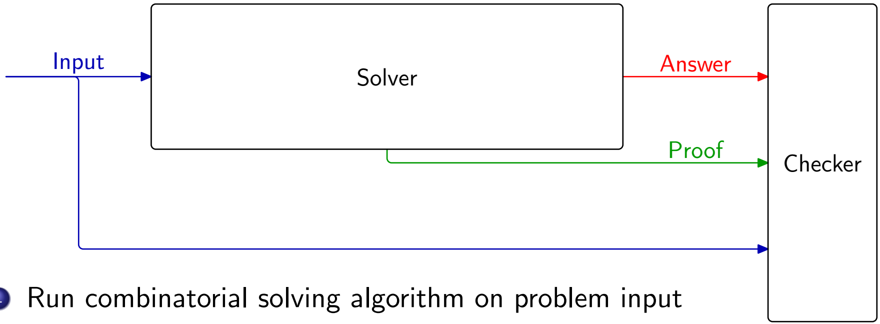

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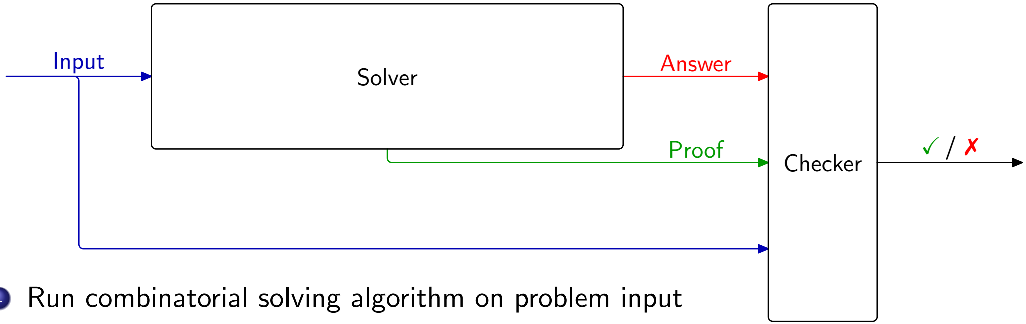

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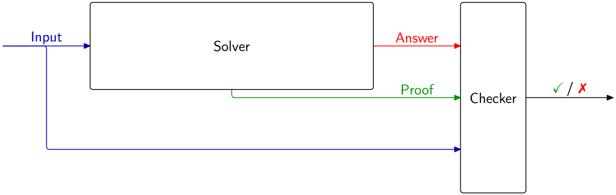
4. Verify that proof checker says answer is correct

Proof System Desiderata

Proof format for certifying solver should be



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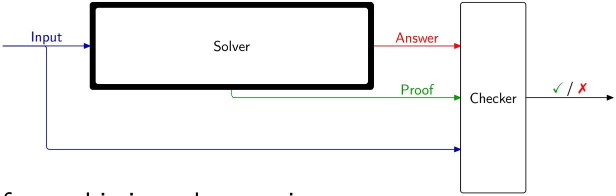

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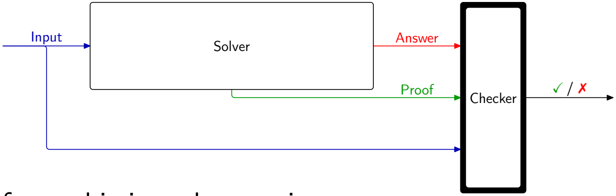

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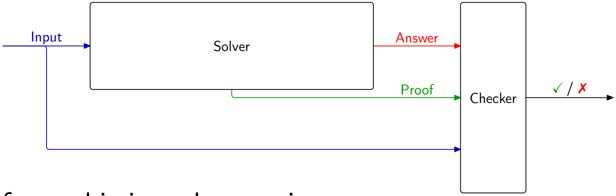

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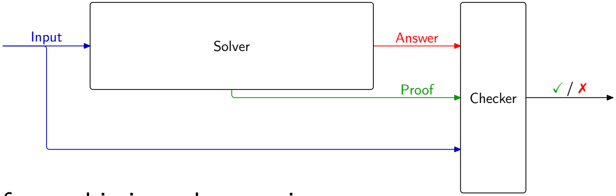

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Interesting problem to try to design suitable proof systems
(also for optimization problems and beyond Boolean format)

Redundance-Based Strengthening

C is redundant with respect to $\mathcal{F}$ if $\mathcal{F}$ and $\mathcal{F} \cup \{C\}$ are equisatisfiable

Want to allow adding such "redundant" constraints

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Redundance-based strengthening ([BT19, GN21], inspired by [JHB12])

C is redundant with respect to $\mathcal{F}$ if and only if there is a substitution ω (mapping variables to truth values or literals), called a witness, for which

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- Proof sketch for interesting direction: If α satisfies $\mathcal{F}$ but falsifies C, then $\alpha \circ \omega$ satisfies $\mathcal{F} \cup \{C\}$
- In a proof, the implication needs to be efficiently verifiable — every $D \in (\mathcal{F} \cup \{C\})\restriction_\omega$ should follow from $\mathcal{F} \cup \{\neg C\}$ either
  1. "obviously" or
  2. by explicitly presented derivation

Example: Deriving $a \leftrightarrow (x \wedge y)$ Using the Redundance Rule

Want to derive

$$2\overline{a} + x + y \geq 2 \qquad a + \overline{x} + \overline{y} \geq 1$$

using condition $\mathcal{F} \cup \{\neg C\} \models (\mathcal{F} \cup \{C\})\restriction_{\omega}$

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1. $\mathcal{F} \cup \{\neg(2\overline{a} + x + y \geq 2)\} \models (\mathcal{F} \cup \{2\overline{a} + x + y \geq 2\})\restriction_{\omega}$
   Choose $\omega = \{a \mapsto 0\}$ — $\mathcal{F}$ untouched; new constraint satisfied

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Example: Deriving $a \leftrightarrow (x \wedge y)$ Using the Redundance Rule

Want to derive

$$2\overline{a} + x + y \geq 2 \qquad a + \overline{x} + \overline{y} \geq 1$$

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   Choose $\omega = \{a \mapsto 1\}$ — $\mathcal{F}$ untouched; new constraint satisfied
   $\neg(a + \overline{x} + \overline{y} \geq 1)$ forces $x \mapsto 1$ and $y \mapsto 1$, hence $2\overline{a} + x + y \geq 2$ remains satisfied after forcing a to be true

Open Problems: Strength of Restricted Redundance Rules?

Adding redundance rule $\Rightarrow$ proof system polynomially equivalent to extended Frege

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1. What is the power of the redundance rule if we forbid new variables?
   For resolution + redundance known that:
   - Pigeonhole principle formulas easy
   - Tseitin formulas easy
2. What is the power of resolution with redundance if we only allow new variables $z \leftrightarrow C$ for previously derived clauses C?
   - Corresponds (kind of) to reasoning in core-guided MaxSAT solvers

Redundance and Dominance Rules in VeriPB (Slightly Simplified)

Redundance-based strengthening, optimization version with objective f [BGMN23]

Add constraint C to derived set $\mathcal{D}$ if exists witness substitution ω such that

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- Applying ω should strictly decrease f
- If so, don't need to show that $(\mathcal{D} \cup \{C\})\restriction_\omega$ implied!

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7. ...
8. Can't go on forever, so finally reach α' satisfying $\mathcal{F} \cup \{C\}$

Soundness of Dominance Rule (Continued)

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- Switch between different orders in same proof

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Yields proof system that is probably stronger than extended Frege [KT24]

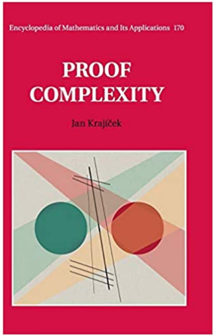
Symmetry-Aware Proof Systems

- With dominance rule, can support fully general symmetry breaking
  - Invent "objective function" that minimizes lexicographic order of satisfying assignment
  - Allows adding lex order constraints forbidding other solutions
  - Other approaches also possible (but beyond the scope of this discussion)
- Modern symmetry handling tools can solve many symmetric hard proof complexity formulas even during preprocessing
- But non-symmetric formulas are presumably still hard?
- Desirable to have lower bounds that remain valid also in the presence of state-of-the-art symmetry handling tools
  - Define "symmetry-aware" versions of resolution, polynomial calculus, cutting planes, ...
  - Develop techniques to prove lower bounds

Handbook of Satisfiability
(Especially chapter 7 ☺)

[BHvMW21]

Proof Complexity
by Jan Krajíček



[Kra19]

Summing up This Course

We focused on some proof systems corresponding to combinatorial solving algorithms:

- Resolution $\longleftrightarrow$ conflict-driven clause learning (CDCL)
- Nullstellensatz and polynomial calculus $\longleftrightarrow$ Gröbner bases
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We mentioned but didn't really go into any details about:

- Sherali–Adams and sums of squares $\longleftrightarrow$ LP and SDP hierarchies
- Stabbing planes $\longleftrightarrow$ integer linear programming
- Extended resolution $\longleftrightarrow$ SAT pre- and inprocessing
- ... and other stronger proof systems

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- Give ideas for new approaches
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Thank you for attending this course!

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